Differential Geometry. Homework 10. Professor: Luis Fernández

- NOTE: You do not need to submit the homework anymore.
- But please do it, check the solutions, and ask any questions you have.
- Further, there are many exercises in this assignment (however, several of them are quite quick and you do not need to write it nicely). I mark with an asterisk the ones that, if you do not have the time, should choose.

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1. Tu, second edition, exercise 18.9.

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2. Tu, second edition, exercises 19.2*, 19.11*.

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3. Tu, second edition, exercise 20.7, 20.10.

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4. Tu, second edition, exercise 21.5*, 21.9*(this is very easy if you use something from a previous assignment).

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5. Let θ denote the 1-form on $\mathbb{R}^3$ given by $\theta = y dz + dx$. Prove that there is no smoothly immersed surface $f : \Sigma \to \mathbb{R}^3$ such that $f^*\theta = 0$. [Hint: compute $d\theta$.]

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6. *Do both parts.

a) State a definition of *orientable* for a smooth manifold.

b) Let M and N be smooth manifolds. Show $M \times N$ is orientable if and only if M and N are both orientable.

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7. *Let $X = x\dfrac{\partial}{\partial x} - x\dfrac{\partial}{\partial y}$ and $Y = (x+y)\dfrac{\partial}{\partial y}$ be vector fields in $\mathbb{R}^2$.

a) Verify that $[X, Y] = 0$ everywhere by direct computation.

b) Find explicit formulas for the flow of X and the flow of Y.

c) Use these to construct coordinates (u, v) near the point $p = (1, 0)$ so that $X = \dfrac{\partial}{\partial u}$ and $Y = \dfrac{\partial}{\partial v}$.

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8. *Recall that if $(M, [\omega^M])$ is an orientable manifold with boundary, then the induced orientation of ∂M is given by $[i_X \omega^M]$, where X is an outward pointing vector field.

Consider the closed unit ball $B^n \subset \mathbb{R}^n$, with the usual orientation of $\mathbb{R}^n$ given by $[dx^1 \wedge \ldots \wedge dx^n]$. Show that the (induced) orientation of $S^{n-1} = \partial B^n$ is given by $[\omega^{S^{n-1}}]$, where

$$\omega^{S^{n-1}} = x^1\ dx^2 \wedge dx^3 \wedge \ldots \wedge dx^n - x^2\ dx^1 \wedge dx^3 \wedge \ldots \wedge dx^n + \cdots + (-1)^{n+1} x^n\ dx^1 \wedge dx^2 \wedge \ldots \wedge dx^{n-1}.$$

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9. *Suppose $n \geq 1$, and let $\alpha : S^n \to S^n$ be the antipodal map: $\alpha(x) = -x$. Show that α is orientation-preserving if and only if n is odd.

[Hint: The previous exercise gives you an orientation form of S^n.]

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10. *Prove that the real projective space $\mathbb{RP}^n$ is orientable if n is odd. [Hint: Use the previous exercise.]

What if n is even?