Differential Geometry. Homework 3. Due February 18th. Professor: Luis Fernández

1. The *symplectic group* is defined as

$$Sp(2n, \mathbb{R}) := \{A \in Gl(2n, \mathbb{R}) : A^t J A = J\}, \quad \text{where} \quad J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Prove that $Sp(2n, \mathbb{R})$ is a manifold, find its dimension, and describe $T_I Sp(2n, \mathbb{R})$ as a subspace of $\mathcal{M}(2n \times 2n, \mathbb{R})$. [NOTE: this group is often denoted $Sp(n, \mathbb{R})$ instead of $Sp(2n, \mathbb{R})$.]

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2. The *special unitary group* is defined as

$$SU(n) := \{A \in U(n) : \det(A) = 1\}.$$

Prove that $SU(n)$ is a manifold, find its dimension, and describe $T_I SU(n, \mathbb{R})$ as a subspace of $\mathcal{M}(2n \times 2n, \mathbb{R})$.

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3. Prove that $\begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}$ is not equal to e^A for any $A \in \mathcal{M}(2n \times 2n, \mathbb{R})$.

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4. Describe the tangent space at the identity as a subspace of $\mathcal{M}(n \times n, \mathbb{C})$ for the group $U(n)$ (defined in Hw 1).

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5. Let $\alpha(t)$ and $\beta(t)$ be smooth curves in a Lie group G such that $\alpha(0) = \beta(0) = e$, and let $\sigma(t) = \alpha(t)\beta(t)$. Prove that

$$\sigma'(0) = \alpha'(0) + \beta'(0).$$

[Hint: Consider the group multiplication map $m : G \times G \to G$, i.e. $m(g, h) = gh$. Let $X, Y \in T_e G$, and show that

$$dm(X, Y) = dm(X, 0) + dm(0, Y) = X + Y.]$$

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6. Let G be a Lie group. Prove that there is a neighbourhood of e where "square roots" are well defined. That is, for every g in that neighbourhood there is h (also in that neighbourhood) such that $h^2 = g$.

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7. Lee, exercise 20-11. (This answers a question posed in class last time.)

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8. Let G be a Lie group, and $c_g : G \to G$ be conjugation by $g \in G$, that is, $c_g(h) = ghg^{-1}$. The *adjoint representation* is the map $\text{Ad} : G \to Gl(\mathfrak{g})$ defined by

$$\text{Ad}(g) = (dc_g)_e.$$

(That is, $\text{Ad}(g)(X) = (dc_g)_e(X)$.)

a) Prove that $\text{Ad}(g^{-1}) = (\text{Ad}(g))^{-1}$; hence the range really is the space $Gl(\mathfrak{g}) = \{\text{Invertible linear maps from } \mathfrak{g} \text{ to } \mathfrak{g}\}$.

b) Prove that if $G = Gl(n, \mathbb{R})$, $\text{Ad}(A)(X) = AXA^{-1}$.

c) Define the map $\text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ as the differential of Ad at the identity (that is, $\text{ad}(X) = d\text{Ad}_e(X)$; note that $\text{ad}(X)$ is an element of $\mathfrak{gl}(\mathfrak{g}) = \{\text{Linear maps from } \mathfrak{g} \text{ to } \mathfrak{g}\}$). Prove that

$$\text{ad}(X)(Y) = [X, Y].$$

[NOTE: to prove this lat part you need the definition of the exponential map, which we may not have time to cover today. If not, prove it when $G = Gl(n, \mathbb{R})$.]