MATH 33 - Analytic Geometry and Calculus III, Sec. E01 – 20088

First test. Time allowed: two hours. Professor Luis Fernández

NAME:____

INSTRUCTIONS: Solve the following exercises. **You must show work** and **justify your answers** in order to receive credit in any of the exercises.

[6] **1.** Write the first 5 terms of the following sequences.

a) $a_n = \dfrac{2n+1}{n+1}$.

b) The sequence defined recursively by $a_1 = 3$, $a_{n+1} = \dfrac{a_n}{a_n - 1}$.

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[20] **2.** Find each limit.

a) $\displaystyle\lim_{n\to\infty} \frac{3n^2+1}{2n^2-9n+5}$

b) $\displaystyle\lim_{n\to\infty} \frac{(2n+1)!}{(2n)!}$

c) $\displaystyle\lim_{n\to\infty} n^2 e^{-n}$

d) $\displaystyle\lim_{n\to\infty} \left(1+\frac{5}{n}\right)^{2n}$

[10] **3.** Find the sum of the series $\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{n+2}$.

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[10] **4.** For the series $\sum_{k=1}^{\infty} \mathrm{Ln}\left(\frac{k}{k+1}\right)$,

a) Find the partial sums $s_n = \sum_{k=2}^{n} a_k$ for $n = 1, 2, 3, 4$.

b) Find a formula for the partial sums s_n for any n.

c) Find $\lim_{n\to\infty} s_n$ to obtain the value of the sum $\sum_{k=1}^{\infty} \mathrm{Ln}\left(\frac{k}{k+1}\right)$.

[30] **5.** Determine whether each series is convergent or divergent.

a) $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{n^3}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{4^n}{3^n + 2^n}$

c) $\displaystyle\sum_{k=1}^{\infty} \frac{k!}{e^{k^2}}$

d) $\displaystyle\sum_{k=1}^{\infty} 3k^2 e^{-k^3}$

e) $\displaystyle\sum_{n=1}^{\infty} \frac{3n+2}{n^2+1}$

f) $\displaystyle\sum_{n=1}^{\infty}\left(\frac{n^2+1}{2n^2+1}\right)^n$

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[10] **6.** Determine whether each series is absolutely convergent, conditionally convergent, or divergent.

a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n e^{1/n}$

b) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$

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[10] **7.** Find the limit of the sequence a_n defined recursively by

$$a_1 = 1, \qquad a_{n+1} = \frac{1}{a_n + 1}.$$

HINT: Use the fact that $\lim_{n\to\infty} a_n = L$, then evidently $\lim_{n\to\infty} a_{n+1} = L$, so you can take the limit of both sides in the definition of the sequence above and then solve the resulting equation for L.

[18] **8.** Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series with $a_n > 0$, for all $n \geq 1$, and $\sum_{n=1}^{\infty} a_n = 0.6$. Let $s_n = a_1 + a_2 + ... + a_n$ be the partial sums. Which of the following statements is true? Which is false? Explain your answers carefully. (Full credit will not be awarded without a clear explanation.)

a) $\lim_{n\to\infty} a_n = 0.6$.

b) $\lim_{n\to\infty} s_n = 0.6$.

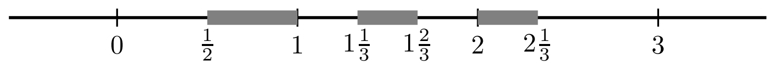
c) $\sum_{n=1}^{\infty} (-1)^n a_n$ converges.

d) If $b_n = a_n + 1$, then $\sum_{n=1}^{\infty} b_n$ converges.

e) $\sum_{n=1}^{\infty} \frac{1}{a_n}$ converges.

f) $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

[10] **9.** The *measure* of a set in the real line is the sum of the lengths of the intervals that form the set. For example, the measure of the set consisting of the shaded part in the picture below is $1/2 + 1/3 + 1/3 = 7/6$.



Consider the collection of sets defined as follows:

$A_1 = [\frac{1}{3}, \frac{2}{3}]$:

$A_2 = [\frac{1}{9}, \frac{2}{9}] \cup [\frac{1}{3}, \frac{2}{3}] \cup [\frac{7}{9}, \frac{8}{9}]$:

$A_3 = [\frac{1}{27}, \frac{2}{17}] \cup [\frac{7}{27}, \frac{8}{27}] \cup A_2 \cup [\frac{19}{27}, \frac{20}{17}] \cup [\frac{25}{27}, \frac{26}{27}]$:

And so on.

a) Find the measure of A_n. **HINT:** From A_1 to A_2 we added two intervals of length $\frac{1}{9}$; from A_2 to A_3 we added four intervals of length $\frac{1}{27}$, etc.

b) What is the limit as $n \to \infty$, of the measure of A_n?

[NOTE: What remains of the interval $[0, 1]$ after removing all the sets A_n is called the *Cantor set*. It has very interesting properties.