Differential Geometry. Homework 1. Due February 4th. Professor: Luis Fernández

1. Let $(\mathbb{R}^2, \phi)$ be the trivial chart of $\mathbb{R}^2$ (i.e. $\phi(x,y) = (x,y)$). Let (V, ψ) be the chart in $\mathbb{R}^2$ defined by $V = \{(x,y) \in \mathbb{R}^2 : x > 0\}$ and $\psi(x,y) = (r,\theta)$ (i.e the point (x,y) in polar coordinates, so $(x,y) = (r\cos\theta, r\sin\theta)$). Find $dx \wedge dy$ in terms of $dr \wedge d\theta$.

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2. Let $\omega \in \Lambda^2(\mathbb{R}^3)$ be defined by $\omega_p(X_p, Y_p) = p \cdot (X_p \times Y_p)$ for any vectors $X_p, Y_p \in T_p\mathbb{R}^3$, where '$\times$' denotes the usual cross product of $\mathbb{R}^3$. Find $d\omega$.

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3. Let $F : \mathbb{RP}^1 \to S^1$ be defined by

$$F([x:y]) = \frac{1}{x^2+y^2}\left(2xy, x^2-y^2\right)$$

(Recall that $[x:y] \in \mathbb{RP}^1$ denotes the straight line in $\mathbb{R}^2$ passing through $(0,0)$ and (x,y).) Show that F is a difeomorphism.

Find dF in local coordinates at a point of the form $[x:1]$. (You have to take coordinate charts for $\mathbb{RP}^1$ and S^1 and express dF in terms of these coordinates.)

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4. Let $F : \mathbb{CP}^1 \to S^2$ be defined by

$$F([z:w]) = \frac{1}{|z|^2+|w|^2}\left(2\ \mathrm{Re}(z\overline{w}), 2\ \mathrm{Im}(z\overline{w}), |w|^2-|z|^2\right)$$

(Recall that $[z:w] \in \mathbb{CP}^1$ denotes the complex subspace of $\mathbb{C}^2$ given by $\{\lambda(z,w) : \lambda \in \mathbb{C}\}$.) Show that F is a difeomorphism.

Now consider the following charts:

- (U_2, φ_2) of $\mathbb{CP}^1$, where $U_2 = \{[u+iv:1] : u,v \in \mathbb{R}\} \subset \mathbb{CP}^2$ and $\phi_2([u+iv:1]) = (u,v)$.
- (V_N, p_N) of S^2, where $V_N = S^2 \setminus \{(0,0,1)\}$ and $p_N(x,y,z) = (x_1,x_2)$ with $x_1 = \dfrac{x}{1-z}$, $x_2 = \dfrac{y}{1-z}$.

Then $\left\{\dfrac{\partial}{\partial u}, \dfrac{\partial}{\partial v}\right\}$ is a basis of $T_{[u+iv:1]}\mathbb{CP}^2$ and $\left\{\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}\right\}$ is a basis of $T_{(x,y,z)}S^2$.

In these bases, find the matrix of dF at a point $[u+iv:1]$. (You can skip finding the actual derivatives because it is long, but indicate which map you have to differentiate and with respect to which variables.)

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5. The matrix group $U(n)$ is defined as

$$U(n) = \{A \in Gl(n,\mathbb{C}) : A^*A = I\},$$

where $A^* := \overline{A}^t$. Prove that $U(n)$ is a submanifold of $Gl(n,\mathbb{C})$.

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6. Let $f : M \to \mathbb{R}^n$ be a C^∞ function, where M is a compact manifold of dimension n. Show that f must have a singular point (i.e. a point where the rank of df is less than n).

[HINT: consider $\|f\|^2 : M \to \mathbb{R}$.]

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7. An element of ΛV is called *homogeneous* if it is in $\Lambda^k V$ for some k (so for example, $dx \wedge dy + dx$ is *not* homogeneous, whereas $dx \wedge dy + dx \wedge dz$ is). It is called *decomposable* if it can be written as $v_1 \wedge v_2 \wedge \cdots \wedge v_k$ for some vectors $v_i \in V$. Show that

a) If dim $V \leq 3$, then every homogeneous element is decomposable.

b) If dim $V > 3$, give an example of an indecomposable homogeneous element.

c) Let $\alpha \in \Lambda^k V$. Is $\alpha \wedge \alpha = 0$?

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8. Recall that the flow of a vector field X on a manifold M is the map σ from a subset of $\mathbb{R} \times M$ to M, such that $\frac{d\sigma_t(p)}{dt} = X_{\sigma_t(p)}$. The Lie derivative with respect to X of a differential form $\alpha \in A(M)$ at a point p is defined as

$$L_X\alpha = \lim_{t\to 0} \frac{(\sigma_t)^*\alpha_{\sigma_t(p)} - \alpha_p}{t}.$$

Prove:

a) If $f \in A^0(M)$, then $L_X f = Xf$. Likewise, the Lie derivative with respect to X of a vector field $X \in \mathfrak{X}(M)$ is given by

$$L_X Y = \lim_{t\to 0} \frac{(\sigma_{-t})_* Y_{\sigma_t(p)} - Y_p}{t}.$$

b) Prove that $L_X Y = [X, Y]$.

[HINT: Take $f \in C^\infty(M)$. Then you want to show that

$$\frac{d}{dt}\Big|_{t=0} (\sigma_{-t})_* Y_{\sigma_f(p)}(f) = X_p(Yf) - Y_p(Xf).$$

This amounts to unraveling all the definitions of push forward, derivative, etc. You can check Warner, Chapter 2, if you get stuck.]

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9. Let X, $Y \in \mathfrak{X}(M)$ with corresponding 1-parameter groups given by θ_t and η_s Let $p \in M$ and let

$$\beta_p(t) = \eta_{-\sqrt{t}} \circ \theta_{-\sqrt{t}} \circ \eta_{\sqrt{t}} \circ \theta_{\sqrt{t}}(p).$$

For $f \in C^\infty(M)$, show that

$$[X, Y]_p(f) = \frac{d}{dt}\Big|_0 f(\beta_p(t)).$$

[HINT: do a Taylor expansion up to order 2 of the C^∞ function $\gamma(t_1, s_1, t_2, s_2) = \eta_{s_2} \circ \theta_{t_2} \circ \eta_{s_1} \circ \theta_{t_1}(p)$.] (NOTE: this is from Warner, exercise 6 of chapter 2.)

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10. Let $\xi \in V$. Prove that the composition

$$\Lambda^p V \xrightarrow{\wedge\xi} \Lambda^{p+1} V \xrightarrow{\wedge\xi} \Lambda^{p+2}$$

is an exact sequence (that is, the image of the first map is the kernel of the second).

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11. Cartan's Lemma: Let $\omega_1, \ldots, \omega_k$ be 1-forms on M^n which are linearly independent pointwise. Let $\theta_1, \ldots, \theta_k$ be 1-forms on M such that

$$\sum_{i=1}^{k} \theta_i \wedge \omega_i = 0.$$

Prove that there exist C^∞ functions A_{ij} on M with $A_{ij} = A_{ji}$ such that

$$\theta_i = \sum_{j=1}^{k} A_{ij}\omega_j.$$

[Hint: Extend the ωs to a basis (locally).]
