(1) $\vec{r}(t) = (2t, 1-3t, 5+4t)$.

a) $\vec{r}\,'(t) = (2, -3, 4)$, and therefore

$$|\vec{r}\,'(t)| = \sqrt{2^2 + (-3)^2 + 4^2} = \sqrt{4+9+16}$$
$$= \sqrt{29}.$$

$\boxed{\text{Length of } \vec{r} \text{ from } t=2 \text{ to } t=5}$

$$= \int_2^5 |\vec{r}\,'(t)|\,dt = \int_2^5 \sqrt{29}\,dt = \sqrt{29}\,t\Big]_2^5$$
$$= \sqrt{29}\,(5-2)$$
$$= \boxed{3\sqrt{29}}$$

b) $$s = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t \sqrt{29}\,du = \sqrt{29}\,u\Big]_0^t = \sqrt{29}\,t$$

Therefore $s = \sqrt{29}\,t$, and therefore,

$$\boxed{t = \frac{s}{\sqrt{29}}}$$

The curve, reparametrized by arc length is

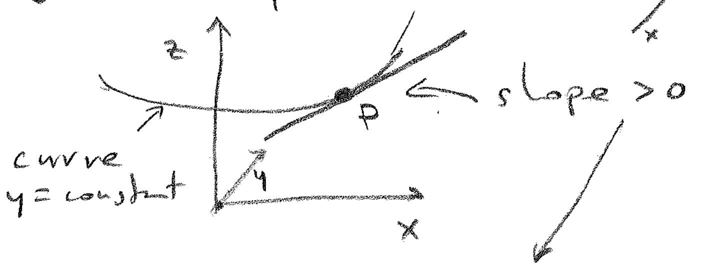
$$\boxed{\vec{r}(s) = \left(\frac{2\cdot s}{\sqrt{29}}, 1 - \frac{3s}{\sqrt{29}}, 5 + \frac{4s}{\sqrt{29}}\right)}$$

c) $\vec{T}(s) = \frac{d\vec{r}}{ds} = \left(\frac{2}{\sqrt{29}}, \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}\right).$

$\frac{d\vec{T}}{ds} = (0,0,0)$, and therefore

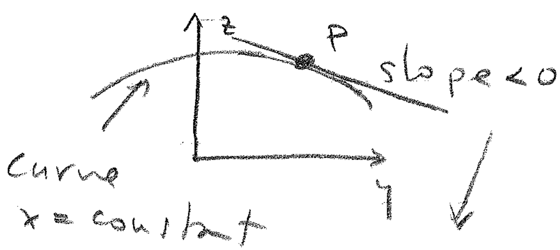
$$\boxed{\kappa = \left|\frac{d\vec{T}}{ds}\right| = 0.}$$

(2) Look at the picture from the negative y-axis:



Therefore, $\boxed{\frac{\partial f}{\partial x} > 0}$

To see $\frac{\partial f}{\partial y}$, look at the picture from the positive x-axis:



Therefore, $\boxed{\frac{\partial f}{\partial y} < 0}$

3

a) $$\lim_{(x,y)\to(0,0)} \frac{y^4}{x^4+3y^4}.$$

If we approach (0,0) via (x,0) (i.e., by the x-axis, y=0) we get

$$\lim_{x\to 0} \frac{0^4}{x^4+3\cdot 0^4} = \underline{\underline{0}}.$$

If we approach via (0,y) (i.e. the y-axis or x=0), we get

$$\lim_{y\to 0} \frac{y^4}{0^4+3y^4} = \lim_{y\to 0} \frac{\cancel{y^4}}{3\cancel{y^4}} = \underline{\underline{\frac{1}{3}}}$$

$\boxed{\text{Since } 0 \neq \frac{1}{3}, \text{ the limit does not exist}}$

b) $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$; write $x = r\cos\theta$
$y = r\sin\theta$,
if $(x,y)\to(0,0)$, then
$r\to 0^+$. We get:

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = \lim_{r\to 0^+} \frac{r^2\cos\theta\sin\theta}{\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}}$$

$$= \lim_{r\to 0^+} \frac{r^2\cos\theta\sin\theta}{\sqrt{r^2\underbrace{(\cos^2\theta+\sin^2\theta)}_{=1}}}$$

$$= \lim_{r\to 0^+} \frac{r^2\cos\theta\sin\theta}{\sqrt{r^2}} = \lim_{r\to 0^+} \frac{r^2\cos\theta\sin\theta}{r}$$

$$= \lim_{r\to 0^+} r\cos\theta\sin\theta = \boxed{0}$$

Therefore, limit $= 0$

4) a) $f(x,y) = xye^{xy}$.

$$\boxed{\frac{\partial f}{\partial x} = ye^{xy} + xy^2 e^{xy}}$$

$$\boxed{\frac{\partial f}{\partial y} = xe^{xy} + x^2 y e^{xy}}$$

b) $f(x,y) = (2x+3y)^{10}$

$$\boxed{\frac{\partial f}{\partial x} = 10(2x+3y)^9 \cdot 2}$$

$$\boxed{\frac{\partial f}{\partial y} = 10(2x+3y)^9 \cdot 3}$$

(5) $f(x,y) = x^3y^5 + 2x^4y$

$$\frac{\partial f}{\partial x} = 3x^2y^5 + 8x^3y$$

$$\frac{\partial f}{\partial y} = 5x^3y^4 + 2x^4$$

a) $\boxed{\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \boxed{6xy^5 + 24x^2y}}$

b) $\boxed{\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \boxed{15x^2y^4 + 8x^3}}$

c) $\boxed{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}} = \boxed{15x^2y^4 + 8x^3}$ by Clairaut's theorem.

d) $\boxed{\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \boxed{20x^3y^3}}$

⑥ $f(x,y) = x\sqrt{y}$.

a) $L(x,y) = f(1,4) + \frac{\partial f}{\partial x}(1,4)\,(x-1) + \frac{\partial f}{\partial y}(1,4)(y-4)$.

$\frac{\partial f}{\partial x} = \sqrt{y}$; $\frac{\partial f}{\partial x}(1,4) = \sqrt{4} = 2$.

$\frac{\partial f}{\partial y} = \frac{x}{2\sqrt{y}}$; $\frac{\partial f}{\partial y}(1,4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$.

$f(1,4) = 1 \cdot \sqrt{4} = 2$.

Therefore,

$$\boxed{L(x,y) = 2 + 2(x-1) + \frac{1}{4}(y-4).}$$

b) The equation of the tangent plane at (1,4,2) is $z = L(x,y)$, where $L(x,y)$ is the linearization at (1,4). Therefore,

$$\boxed{z = 2 + 2(x-1) + \frac{1}{4}(y-4)}$$

is the equation.

⑦ $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

If $t=3$, then $x = g(3) = 2$
$y = h(3) = 7$.

$\frac{\partial f}{\partial x}(2,7) = 6$; $\frac{\partial f}{\partial y}(2,7) = -8$.

$\frac{dx}{dt}(3) = g'(3) = 5$. $\frac{dy}{dt}(3) = h'(3) = -4$.

Substituting in the first line,

$$\frac{dz}{dt}(3) = 6 \cdot 5 + (-8)\cdot(-4)$$
$$= 30 + 32$$
$$= \boxed{62}$$

8) $R = \ln(u^2+v^2+w^2)$

$u = x+2y, \quad v = 2x-y, \quad w = 2xy.$

Using the chain rule,

$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{2u}{u^2+v^2+w^2}\cdot 1 + \frac{2v}{u^2+v^2+w^2}\cdot 2 + \frac{2w}{u^2+v^2+w^2}\cdot 2y.$$

If $x=1, y=1$, then $u = 1+2=3$

$v = 2-1=1$

$w = 2\cdot1\cdot1=2$. Therefore,

$$\frac{\partial R}{\partial x}(1,1) = \frac{2\cdot3}{3^2+1^2+2^2} + \frac{2\cdot1}{3^2+1^2+2^2}\cdot 2 + \frac{2\cdot2}{3^2+1^2+2^2}\cdot 2\cdot1$$

$$= \boxed{\frac{18}{14} = \frac{9}{7}}$$

$$\frac{\partial R}{\partial y} = \frac{2u}{u^2+v^2+w^2}\cdot 2 + \frac{2v}{u^2+v^2+w^2}\cdot(-1) + \frac{2w}{u^2+v^2+w^2}\cdot 2x$$

$$\Rightarrow \frac{\partial R}{\partial y}(1,1) = \frac{2\cdot3}{3^2+1^2+2^2}\cdot 2 + \frac{2\cdot1}{3^2+1^2+2^2}(-1) + \frac{2\cdot2}{3^2+1^2+2^2}\cdot 2\cdot1$$

$$= \frac{18}{14} = \boxed{\frac{9}{7}}$$

9 $f(x,y) = \sin(2x+3y)$

a) $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$

$= \left(\cos(2x+3y)\cdot 2, \cos(2x+3y)\cdot 3\right).$

$\nabla f(-6,4) = \left(\cos(2\cdot(-6)+3\cdot 4)\cdot 2, \cos(2\cdot(-6)+3\cdot 4)\cdot 3\right)$

$= (2\cos 0, 3\cos 0) = (2,3).$

b) $D_{\vec{u}} f(-6,4) = \nabla f(-6,4) \cdot \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$= (2,3)\cdot\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \boxed{\frac{2\sqrt{3}+3}{2}}$

(10) Maximum rate of change at $(2,4)$

is $|\nabla f(2,4)|$, and it occurs in the direction of $\nabla f(2,4)$ [This is a theorem].

$f(x,y) = \frac{y^2}{x}$.

$\nabla f = \left(-\frac{y^2}{x^2}, \frac{2y}{x}\right)$; $\nabla f(2,4) = \left(-\frac{16}{4}, \frac{\cancel{2}\cdot 4}{\cancel{2}}\right)$
$= (-4, 4)$.

Maximum rate of change is
$|(-4,4)| = \sqrt{(-4)^2 + 4^2} = \sqrt{32} = \boxed{4\sqrt{2}}$

It happens in the direction of $\nabla f = (-4,4)$, that is,

$D_{\vec{u}} f(2,4) = 4\sqrt{2}$,

where $\vec{u} = \frac{(-4,4)}{4\sqrt{2}} = \boxed{\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)}$