Ratios and rates

Remembe: The *ratio* of a quantity a to a quantity b is the fraction $\frac{a}{b}$ (as usual, written in lowest terms).

For example, the ratio of 3 to 12 is $\frac{3}{12} = \frac{1}{4}$.

For example, the ratio of $\frac{5}{2}$ to $\frac{3}{4}$ is $\frac{\frac{5}{2}}{\frac{3}{4}} = \frac{5}{2} \cdot \frac{4}{3} = \frac{20}{6} = \frac{10}{3}$.

If the quantities have decimals, you will get a fraction with decimals in the numerator and/or denominator. Then move the decimal point to the right *simultaneously* as many steps as you need so that both numerator and denominator are whole numbers. For example, the ratio of 2.1 to 0.7 is $\frac{2.1}{0.7} = \frac{21.}{7.} = \frac{3}{1}$.

In a ratio, both quantities must be measured *in the same units!*

Especially if they are different, it is often useful to write the *units* that are being used in both the numerator and denominator of the fraction. Then, to get the ratio, rewrite both in the same units, cross the common units out and simplify the fraction:

For example, if I sleep 8 hours out of every 24 hours, that means that I spend $\frac{8 \text{ hours}}{24 \text{ hours}} = \frac{8}{24} = \frac{1}{3}$ of the time sleeping.

If I spend 10 minutes out of every 1 hour looking at my phone, then the ratio is $\frac{10 \text{ minutes}}{1 \text{ hour}} = \frac{10 \text{ minutes}}{60 \text{minutes}} \frac{10}{60} = \frac{1}{6}$ (because 1 hour = 60 minutes; we have to measure both quantities in minutes).

It is often useful to write the *units* that are being used in both the numerator and denominator of the fraction, then rewrite both in the same units, cross the common units out and simplify the fraction:

For example, the ratio of 4 inches to 2 feet is (remember that 1 foot = 12 inches) $\frac{4 \text{ inches}}{2 \text{ feet}} = \frac{4 \text{ inches}}{24 \text{ inches}} = \frac{4}{24} = \frac{1}{6}$.

Ratios that have different units in the numerator and denominator are called *rates*.

For example, if I drive 83 miles in 2 hours, then the rate of miles I covered to the hours I spent driving is $\frac{83 \text{ miles}}{2 \text{ hours}}$.

Normally this is written as $\frac{83}{2} \ \frac{\text{miles}}{\text{hour}}$, or as 41.5 miles/hour.

Working with ratios or with rates is the same, only that with rates you need to keep track of the units.

Exercises: Find the following ratios.

1. 14 to 3. **2.** 5 to 16 **3.** 12 to 42

4. $\frac{2}{3}$ to $\frac{1}{2}$. **5.** $1\frac{2}{3}$ to 5 **6.** $\frac{1}{2}$ to $\frac{1}{3}$

7. 0.7 to 1.4. **8.** 0.16 to 0.4 **9.** 0.5 to 5

Proportions A proportion is a statement that two ratios (or rates) are equal. In other words, it is a statement of the form

$$\frac{a}{b} = \frac{c}{d}.$$

The main property of a proportion is that **the cross-products are equal**. So for example, to check if $\frac{10}{4} = \frac{15}{6}$ is a proportion, we cross multiply and check if they are equal: $10 \cdot 6 = 60$ and $15 \cdot 4 = 60$. Since they are equal, the proportion is true.

Example: Is the proportion $\frac{3}{4} = \frac{5}{2}$ true? Check the cross-products: $3 \cdot 2 = 6$ and $5 \cdot 4 = 20$. Since 6 is not equal to 20, this proportion is *not* true.

In a proportion there is often a number whose value we do not know and we want to find it so that the proportion is true. This is called *solving a proportion*. To do this, cross multiply and then divide both sides by the number multiplying the variable.

For example: Find x so that $\frac{x}{20} = \frac{3}{4}$: cross multiply to get $4x = 20 \cdot 3$, so $4x = 60$. Divide both sides by 4 and simplify:

$$\frac{4x}{4} = \frac{60}{4} \implies x = 15.$$

For example: Find x so that $\frac{3}{x} = \frac{5}{2}$: cross multiply to get $5x = 2 \cdot 3$, so $5x = 6$. Divide both sides by 5 and simplify:

$$\frac{5x}{5} = \frac{6}{5} \implies x = \frac{6}{5}.$$

Exercises: Solve the following proportions.

10. $\frac{x}{4} = \frac{15}{20}$ **11.** $\frac{x}{3} = \frac{8}{12}$ **12.** $\frac{x}{2} = \frac{15}{6}$

13. $\frac{5}{y} = \frac{25}{35}$ **14.** $\frac{4}{y} = \frac{5}{2}$ **15.** $\frac{5}{3} = \frac{x}{6}$

16. $\frac{b}{4} = \frac{15}{20}$ **17.** $\frac{b}{3} = \frac{8}{12}$ **18.** $\frac{5}{7} = \frac{6}{b}$

19. $\frac{7}{4} = \frac{a}{20}$ **20.** $\frac{a}{6} = \frac{8}{5}$ **21.** $\frac{4}{a} = \frac{9}{4}$

Percent problems Problems involving percents are easily solved using proportion. The statement

$$A \text{ is } P \text{ percent of } B$$

can be written as

$$\frac{A}{B} = \frac{P}{100}.$$

Thus, if we want to find any of A, B, or P given the other quantities, we can write a proportion and solve it.

The only important thing to remember is what to write where in the proportion. In general,

- The quantity A is right before the 'is' (this is the number we get after we do the percent).
- The quantity B is right after the 'of' (this is what we are taking the percent of).
- The quantity P is the percent (so it always appears as P percent).

Example: 6 is 15% of what number?

- The quantity before the 'is' is 6 (the quantity we obtain when we do the percent of some number), so $A = 6$.
- $P = 15\%$.
- B is the unknown quantity (the quantity we are taking the percent of). We obtain the proportion $\frac{6}{B} = \frac{15}{100}$. Solve it as usual: $6 \cdot 100 = 15 \cdot B$, so $600 = 15B$, and dividing both sides by 15 we obtain $B = \frac{600}{15} = 40$. So we obtain the answer: 6 is 15% of 40.

Example: What percent of 20 is 36?

- The quantity before the 'is' is 36 (the quantity we obtain when we do the percent), so $A = 36$.
- The quantity after 'of' is 20, (the quantity we are taking the percent of), so $B = 20$.
- The percent P is the unknown variable.

We obtain the proportion $\frac{36}{20} = \frac{P}{100}$. Solve it as usual: $36 \cdot 100 = 20 \cdot P$, so $3600 = 20P$, and dividing both sides by 20 we obtain $P = \frac{3600}{20} = 180$. So we obtain the answer: 36 is 180% of 20.

Exercises:

22. What percent of 29 is 3? **23.** What percent of 33.5 is 21? **24.** How much is 41% of 78?

25. 58% of what number is 63.4? **26.** 9 is 15% of what number? **27.** 108 is 90% of what number?

28. What percent of 25 is 12? **29.** 15 is 12% of what number? **30.** What is 80% of 34?

31. What percent of 18 is 9? **32.** 132 is 11% of what number? **33.** What is 21% of 16?

34. What percent of 25 is 35? **35.** 11 is 220% of what number? **36.** What is 180% of 32?