Least Common Multiple, Greatest Common Factor, Prime Factorization. Prof. L. Fernández

Least Common Multiple

Definitions

Recall:

- A number m is a **multiple** of a number a if m equals a times a whole number.

  For example, the multiples of 5 are $5, 10, 15, 20, 25, 30, \ldots$, and the multiples of 3 are $3, 6, 9, 12, 15, 18, \ldots$ A number has infinitely many multiples.
- A **common multiple** of two or more numbers is a number that is a multiple of all of them.

  For example, 15 is a common multiple of 5 and 3 because it is a multiple of 5 and also a multiple of 3.

  Two numbers have infinitely many common multiples. For example, the common multiples of 5 and 3 are $15, 30, 45, 60, \ldots$.
- The **Least Common Multiple (LCM)** of two or more numbers is the smallest common multiple of the given numbers.

  For example, the LCM of 3 and 5 is 15.

How to find the LCM - method 1: just use the definition.

For each number, write down the list of its multiples and find the first number that appears in all the lists (eventually, one common multiple will appear). **Drawback:** This method is easy, but the lists can be **very** long!

Example: Find the LCM of 8 and 6:

Multiples of 8: $8, 16, 24, 32, 40, 48, \ldots$

Multiples of 6: $6, 12, 18, 24, 30, 36, 42, 48, \ldots$

The first number that appears in both lists is 24. Therefore the LCM of 8 and 6 is 24.

(Note that 48 is also a common multiple, but we are only looking for the smallest.)

Use the method above to find the LCM of the following numbers:

1. LCM of 6 and 9. **2.** LCM of 4 and 6. **3.** LCM of 12 and 8. **4.** LCM of 6, 4 and 9.

5. LCM of 20 and 25. **6.** LCM of 3 and 9. **7.** LCM of 8 and 4. **8.** LCM of 10, 15 and 4.

How to find the LCM - method 2: use the prime number decomposition of the numbers.

This method works well in all cases, and does not take long:

(a) Write the prime number factorization of all the numbers (review of this is done below).

(b) From all the factorizations, make a list of each factor raised to the highest exponent it has in the factorizations.

(c) Multiply the numbers of the list you got in (b).

Example: Find the LCM of 8 and 6.

(a) $8 = 2^3$. $6 = 2 \cdot 3$.

(b) Factors: 2 with exponent 3 (2 appears in both factorizations, and the highest exponent is 3), and 3 with exponent 1. That gives $2^3 = 8$ and $3^1 = 3$.

(c) Multiply: $8 \cdot 3 = 24$. So 24 is the LCM of 8 and 6.

Use this method to find the LCM of the following numbers:

9. LCM of 6 and 9. **10.** LCM of 4 and 6. **11.** LCM of 12 and 8. **12.** LCM of 6, 4 and 9.

13. LCM of 20 and 25. **14.** LCM of 3 and 9. **15.** LCM of 8 and 4. **16.** LCM of 10, 15 and 4.

17. LCM of 18 and 15. **18.** LCM of 24 and 30. **19.** LCM of 10 and 25. **20.** LCM of 6, 9 and 4.

21. LCM of 30 and 35. **22.** LCM of 20 and 24. **23.** LCM of 11 and 55. **24.** LCM of 15, 25 and 40.

Greatest Common Factor (GCF)

Definitions:

- A **divisor** or **factor** of a number is a number that divides it exactly (no remainder).

  For example, 4 is a divisor, or a factor, of 8 because $8 \div 4 = 2$ exactly. However, 5 is not because the division $8 \div 5$ has a remainder.
- The **divisors** or **factors** of a number are all the numbers that divide it exactly.

  For example, the divisors or factors of 8 are $1, 2, 4$ and 8.
- A **common divisor** or **factor** of two or more numbers is a number that is a divisor or factor of all of them.

  For example, the common divisors of 12 and 18 are $1, 2, 3$ and 6.
- The **Greatest Common Factor (GCF)** (also called Greatest Common Divisor) of two or more numbers is the greatest number that is a common factor of the given numbers.

  For example, the GCF of 8, 24 and 20 is 4, because the factors of 8 are $1, 2, 4, 8$. $1, 2$ and 4 are also factors of 24 and 20, but 8 is not a factor of 20, so the greatest of all the common factors is 4.

How to find the GCF - method 1: just use the definition

For each number, write all its factors, and then take the greatest number that appears in all the list.

Drawback: this method can take very long if a number has many factors.

Example: Find the GCF of 15 and 25.

Factors of 15: $1, 3, 5, 15$. Factors of 25: $1, 5, 25$. The common factors are 1 and 5. The GCF is therefore 5.

In the following exercises, find the GCF using this method.

25. GCF of 6 and 9. **26.** GCF of 4 and 6. **27.** GCF of 16 and 8. **28.** GCF of 6, 12 and 9.

29. GCF of 20 and 30. **30.** GCF of 14 and 9. **31.** GCF of 8 and 12. **32.** GCF of 6, 12 and 18.

How to find the GCF - method 2: use the prime number decomposition of the numbers

(a) Write the prime number factorization of all the numbers (review of this is done below).

(b) From all the factorizations, make a list of **only the common factors** raised to the **lowest** exponent it has in the factorizations.

(c) Multiply the numbers of the list you got in (b).

Example: Find the GCF of 24 and 60.

(a) $24 = 2^3 \cdot 3$. $6 = 2^2 \cdot 3 \cdot 5$.

(b) **Common** factors: 2, with lowest exponent 2, and 3 with lowest exponent 1. That gives $2^2 = 4$ and $3^1 = 3$.

(c) Multiply: $4 \cdot 3 = 12$. So 12 is the LCM of 24 and 60.

In the following exercises, find the GCF using this method.

33. GCF of 6 and 9. **34.** GCF of 4 and 6. **35.** GCF of 16 and 8. **36.** GCF of 6, 12 and 9.

37. GCF of 28 and 15. **38.** GCF of 14 and 9. **39.** GCF of 8 and 11. **40.** GCF of 6, 12 and 18.

41. GCF of 60 and 35. **42.** GCF of 36 and 60. **43.** GCF of 2 and 12. **44.** GCF of 54, 18.

45. GCF of 42 and 70. **46.** GCF of 49 and 63. **47.** GCF of 90 and 12. **48.** GCF of 24, 36 and 18.

Addition and subtraction of fractions, when denominators are large

When we studied addition and subtraction of fractions we always used the product of the denominators as the common denominator. When the denominators are large, the common denominator can be too big!! And then the final simplification is quite hard.

For example, $\dfrac{3}{20} + \dfrac{4}{15} = \dfrac{45}{300} + \dfrac{80}{300} = \dfrac{125}{300} = \dfrac{5}{12}$ (big numbers to simplify!)

Instead, we can use the LCM of the denominators as the common denominator. The LCM of all the denominators is often called **Least Common Denominator (LCD)**.

For example, for $\dfrac{3}{20} + \dfrac{4}{15}$ the LCD of is 60. Thus we have $\dfrac{3}{20} + \dfrac{4}{15} = \dfrac{9}{60} + \dfrac{16}{60} = \dfrac{25}{60} = \dfrac{5}{12}$.

For the following exercises, write the fractions using the Least Common Denominator, then add or subtract, as indicated. Make sure to write the final answer in lowest terms.

49. $\frac{4}{20} + \frac{5}{12} = \frac{12}{60} + \frac{25}{60} = \frac{37}{60}$ (ex.)

50. $\frac{7}{30} + \frac{9}{20} = \_\_\_ + \_\_\_ =$

51. $\frac{5}{18} - \frac{5}{24} = \_\_\_ + \_\_\_ =$

52. $\frac{17}{40} - \frac{7}{20} = \_\_\_ + \_\_\_ =$

53. $\frac{4}{35} + \frac{3}{30} = \_\_\_ + \_\_\_ =$

54. $\frac{9}{28} - \frac{2}{21} = \_\_\_ + \_\_\_ =$

55. $\frac{5}{21} - \frac{5}{24} = \_\_\_ + \_\_\_ =$

56. $\frac{17}{45} - \frac{7}{20} = \_\_\_ + \_\_\_ =$

57. $\frac{5}{8} + \frac{5}{6} - \frac{1}{4} = \_\_\_ + \_\_\_ + \_\_\_ =$

58. $\frac{5}{3} - \frac{1}{4} + \frac{5}{6} = \_\_\_ + \_\_\_ + \_\_\_ =$

59. $\frac{7}{10} + \frac{1}{6} + \frac{2}{15} = \_\_\_ + \_\_\_ + \_\_\_ =$

60. $\frac{7}{2} + \frac{1}{3} + \frac{2}{5} = \_\_\_ + \_\_\_ + \_\_\_ =$

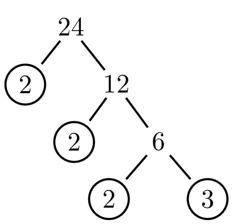
Prime number factorization

Recall: A number is **prime** if it has no factors other than 1 and itself.

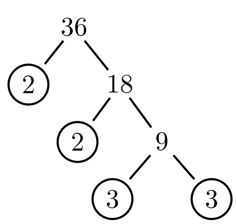
For example, $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, \ldots$ are all prime. We could keep going for ever with the list, as there are infinitely many primes.

Every whole number can be written in a unique way as a product of prime numbers. For example, $24 = 2 \cdot 2 \cdot 2 \cdot 3$. Instead of repeating a factor, we use exponents to make it shorter, so we would write $24 = 2^3 \cdot 3$. This is called the **prime number factorization** of a number.

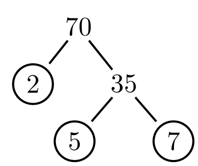
To find the prime number factorization of a number the only method is trial and error. Start checking whether 2 is a factor. If it is, divide the number by 2 and check if the result also has 2 as a factor. If it is, divide again, check again, etc. If 2 is not a factor, try 3, then 5, etc. It is often written in a tree. For example



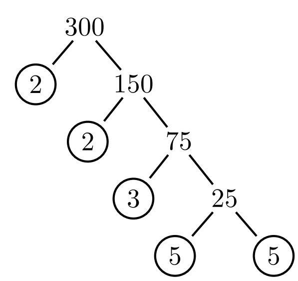
Gives $24 = 2^3 \cdot 3$.



Gives $36 = 2^2 \cdot 3^2$.



Gives $70 = 2 \cdot 5 \cdot 7$.



Gives $300 = 2^2 \cdot 3 \cdot 5^2$.

Note that **all the numbers at the end of the tree have to be prime**. Otherwise, they can branch further.

In the following exercises, find the prime number factorization of the given numbers (just a few here – there are many more in the GCF and LCD worksheets).

61. 16

62. 60

63. 54

64. 15

65. 142

66. 320