Differential Geometry. Homework 2. Due February 9th. Professor: Luis Fernández

1. Since $Gl(n,\mathbb{R})$ is an open submanifold of $\mathcal{M}(n \times n, \mathbb{R}) \cong \mathbb{R}^{n^2}$, one can identify $T_A Gl(n,\mathbb{R})$ with $\mathcal{M}(n \times n, \mathbb{R})$. Consider the map $\det : Gl(n,\mathbb{R}) \to \mathbb{R}$. Prove that if $X \in T_A Gl(n,\mathbb{R})$ (so X is an $n \times n$ matrix), then $\det_{*A}(X) = \det(A)\,\mathrm{Tr}(A^{-1}X)$ (where "Tr" is the trace).

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2. Consider the set

$$M = \{(L_1, L_2) : \{0\} \subset L_1 \subset L_2 \subset \mathbb{R}^3\}.$$

where L_1 y L_2 are vector subspaces of $\mathbb{R}^3$ of dimensions 1 and 2 respectively (in other words, the inclusions in $\{0\} \subset L_1 \subset L_2 \subset \mathbb{R}^3\}$ are strict). Show that M can be given the structure of a manifold and find its dimension.

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3. The three sphere S^3 can be written as the set of pairs $(z,w) \in \mathbb{C}^2$ such that $|z|^2 + |w|^2 = 1$. Hence we can consider the map $\pi : S^3 \subset \mathbb{R}^4 \to \mathbb{CP}^1$ given by

$$\pi(z,w) = [z : w],$$

where $[z : w]$ are homogeneous coordinates in $\mathbb{CP}^2$, so they denote the complex line in $\mathbb{C}^2$ spanned by (z,w).
Prove that for any $q \in \mathbb{CP}^1$, the set $\pi^{-1}(q)$ is isomorphic to an $S^1 \subset S^3$. Then prove that $(\pi, S^3, \mathbb{CP}^1, S^1)$ is a fibre bundle. (For completeness, the definition of fiber bundle is given below.) Some remarks:

1) This generalizes to $(\pi, S^{2n+1}, \mathbb{CP}^n, S^1)$ (your proof probably works for this general case essentially with no change). These are called the *Hopf fibrations*.
2) Combining this with exercise 4 of homework 1, this shows that S^3 is a fiber bundle over S^2 with fiber S^1. In particular, if we remove a point from S^2 and its corresponding fiber in S^3, topologically we obtain a product bundle over a disk with fibre S^1 (in other words, a solid torus). Hence, S^3 minus a circle gives a solid torus. Try to visualize this. Can you do a picture? Also, try to visualize the fibration itself.

A *fiber bundle* over M with fiber F is a tuple (π, E, M, F), with E, M, F manifolds and $\pi : E \to M$ smooth such that

(i) π is onto.
(ii) For every $p \in M$ there is a in open $U \ni p$ and diffeomorphisms $\psi : \pi^{-1}(U) \to U \times F$ (called "trivializations") such that $\pi_U \circ \psi = \pi$, where $\pi_U : U \times F \to U$ is the projection into the first factor.

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4. Hodge star operator. This is from Warner, Chapter 2, Exercise 13:
Let V be an n-dimensional real inner product space, with the inner product denoted by $\langle\ ,\rangle$. We extend the inner product from V to all of $\Lambda(V)$ by setting the inner product of elements which are homogeneous of different degrees equal to zero, and by setting

$$\langle w_1 \wedge \cdots \wedge w_p, v_1 \wedge \cdots \wedge v_p\rangle = \det\langle w_i, v_j\rangle$$

(that is, the determinant of the matrix whose ij entry is $\langle w_i, v_j\rangle$) and then extending bilinearly to all of $\Lambda^p(V)$. Prove that if $\{e_1, \dots, e_n\}$ is an orthonormal basis of V, then the basis

$$\{e_{1_i} \wedge \cdots \wedge e_{i_k}\}, \quad \text{where } 1 \le i_1 < \dots, < i_k \le n, \text{ and } k = 0, 1, \dots, n$$

is an orthonormal basis for $\Lambda(V)$.
Since $\Lambda^n(V)$ is one-dimensional, $\Lambda^n(V) \setminus \{0\}$ has two components. An orientation on V is a choice of a component of $\Lambda^n(V) \setminus \{0\}$. If V is an oriented inner product space, then there is a linear transformation

$$\star : \Lambda(V) \to \Lambda(V),$$

called star, which is well-defined by the requirement that for any orthonormal basis $\{e_1, \ldots, e_n\}$ of V (in particular, for any re-ordering of a given basis),

$$\star(1) = \pm e_1 \wedge \cdots \wedge e_n \qquad\qquad \star(e_1 \wedge \cdots \wedge e_n) = \pm 1$$

$$\star(e_1 \wedge \cdots \wedge e_p) = \pm e_{p+1} \wedge \cdots \wedge e_n$$

where one takes "+" if $e_1 \wedge \cdots \wedge e_n$ en lies in the component of $\Lambda^n(V) \setminus \{0\}$ determined by the orientation and "−" otherwise. Observe that

$$\star : \Lambda^p(V) \to \Lambda^{n-p}(V).$$

Prove that on $\Lambda^p(V)$,

$$\star\star = (-1)^{p(n-p)}.$$

Also prove that for arbitrary $v, w \in \Lambda^p(V)$, their inner product is given by

$$\langle v, w \rangle = \star(w \wedge \star v) = \star(v \wedge \star w).$$

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5. Prove that the map $\pi : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{RP}^n$ given by $\pi(x_1, \ldots, x_n) = [x_1 : \cdots : x_n]$ is a submersion, and the kernel of π_{*p} is the subspace of $T_p\mathbb{R}^{n+1}$ generated by p.
[HINT: you can factor it through S^n, that is, write it as a map from $\mathbb{R}^{n+1} \setminus \{0\}$ to S^n composed with the projection from S^n to $\mathbb{RP}^n$ (which is a local diffeomorphism).

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6. (This is in Lee 1, ex. 21-14) The Plücker embedding: Consider the map $\rho : Gr(k, \mathbb{R}^n) \to \mathbb{P}(\Lambda^k\mathbb{R}^n)$ given by

$$\rho(P) = [v_1 \wedge \cdots v_k] \;\text{ if }\; S = \text{span}\{v_1, \ldots, v_k\}.$$

Prove that ρ is well defined and is a smooth embedding whose image is the set of equivalence classes of nonzero decomposable elements of $\Lambda^k(\mathbb{R}^n)$.
[Hint: Let $P \in Gr(k, \mathbb{R}^n)$, and suppose that $v_1, \ldots, v_k$ span P. Consider a curve $v(t)$ in $\mathbb{R}^n$ with $v(0) = v_1$ and $v'(0) = X \notin P$. Then the curve $\text{span}\{v(t), v_2, \ldots, v_k\}$ is a nonconstant curve in $Gr(k, \mathbb{R}^n)$ and determines a nonzero element of $T_P Gr(k, \mathbb{R}^n)$. You only need to prove that the derivative with respect to t of the curve in $\mathbb{P}(\Lambda^k\mathbb{R}^n)$ given by $c(t) := \rho(\,\text{span}(\{v(t), v_2, v_k)\,\}$ is not zero. To do this you can write $c(t)$ as $v(t) \wedge v_2 \wedge v_k$ composed with the map π from the previous exercise.]