Linear equations

An **equation** is a mathematical statement with an "=" and one of more variables (that is, unknown numbers) represented by letters.

Examples: $2x + 3 = 7$, $3x - 5 = 6x + 7$, $2x + y = 12$, $x^2 - 6x + 8 = 0$ are all equations.

A **solution** of an equation is a value of the variables that make the equation an *equality* (that is, a *true* statement with an "=" and no variables).

For example: Is $x = 2$ a solution of $2x + 3 = 7$? Let us substitute $x = 2$ in the equation. To make it easier, let us substitute on each side of the equation separately:

- Substitute $x = 2$ on the left hand side (LHS) of the equation ($2x+3$) and simplify: LHS$= 2\cdot(2)+3 = 4+3 = 7$.
- Substitute $x = 2$ on the right hand side (RHS) of the equation (7) and simplify: we just get RHS$= 7$ (there is no x).

Since the LHS is equal to the RHS (both sides equal 7), $x = 2$ *is* a solution of $2x + 3 = 7$.

For example: Is $x = 3$ a solution of $3x - 5 = 6x + 7$?

- Substitute $x = 2$ on LHS: $3 \cdot (3) - 5 = 9 - 5 = 4$.
- Substitute $x = 2$ on HS: $6 \cdot (3) + 7 = 18 + 7 = 25$.

Since the LHS and the RHS are not equal, $x = 3$ *is not* a solution of $3x - 5 = 6x + 7$.

For example: Is $x = (-4)$ a solution of $3x - 5 = 6x + 7$?

- Substitute $x = (-4)$ on LHS: $3 \cdot (-4) - 5 = (-12) - 5 = (-17)$.
- Substitute $x = (-4)$ on HS: $6 \cdot (-4) + 7 = (-24) + 7 = (-17)$.

Since the LHS and the RHS are equal (they are both (-17)), $x = (-4)$ *is* a solution of $3x - 5 = 6x + 7$.

<u>Exercises</u>: Determine whether the given value of the variable is a solution of the given equation.

1. $x = 2$, of $2x + 6 = 10$.
2. $x = 4$, of $2x - 2 = -5$.
3. $x = (-2)$, of $2x + 6 = 3x + 8$.
4. $x = 5$, of $4(x - 2) = 4x - 8$.
5. $x = (-1)$, of $2(x + 4) = 3(x + 3)$.
6. $x = (-2)$, of $x^2 + 6x = 2x - 4$.
7. $x = (-1)$, of $-2x + 5 = x + 8$.
8. $x = 3$, of $4x - 2 = 4x + 3$.
9. $x = 4$, of $3x + 6 = 5x - 2$.

Manipulating linear equations

Mathematical expression often appear as a sum of smaller expressions. For example, $3x - 4$ is the sum of $3x$ and (-4) (because $3x - 4 = 3x + (-4)$). These smaller expressions are called *terms*.

Examples: $3x - 4y + 5$ has three terms: $3x$, $-4y$ and 5.

$2(x + 3) + 8$ has two terms: $2(x + 3)$ and 8.

$3(x + 4)$ has only one term: $3(x + 4)$.

Terms of the form "number · variable" are called *linear terms*. For example, $2x$, $-7x$, $63y$, and $12s$ are all linear terms. The number multiplying the variable in a linear term (with the sign included) is called the *coefficient*.

Terms in an expression that have the same variables raised to the same exponent (or no variables at all) are called *like terms*. For example, $2x$, $-6x$, and $99x$ are like terms, and so are 3, 66, and -12, whereas $5y$ and $3x$ are not like terms, and neither are $2x^2$ and $12x$.

Like terms can be combined into a single term by simply adding the coefficients.

For example, $3x + 6x = 9x$.

Another example: $5x + 4 + 7x + 2 = (5x + 7x) + (4 + 2) = 12x + 6$.

Note that *unlike* terms *cannot* be combined. For example, unless I know the value of x, I cannot combine x and 4 in the expression $x + 4$.

One more example: $5x - 4 - 2x + 3 = (5x - 2x) + (-4 + 3) = 3x - 1$.

<u>Exercises</u>: Combine like terms.

10. $3x + 7x$
11. $-5x + 7x + 3 - 2$
12. $6x - 6 + 4x - 6 + 9 - 5x + 2$
13. $3x^2 + 5x^2$
14. $6x - 5 - 4x - 8 - 2$
15. $-3x - 4x + 4x - 5 + 2 + 2$
16. $3x + 4 - 4x + 5 + x - 9$
17. $12x - 5x - 3 - 22$
18. $6x^2 - 6 + 4x^2 - 6x + 9 - 5x$

Solving linear equations

An equation is called *linear* if it has no exponents. To solve a linear equation, we manipulate the equation in order to simplify it and obtain the value of the variable. The rules to manipulate equations are as follows:

1. You can add **the same** number or expression to **both sides** of the equation.
2. You can subtract **the same** number or expression from **both sides** of the equation.
3. You can multiply **both sides** of the equation by **the same** number or expression.
4. You can divide **both sides** of the equation by **the same** number or expression.

You can use *any* of these rules as many times as you want. Just remember: it is easier to use rules 1 and/or 2 first, and then finish with 3 and/or 4.

Example: Solve the equation $3x + 4 = 10$.

- Subtract 4 from both sides of the equation and combine like terms: $3x + 4 - 4 = 10 - 4 \Rightarrow 3x = 6$.
- Divide both sides by 3: $3x \div 3 = 6 \div 3 \Rightarrow x = 2$ (because $3x \div 3 = x$).

Therefore the solution is $x = 2$. It easily checks: $3 \cdot (2) + 4 = 6 + 4 = 10$.

The goal is to leave x alone in one of the sides of the equation to end up with an expression of the form $x =$ number or number $= x$. The general idea to do this is:

a. If necessary, combine like terms in each side of the equation separately.

b. Subtract the constant term of the LHS from **both** sides of the equation. This way the LHS will have no constant term (you *moved it* to the other side).

c. Subtract the linear term of the RHS from **both** sides. This way the RHS will have no linear term (you *moved it* to the other side).

d. Divide **both** sides by the coefficient of the linear term in the LHS.

Example: Solve the equation $3x + 6 = x + 11$.

- Subtract 6 (the constant term of the LHS) from both sides of the equation and combine like terms:
  $3x + 6 - 6 = x + 11 - 6 \Rightarrow 3x = x + 5$.
- Subtract x (the linear term of the RHS) from both sides of the equation and combine like terms:
  $3x - x = x + 11 - x \Rightarrow 2x = 11$.
- Divide both sides by 2 (the coefficient of the linear term on the LHS): $2x/2 = 11/2 \Rightarrow x = 11/2$

Therefore the solution is $x = \frac{11}{2}$.

Example: Solve the equation $\dfrac{3x}{2} + 5 = 11$.

- Since the linear term on the LHS has a denominator (2), multiply both sides by 2 and simplify:
  $2 \cdot \dfrac{3x}{2} = 2 \cdot 6 \Rightarrow \dfrac{6x}{2} = 12 \Rightarrow 3x = 12$
- Divide both sides by 3 (the coefficient of the linear term on the LHS): $3x/3 = 12/3 \Rightarrow x = 4$.

Therefore the solution is $x = 4$.

<u>Exercises</u>: Solve the following equations.

19. $2x + 7 = 15$
20. $5x - 4 = 18$
21. $7x - 5 = 12$
22. $6 - 2x = 14$
23. $-8 - 7x = -1$
24. $-5x + 7 = 12$
25. $6x - 5 = 2x - 13$
26. $4x + 2 = 2 + x$
27. $4 = 2 + x$
28. $6 = -5 + 5x$
29. $4 = 2 - x$
30. $5 = -11 - x$
31. $4 - x = 8 + x$
32. $6 + 5x = -5 - 5x$
33. $2x - 7 = -5x + 2$
34. $12x = -11 - 3x$
35. $6x + 5 = -55 - 4x$
36. $6 - x = 4 - 2x$
37. $2x - 9 = 3x + 7$
38. $6 = -5x$
39. $-x + 6 = -6x - 4$
40. $6 + 5x - 4 = -2x - 8 + 5x$
41. $x - 7 + 5x = 6 + x + 2$
42. $2x + 7 = -11 - x$
43. $\dfrac{x}{4} = 2$
44. $\dfrac{x}{5} = -3$
45. $\dfrac{3x}{4} = 6$
46. $\dfrac{x}{5} + 6 = 9$
47. $\dfrac{5x}{2} = 15$
48. $\dfrac{-4x}{3} = -16$
49. $\dfrac{-2x}{3} = 8$
50. $\dfrac{x}{2} - 4 = 7$
51. $\dfrac{5x}{3} + 6 = -5$