MATH 33 - Analytic Geometry and Calculus III, Sec. E01 – 20088

**First test. Time allowed: two hours.** Professor Luis Fernández

**NAME:** SOLUTION

**INSTRUCTIONS:** Solve the following exercises. **You must show work** and **justify your answers** in order to receive credit in any of the exercises.

[6] **1.** Write the first 5 terms of the following sequences.

**a)** $a_n = \dfrac{2n+1}{n+1}$. $a_1 = \dfrac{2+1}{1+1} = \boxed{\dfrac{3}{2}}$; $a_2 = \dfrac{4+1}{3} = \boxed{\dfrac{5}{3}}$; $a_3 = \boxed{\dfrac{7}{4}}$ $a_4 = \boxed{\dfrac{9}{5}}$ $a_5 = \boxed{\dfrac{11}{6}}$

**b)** The sequence defined recursively by $\boxed{a_1 = 3}$ $a_{n+1} = \dfrac{a_n}{a_n - 1}$. $\boxed{a_2 = \dfrac{3}{3-1} = \dfrac{3}{2}}$; $a_3 = \dfrac{\frac{3}{2}}{\frac{3}{2}-1} = \dfrac{\frac{3}{2}}{\frac{1}{2}} = \boxed{3}$ $\boxed{a_4 = \dfrac{3}{2}}$ $\boxed{a_5 = 3}$

[20] **2.** Find each limit.

**a)** $\lim\limits_{n\to\infty} \dfrac{3n^2+1}{2n^2-9n+5} = \lim\limits_{n\to\infty} \dfrac{\frac{3n^2}{n^2}+\frac{1}{n^2}}{\frac{2n^2}{n^2}-\frac{9n}{n^2}+\frac{5}{n^2}} = \lim\limits_{n\to\infty} \dfrac{3+\frac{1}{n^2}}{2-\underbrace{\frac{9}{n}}_{\to 0}+\underbrace{\frac{5}{n^2}}_{\to 0}} = \boxed{\dfrac{3}{2}}$

**b)** $\lim\limits_{n\to\infty} \dfrac{(2n+1)!}{(2n)!} = \lim\limits_{n\to\infty} \dfrac{\cancel{(2n)!}\cdot(2n+1)}{\cancel{(2n)!}} = \lim\limits_{n\to\infty}(2n+1) = \boxed{\infty}$

$\boxed{\text{(diverges)}}$

**c)** $\lim\limits_{n\to\infty} n^2e^{-n} = \lim\limits_{n\to\infty} \dfrac{n^2 \to \infty}{e^n \to \infty} \overset{\text{L'Hopital}}{=} \lim\limits_{n\to\infty} \dfrac{2n \to \infty}{e^n \to \infty} \overset{\text{L'Hopital}}{=} \lim\limits_{n\to\infty} \dfrac{2}{e^n \to \infty} = \boxed{0}$

**d)** $\lim\limits_{n\to\infty} \left(1+\dfrac{5}{n}\right)^{2n} = \lim\limits_{n\to\infty} e^{\operatorname{Ln}\left(1+\frac{5}{n}\right)^{2n}} = e^{\lim\limits_{n\to\infty} 2n\cdot \operatorname{Ln}\left(1+\frac{5}{n}\right)}$

$\lim\limits_{n\to\infty} 2n\cdot\operatorname{Ln}\left(1+\frac{5}{n}\right) = \lim\limits_{n\to\infty} \dfrac{2\operatorname{Ln}\left(1+\frac{5}{n}\right)}{\frac{1}{n}} \overset{\text{L'Hopital}}{=}$

$= \lim\limits_{n\to\infty} \dfrac{2\,\frac{1}{\left(1+\frac{5}{n}\right)}\cdot\left(\cancel{-}\frac{5}{\cancel{n^2}}\right)}{\cancel{-}\frac{1}{\cancel{n^2}}} = \lim\limits_{n\to\infty} \dfrac{10}{1+\underbrace{\frac{5}{n}}_{\to 0}} = 10.$ So, lim is $\boxed{e^{10}}$

[10] **3.** Find the sum of the series $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{n+2} = \frac{1^{st}}{1-r}$

$$= \frac{\left(\frac{3}{4}\right)^2}{1-\frac{3}{4}} = \frac{\frac{9}{16}}{\frac{1}{4}} = \boxed{\frac{9}{4}}$$

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[10] **4.** For the series $\sum_{k=1}^{\infty} \text{Ln}\left(\frac{k}{k+1}\right)$,

**a)** Find the partial sums $s_n = \sum_{k=2}^{n} a_k$ for $n = 1, 2, 3, 4$.

$$S_1 = \boxed{\text{Ln}\left(\frac{1}{2}\right)}; \quad S_2 = \text{Ln}\left(\frac{1}{2}\right) + \text{Ln}\left(\frac{2}{3}\right) = \text{Ln}\,1 - \cancel{\text{Ln}\,2} + \cancel{\text{Ln}\,2} - \text{Ln}\,3 = \boxed{-\text{Ln}\,3}$$

$$S_3 = S_2 + \text{Ln}\left(\frac{3}{4}\right) = -\cancel{\text{Ln}\,3} + \cancel{\text{Ln}\,3} - \text{Ln}\,4 = \boxed{-\text{Ln}\,4}$$

**b)** Find a formula for the partial sums $s_n$ for any $n$.

$$S_4 = S_3 + \text{Ln}\,\frac{4}{5} = -\text{Ln}\,4 + \text{Ln}\,4 - \text{Ln}\,5 = \boxed{-\text{Ln}\,5}$$

$$\text{Ln}\left(\frac{1}{2}\right) + \text{Ln}\left(\frac{2}{3}\right) + \cdots + \text{Ln}\left(\frac{n-1}{n}\right) + \text{Ln}\left(\frac{n}{n+1}\right)$$

$$= \cancel{\text{Ln}\,1}^{\,0} - \cancel{\text{Ln}\,2} + \cancel{\text{Ln}\,2} - \cancel{\text{Ln}\,3} + \cdots + \cancel{\text{Ln}(n-1)} - \cancel{\text{Ln}\,n} + \cancel{\text{Ln}\,n} - \text{Ln}(n+1)$$

$$= \boxed{-\text{Ln}(n+1)}$$

**c)** Find $\lim_{n\to\infty} s_n$ to obtain the value of the sum $\sum_{k=1}^{\infty} \text{Ln}\left(\frac{k}{k+1}\right)$.

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(-\text{Ln}(n+1)\right) = \boxed{-\infty}$$

$\boxed{\text{(diverges)}}$

[30] **5.** Determine whether each series is convergent or divergent.

a) $\sum_{n=1}^{\infty} \frac{n-1}{n^3}$ ; $\frac{n-1}{n^3} \leq \frac{n}{n^3} = \frac{1}{n^2}$, which converges since it is a p-series with $p=2>1$.

$\boxed{\text{Converges by the comparison test}}$

b) $\sum_{n=1}^{\infty} \frac{4^n}{3^n+2^n}$

$$\lim_{n\to\infty} \frac{4^n}{3^n+2^n} = \lim_{n\to\infty} \frac{\left(\frac{4}{3}\right)^n}{\left(\frac{3}{3}\right)^n+\left(\frac{2}{3}\right)^n} = \lim_{n\to\infty} \frac{\overbrace{\left(\frac{4}{3}\right)^n}^{\to\infty}}{1+\underbrace{\left(\frac{2}{3}\right)^n}_{\to 0}} = \infty.$$

Since $\lim_{n\to\infty} \frac{4^n}{3^n+2^n} \neq 0$, the series cannot converge : $\boxed{\text{DIVERGENT}}$

c) $\sum_{k=1}^{\infty} \frac{k!}{e^{k^2}}$ Use the ratio test:

$$\lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \frac{\frac{(k+1)!}{e^{(k+1)^2}}}{\frac{k!}{e^{k^2}}} = \lim_{k\to\infty} \frac{(k+1)!\ e^{k^2}}{k!\ e^{(k+1)^2}} = \lim_{k\to\infty} \frac{\cancel{k!}\cdot(k+1)\,\cancel{e^{k^2}}}{\cancel{k!}\ e^{\cancel{k^2}+2k+1}}$$

$$= \lim_{k\to\infty} \frac{(k+1)}{e^{2k+1}} \overset{\text{L'Hôpital}}{=} \lim_{k\to\infty} \frac{1}{2\underbrace{e^{2k+1}}_{\to\infty}} = 0 < 1.$$

$\boxed{\text{Convergent by the ratio test}}$

d) $\sum_{k=1}^{\infty} 3k^2 e^{-k^3}$

Integral test: $\int_1^\infty 3x^2 e^{-x^3}\,dx = \int_1^\infty e^{-u}\,du = \lim_{N\to\infty} -e^{-u}\Big]_1^N$

$u = x^3$
$du = 3x^2\,dx$

$= \lim_{N\to\infty} \underbrace{e^{-N}}_{\to 0} + e^{-1} = \boxed{e^{-1}}$

$\boxed{\text{CONVERGENT BY INTEGRAL TEST}}$

e) $\sum_{n=1}^{\infty} \frac{3n+2}{n^2+1}$ Use limit comparison test; compare to $\frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\frac{3n+2}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{(3n+2)n}{n^2+1} = \lim_{n\to\infty} \frac{3n^2+2n}{n^2+1} = \boxed{3}.$$

Since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{3n+2}{n^2+2}$.

$\boxed{\text{DIVERGES}}$

f) $\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$ Use root test:

$$\lim_{n\to\infty}\sqrt[n]{\left(\frac{n^2+1}{2n^2+1}\right)^{n}} = \lim_{n\to\infty}\frac{n^2+1}{2n^2+1} = \frac{1}{2} < 1.$$

$\boxed{\text{CONVERGES by root test}}$

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[10] **6.** Determine whether each series is absolutely convergent, conditionally convergent, or divergent.

a) $\sum_{n=1}^{\infty}(-1)^n e^{1/n}$ $\lim_{n\to\infty}(-1)^n e^{1/n}$ does not exist.

alternates $e^0 = 1$

Therefore, the series $\boxed{\text{DIVERGES}}$

b) $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ Alternating series,

1) Decreasing: $\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}}$ (since denominator is greater). ✓

2) $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ ✓

Therefore series converges (by alternating series test).

Since $\sum\left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum\frac{1}{\sqrt{n}}$ diverges (p series w/ $p = 1/2 < 1$),

series does not converge absolutely, so it is $\boxed{\text{CONDITIONALLY CONVERGENT}}$

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[10] **7.** Find the limit of the sequence $a_n$ defined recursively by

$$a_1 = 1, \quad a_{n+1} = \frac{1}{a_n + 1}.$$

**HINT:** Use the fact that $\lim_{n\to\infty} a_n = L$, then evidently $\lim_{n\to\infty} a_{n+1} = L$, so you can take the limit of both sides in the definition of the sequence above and then solve the resulting equation for $L$.

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\frac{1}{a_n+1} = \frac{1}{L+1}.$$

$\therefore$ $L = \frac{1}{L+1}$, so $L^2 + L - 1 = 0$, so $L = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1\pm\sqrt{5}}{2}$

Since $a_n > 0$, $L$ must be $> 0$, so $\boxed{L = \frac{-1+\sqrt{5}}{2}}$

[18] **8.** Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series with $a_n > 0$, for all $n \geq 1$, and $\sum_{n=1}^{\infty} a_n = 0.6$. Let $s_n = a_1 + a_2 + ... + a_n$ be the partial sums. Which of the following statements is true? Which is false? Explain your answers carefully. (Full credit will not be awarded without a clear explanation.)

**a)** $\lim_{n\to\infty} a_n = 0.6$. $\boxed{\text{FALSE}}$ $\lim_{n\to\infty} a_n = 0$, because series converges.

**b)** $\lim_{n\to\infty} s_n = 0.6$. $\boxed{\text{TRUE}}$ $0.6 = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} s_n$.

by definition

**c)** $\sum_{n=1}^{\infty} (-1)^n a_n$ converges. $\boxed{\text{TRUE}}$ The series $\sum_{n=1}^{\infty} (-1)^n a_n$ is absolutely convergent since $\sum_{n=1}^{\infty} |(-1)^n a_n| = \sum_{n=1}^{\infty} a_n$, which converges.

Absolute convergence implies convergence, so $\sum (-1)^n a_n$ is convergent.

**d)** If $b_n = a_n + 1$, then $\sum_{n=1}^{\infty} b_n$ converges. $\boxed{\text{FALSE}}$

If $t_n = b_1 + b_2 + ... + b_n$ are the partial sums, then

$t_n = a_1 + 1 + a_2 + 1 + \cdots + a_n + 1 = a_1 + a_2 + \cdots + a_n + n = s_n + n$,

and $\lim_{n\to\infty} t_n = \lim_{n\to\infty} (\underbrace{s_n}_{\to 0.6} + \underbrace{n}_{\to\infty}) = \boxed{\infty}$.

**e)** $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

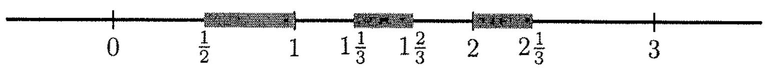
$\boxed{\text{FALSE}}$ Since $\sum a_n$ converges, $\lim_{n\to\infty} a_n = 0$,

so $\lim_{n\to\infty} \frac{1}{a_n} = \infty$, so series DIVERGES.

**f)** $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

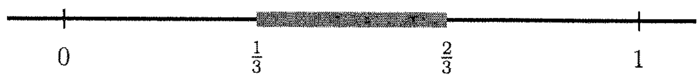
$\boxed{\text{TRUE}}$ $\frac{a_n}{n} \leq a_n$. Since $\sum a_n$ converges, so does $\sum \frac{a_n}{n}$.

[10] **9.** The *measure* of a set in the real line is the sum of the lengths of the intervals that form the set. For example, the measure of the set consisting of the shaded part in the picture below is $1/2 + 1/3 + 1/3 = 7/6$.

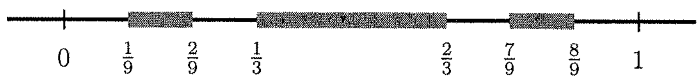



Consider the collection of sets defined as follows:

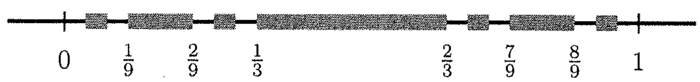
$A_1 = [\frac{1}{3}, \frac{2}{3}]$:




$A_2 = [\frac{1}{9}, \frac{2}{9}] \cup [\frac{1}{3}, \frac{2}{3}] \cup [\frac{7}{9}, \frac{8}{9}]$:




$A_3 = [\frac{1}{27}, \frac{2}{17}] \cup [\frac{7}{27}, \frac{8}{27}] \cup A_2 \cup [\frac{19}{27}, \frac{20}{17}] \cup [\frac{25}{27}, \frac{26}{27}]$:




And so on.

**a)** Find the measure of $A_n$. **HINT:** From $A_1$ to $A_2$ we added two intervals of length $\frac{1}{9}$; from $A_2$ to $A_3$ we added four intervals of length $\frac{1}{27}$, etc.

Measure of $A_1 = \frac{1}{3}$

Measure of $A_2 = \frac{1}{3} + \frac{2}{9}$

" " $A_3 = \frac{1}{3} + \frac{2}{9} + \frac{4}{27}$

" " $A_4 = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{2^3}{3^4}$

" " $A_n = \frac{1}{3} + \frac{2}{9} + \dots + \frac{2^{n-1}}{3^n} = \sum_{k=1}^{n} \frac{2^{k-1}}{3^k}$

**b)** What is the limit as $n \to \infty$, of the measure of $A_n$?

[NOTE: What remains of the interval $[0, 1]$ after removing all the sets $A_n$ is called the *Cantor set*. It has very interesting properties.]

$$A_n = \sum_{k=1}^{n} \frac{2^{k-1}}{3^k} = \frac{1}{3} \sum_{k=1}^{n} \frac{2^{k-1}}{3^{k-1}} = \frac{1}{3} \sum_{k=1}^{n} \left(\frac{2}{3}\right)^{k-1}$$

geometric series.

$$= \frac{1}{3} \cdot \frac{\frac{2}{3} - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}} \xrightarrow{n \to \infty} \frac{1}{3} \cdot \frac{\frac{2}{3}}{1 - \frac{2}{3}} = \boxed{1}$$

$\boxed{\text{Limit is } 1}$