1 a) Use ratio test:

$$a_n = \frac{x^n}{n 2^n};$$

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\dfrac{|x|^{n+1}}{(n+1)2^{n+1}}}{\dfrac{|x|^n}{n2^n}} =$$

$$= \lim_{n\to\infty} \frac{|x|^{\cancel{n+1}} \cdot n \cancel{2^n}}{\cancel{|x|^n} \cdot (n+1) 2^{\cancel{n+1}}} = \frac{|x|}{2} \lim_{n\to\infty} \frac{n}{n+1}$$

$$= \frac{|x|}{2}.$$

Therefore, the seris is absolutely convergent when $\frac{|x|}{2} < 1$, or $|x| < 2$, and therefore the radiu of convergence is:

$$\boxed{R = 2}$$

b) The interval of convergence is either $[-2, 2]$ or $(-2, 2]$ or $[-2, 2)$ or $(-2, 2)$, so we have to check at $x = 2$ and $x = -2$.

[1b cont]

At $x=2$,

$$\sum_{n=1}^{\infty} \frac{\cancel{2^n}}{n\cancel{2^n}} = \sum_{n=1}^{\infty} \frac{1}{n} \text{ divergent (harmonic series)}$$

At $x=-2$, $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n \cancel{2^n}}{n\cancel{2^n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$,

which converges by the alternating series test.

Therefore, the series converges at $x=-2$ and diverges at $x=2$, and therefore the interval of convergence is

$$\boxed{[-2, 2)}$$

(2) $f(x) = \sum_{n=0}^{\infty} c_n \left(x - \frac{\pi}{2}\right)^n$, with $c_n = \dfrac{f^{(n)}\left(\frac{\pi}{2}\right)}{n!}$

$f(x) = \cos x$. $f\left(\frac{\pi}{2}\right) = 0 \to c_0 = 0$

$f'(x) = -\sin x$; $f'\left(\frac{\pi}{2}\right) = -1 \to c_1 = -1$

$f''(x) = -\cos x$; $f''\left(\frac{\pi}{2}\right) = 0 \to c_2 = 0$

$f'''(x) = \sin x$; $f'''\left(\frac{\pi}{2}\right) = 1 \to c_3 = \frac{1}{3!}$

$f^{(4)}(x) = \cos x$; $f^{(4)}\left(\frac{\pi}{2}\right) = 0 \to c_4 = 0$

⋮ etc (repeats from here).

$\left(c_5 = -\frac{1}{5!},\ c_6 = 0,\ c_7 = \frac{1}{7!}\right.$, etc$\left.\right)$.

$$\Rightarrow \boxed{\begin{aligned} \cos x &= -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!} + \cdots \\ &= \sum_{k=0}^{\infty} (-1)^{k+1} \frac{x^{2k+1}}{(2k+1)!} \end{aligned}}$$

(3) a) $f(x) = x^{1/2} = \sqrt{x} \quad f(4) = \sqrt{4} = 2.$

$f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}} \ ; \ f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$

$f''(x) = -\frac{1}{4} x^{-3/2} = \frac{-1}{4\sqrt{x^3}} \ ; \ f''(4) = \frac{-1}{4 \cdot 2^3} = -\frac{1}{32}$

$$\Rightarrow \boxed{T_2(x) = 2 + \frac{1}{4}(x-2) - \frac{1}{64}(x-2)^2}$$

$$\frac{f''(4)}{2!} = \frac{-\frac{1}{32}}{2} = -\frac{1}{64}$$

b) Taylor's theorem says, in this case:

If $R_2(x) = f(x) - T_2(x)$,

then $|R_2(x)| \leq \frac{M}{(2+1)!} |x-4|^{2+1}$, where

M is such that $|f^{(3)}(x)| \leq M$ for x in the interval $[4, 5]$.

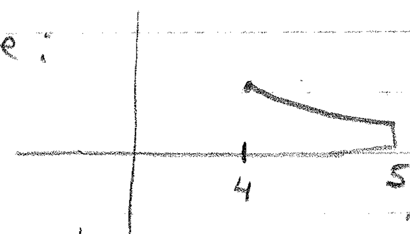
$f^{(3)}(x) = \frac{3}{8} x^{-5/2} = \frac{3}{8} \frac{1}{(\sqrt{x})^5}.$

(3b cont) Note that $f^{(3)}(x) = \frac{3}{8}\frac{1}{(\sqrt{x})^5}$

is decreasing, so if $x \geq 4$,

$$|f^{(3)}(x)| \leq |f^{(3)}(4)| = \frac{3}{8}\frac{1}{(\sqrt{4})^5} = \frac{3}{256}$$

(see picture:



Thus we can take $M = \frac{3}{256}$,

and we obtain the error estimate

$$|R_2(x)| \leq \frac{\frac{3}{256}}{3!}|x-4|^3,$$

so $\boxed{|R_2(x)| \leq \frac{1}{512}|x-4|^3}$

(4) Binomial series: $(1+x)^p = \sum_{k=0}^{\infty} \binom{p}{k} x^k$,

where $\binom{p}{0} = 1$, and $\binom{p}{k} = \frac{\overbrace{p\cdot(p-1)\cdots(p-k+1)}^{k \text{ terms}}}{k!}$

In our case,

$$\frac{1}{(1+x)^2} = (1+x)^{-2} = \sum_{k=0}^{\infty} \binom{-2}{k} x^k.$$

Let us find $\binom{-2}{k}$.

$$\binom{-2}{0} = 1. \quad \binom{-2}{1} = \frac{(-2)}{1!} = -2.$$

$$\binom{-2}{2} = \frac{(-2)(-3)}{2!} = 3 \ ; \binom{-2}{3} = \frac{(-\cancel{2})(-\cancel{3})(-4)}{1\cdot\cancel{2}\cdot\cancel{3}} = -4$$

$$\binom{-2}{4} = \frac{(-\cancel{2})(-\cancel{3})(-\cancel{4})(-5)}{1\cdot\cancel{2}\cdot3\cdot\cancel{4}} = 5.$$

In general,

$$\binom{-2}{k} = \frac{\overbrace{(-2)(-3)\cdots(-2-k+1)}^{k \text{ factors}}}{k!} = \frac{(-1)^k(\cancel{2}\cdot\cancel{3}\cdots\cancel{k}(k+1))}{1\cdot\cancel{2}\cdot\cancel{3}\cdots\cancel{k}}$$

$$= \boxed{(-1)^k\cdot(k+1)}$$

(4 cont) Therefore,

$$\boxed{\frac{1}{(x+1)^2} = \sum_{k=0}^{\infty} (-1)^k (k+1)\, x^k}$$

(5) Since O is the center and it passes through P the radius must be the distance between O and P:

$$d(O,P) = \sqrt{(5-3)^2 + (-1-(-2))^2 + (2-4)^2}$$
$$= \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{9} = \underline{\underline{3}}$$

The equation is, therefore,

$$\boxed{(x-3)^2 + (y+2)^2 + (z-4)^2 = 9}$$

6 a) $\boxed{\vec{a}} = \vec{AB} = B - A = \boxed{(1, 1, 1).}$

$\boxed{\vec{b}} = \vec{BC} = C - B = \boxed{(-2, 0, -2)}$

b) $\boxed{|\vec{a}| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}}$

$\boxed{|\vec{b}| = \sqrt{(2)^2 + 0^2 + (-2)^2} = \sqrt{8}}$

c) If θ is the angle between $\vec{a}$ & $\vec{b}$,

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|} = \frac{(1\cdot(-2) + 1\cdot 0 + 1\cdot(-2))}{\sqrt{3}\,\sqrt{8}}$$

$$= \boxed{\frac{-4}{\sqrt{3}\,\sqrt{8}}}$$

$$\theta = \arccos\left(\frac{-4}{\sqrt{24}}\right) \approx 144.74^\circ$$

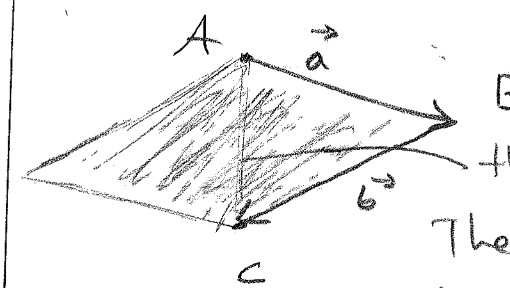
or

$$\approx 2.53$$

d) $\text{proj}_{\vec{a}} \vec{b} = \dfrac{(\vec{a} \cdot \vec{b})}{|\vec{a}|^2} \cdot \vec{a}$

$= \dfrac{-4}{(\sqrt{3})^2} \cdot (1,1,1) = \boxed{\left(\dfrac{-4}{3}, \dfrac{-4}{3}, \dfrac{-4}{3}\right)}$

e)



this area is $|\vec{a} \times \vec{b}|$

The area of the triangle is therefore $\frac{1}{2} |\vec{a} \times \vec{b}|$.

Area of triangle $= \frac{1}{2} |(-2, 0, 2)|$

$= \frac{1}{2} \sqrt{(-2)^2 + 0^2 + 2^2}$

$= \frac{1}{2} \sqrt{8} = \boxed{\sqrt{2}}$

7

a) $\boxed{\vec{a} \times \vec{b} = (-7, 2, -3)}$

b) $|\vec{a} \times \vec{b}| = \sqrt{(-7)^2 + 2^2 + (-3)^2}$

$= \sqrt{49 + 4 + 9} = \sqrt{62}$

Unit vector $\perp$ to $\vec{a}$ & $\vec{b}$: $\dfrac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|} = \boxed{\dfrac{(-7, 2, -3)}{\sqrt{62}}}$

c) Volume of parallelepiped determined by $\vec{a}$, $\vec{b}$ & $\vec{c}$ is

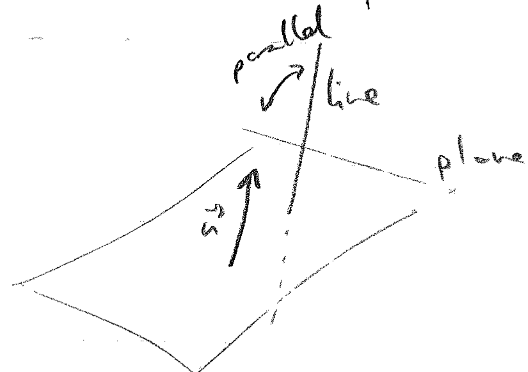
$|\vec{c} \cdot (\vec{a} \times \vec{b})|$

$= |(3, -2, -3) \cdot (-7, 2, -3)|$

$= |-21 - 4 + 9| = \boxed{16}$.

8) a) $\boxed{1\cdot(x-2) - 2\cdot(y+3) - 1\cdot(z-4) = 0}$

b) If line is perpendicular to the plane, and plane is perpendicular to $\vec{n}$, then line is parallel to $\vec{n}$:



Thus the equation is

$$\boxed{\frac{x-1}{1} = \frac{y-1}{-2} = \frac{z+3}{-1}}$$

c) Let us find a vector perpendicular to the plane through P, Q, R.

$\overrightarrow{PQ}$ and $\overrightarrow{QR}$ are parallel to the plane

$\Rightarrow$ $\overrightarrow{PQ} \times \overrightarrow{QR}$ are $\perp$ to the plane.

$\vec{PQ} = (-1, 4, -7)$

$\vec{QR} = (-1, 0, 4)$

$\vec{PQ} \times \vec{QR} = (16, 11, 4)$

Thus, the plane is perpendicular to $(16, 11, 4)$, and passes through $R = (0, 1, 1)$, so its equation is

$$\boxed{16x + 11(y-1) + 4(z-1) = 0}$$

Check: At P, $x = 2$, $y = -3$, $z = 4$,

$16 \cdot 2 + 11 \cdot (-4) + 4 \cdot 3 = \underline{0}$ ✓

At Q, $x = 1$, $y = 1$, $z = -3$,

$16 \cdot 1 + 11 \cdot 0 + 4(-4) = 0$ ✓

(9) It is quite clear (see picture!) that the distance from the plane to P_1 is the length of the projection of $\overrightarrow{P_0P_1}$ over $\vec{n}$. So we need to find

$$|\text{proj}_{\vec{n}}\, \overrightarrow{P_0P_1}| = \left| \frac{(\vec{n} \cdot \overrightarrow{P_0P_1})}{|\vec{n}|^2} \cdot \vec{n} \right|$$

$$= \frac{|(\vec{n} \cdot \overrightarrow{P_0P_1})|}{|\vec{n}|}.$$

$P_1 = (2, 3, 4)$. $P_0 = (1, -1, 2)$ (a point on the plane)

$\overrightarrow{P_0P_1} = P_1 - P_0 = (1, 4, 2)$.

$\vec{n} = (2, -3, 1)$. $\vec{n} \cdot \overrightarrow{P_0P_1} = -8$.

$|\vec{n}| = \sqrt{4+9+1} = \sqrt{14}$. Therefore,

$$\boxed{\text{dist}(\text{plane}, P) = \frac{8}{\sqrt{14}}}$$