THE TOPOLOGY OF THE SPACE OF HARMONIC 2-SPHERES IN THE M-SPHERE

LUIS FERNÁNDEZ

Abstract. We show that various natural topologies and analytic structures that can be given to the space of harmonic maps from the 2-sphere to the m-sphere are all equal.

1. Introduction

Let $\mathrm{Hol}_\delta(S^2, \mathbb{CP}^N)$ denote the set of holomorphic maps of degree δ from S^2 to $\mathbb{CP}^N$. Regarding S^2 as $\mathbb{C} \cup \{\infty\}$, a map $\psi \in \mathrm{Hol}_\delta(S^2, \mathbb{CP}^N)$ can always be written uniquely, for all $z \in \mathbb{C}$, as $\psi(z) = [\boldsymbol{f}(z)]$, where $\boldsymbol{f}(z) = (f_0, \ldots, f_N)$ is an N-tuple of polynomials in z without nontrivial common factors and maximum degree δ. Then $\boldsymbol{f}(\infty)$ can be interpreted as a limit. Let

$$\mathrm{Res}_\delta = \{\boldsymbol{f} = (f_0, \ldots, f_N) \in \mathbb{C}[z]^N_\delta : \text{the } f_i \text{ have a nontrivial common factor}\}$$

and let $\mathbb{C}[z]^N_{\delta-1} \subset \mathbb{C}[z]^N_\delta$ be the subspace of N-tuples of polynomials of degree $\delta - 1$. Using the usual identification topology (which we will call the *Euclidean* topology, or E topology for short) induced by the projection $C[z]^N \to \mathbb{P}(\mathbb{C}[z]^N)$ given by $\boldsymbol{P} \to [\boldsymbol{P}]$, both Res_δ and $\mathbb{C}[z]^N_{\delta-1}$ are closed subvarieties of $\mathbb{C}[z]^N_\delta$ (Res_δ is for example the zero locus of the generalized resultant defined in [5]). We thus have a bijective map

$$\begin{aligned} \mathcal{P} : \mathrm{Hol}_\delta(S^2, \mathbb{CP}^N) &\to \mathbb{P}(\mathbb{C}[z]^N \setminus (\mathrm{Res}_\delta \cup \mathbb{C}[z]^N_{\delta-1})) \\ \psi &\to [\boldsymbol{f}] \end{aligned}$$

Give $\mathbb{P}(\mathbb{C}[z]^N \setminus (\mathrm{Res}_\delta \cup \mathbb{C}[z]^N_{\delta-1}))$ the subspace topology, and $\mathrm{Hol}_\delta(S^2, \mathbb{CP}^N)$ the compact-open (C-O for short) topology. Then we have

Proposition 1. *The map $\mathcal{P}$, with the topologies specified above, is a homeomorphism.*

For the proof we will use the following definition and lemma, which will be very useful in subsequent proofs. Let V_δ be the set of possible $(2\delta + 1)$-tuples of points in the graphs of projectivized polynomial vectors, i.e.

$$\mathrm{Va}_\delta = \Big\{((z_0, [\boldsymbol{f}(z_0)]), \ldots, (z_{2n}, [\boldsymbol{f}(z_{2n})])) \in (\mathbb{C} \times \mathbb{CP}^N)^{2\delta+1} : z_i \neq z_j \text{ for } i \neq j, \\ \boldsymbol{f} \in \mathbb{P}(\mathbb{C}[z]^N \setminus (\mathrm{Res}_\delta \cup \mathbb{C}[z]^N_{\delta-1}))\Big\}$$

We have

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Lemma 1. *The map*

$$\begin{array}{rcl} \mathrm{Pol}_\delta : \mathrm{Va}_\delta & \to & \mathbb{P}(\mathbb{C}[z]^N \setminus (\mathrm{Res}_\delta \cup \mathbb{C}[z]^N_{\delta-1})) \\ ((z_0, [\boldsymbol{f}(z_0)]), \dots, (z_{2n}, [\boldsymbol{f}(z_{2n})]) & \to & \boldsymbol{g} \text{ such that } [\boldsymbol{g}(z_k)] = [\boldsymbol{f}(z_k)],\ 0 \le k \le 2\delta+1 \end{array}$$

is well defined, surjective and algebraic.

Proof. To prove that it is well defined we show that $[\boldsymbol{f}] \in \mathbb{P}(\mathbb{C}[z]^N \setminus (\mathrm{Res}_\delta \cup \mathbb{C}[z]^N_{\delta-1}))$ is completely determined by its values at $2\delta+1$ distinct points $\{z_0, \dots, z_{2\delta+1}\}$ as follows. Note that $[\boldsymbol{f}(z)] = [\boldsymbol{g}(z)]$ if and only if the determinants of the 2×2 minors of the matrix

$$\begin{pmatrix} \boldsymbol{f}(z) \\ \boldsymbol{g}(z) \end{pmatrix}$$

are all identically zero. These determinants are polynomials of degree $2\delta+1$, so they will be identically zero if and only if they vanish at $2\delta+1$ points. Therefore g is given uniquely by the equations given by the vanishing of all the 2×2 minors of

$$\begin{pmatrix} \boldsymbol{f}(z_k) \\ \boldsymbol{g}(z_k) \end{pmatrix} \tag{1}$$

for $0 \le k \le 2\delta$. These equations are linear in the coefficients of the components of $\boldsymbol{g}$, so they are given by a formula involving only matrix operations (it is not hard to find the explicit formula, but we will not need it). Therefore Pol_δ is algebraic.

Finally is clear from the definition of Va_δ that Pol_δ is surjective.

□

Proof of Proposition 1. Fix $z_0, \dots, z_{2\delta}$ distinct points in $\mathbb{C}$. Then the map

$$(z_0, e_{z_0}) \times \cdots \times (z_{2\delta}, e_{z_{2\delta}}) : \mathrm{Hol}_\delta(S^2, \mathbb{CP}^N) \to \mathrm{Va}_\delta,$$

where $e_z : \mathrm{Hol}_\delta(S^2, \mathbb{CP}^N) \to \mathbb{CP}^N$ is the evaluation map defined by $e_z(\psi) = \psi(z)$ is continuous because each e_{z_i} is continuous [2, p. 259]. Therefore $\mathcal{P}$ is continuous since it is the composition of this continuous map and Pol_δ, which is also continuous by Lemma 1.

Since $\mathcal{P}$ is bijective, the collection $\mathcal{P}^{-1}(E) = \{\mathcal{P}^{-1}(U) : U \in E\}$ defines a topology in $\mathrm{Hol}_\delta(S^2, \mathbb{CP}^N)$ with the property that the evaluation map

$$\begin{array}{rcl} e : S^2 \times \mathrm{Hol}_\delta(S^2, \mathbb{CP}^N) & \to & \mathbb{CP}^N \\ (z, \psi) & \to & \psi(z) \end{array}$$

is continuous. The compact open topology is the coarsest topology with this property [3, p. 223], so if U is open in the C-O topology then $U \in \mathcal{P}^{-1}(E)$, so $\mathcal{P}(U) \in E$, so $\mathcal{P}$ is an open map.

□

Corollary 1. *The E topology and the C-O topology in* $\mathrm{HH}_d^{(\le n)}(S^2, \mathcal{Z}_n)$ *coincide.*

Proof. The compact open topology in $\mathrm{HH}_d^{(\le n)}(S^2, \mathcal{Z}_n)$ is the same as the relative topology as a subset of $\mathrm{Hol}_{2d}(S^2, \mathbb{CP}^{N_n})$ (see e.g. [2]), which by Proposition 1 coincides with the relative topology as a subset of $\mathbb{P}(\mathbb{C}[z]^N \setminus (\mathrm{Res}_{2d} \cup \mathbb{C}[z]^N_{2d-1}))$.

□

Let $\mathrm{Harm}_d^{(\le 2n),+}(S^2, S^{2n}) = \mathrm{Harm}_d^{(\le 2(n-1))}(S^2, S^{2n}) \cup \mathrm{Harm}_d^{f,+}(S^2, S^{2n})$. Then

Proposition 2. *The map* $\Pi : \mathrm{HH}_d^{(\le n)}(S^2, \mathcal{Z}_n) \to \mathrm{Harm}_d^{(\le 2n),+}(S^2, S^{2n})$ *is continuous and closed.*

Proof. Since $\pi : \mathcal{Z}_n \to S^{2n}$ is continuous, so is Π (see [2, p. 259]).

To prove that Π is closed we will use the same proof as in [1, Lemma 3.1] and [4, Theorem 2.3], with very slight modifications. Let C be a closed subset of $\mathrm{HH}_d^{(\leq n)}(S^2, \mathcal{Z}_n)$ and let $\overline{C}$ denote the closure of C in $\mathbb{P}(\mathbb{C}[z]_{2d}^{N_n+1})$. Let y be a limit point of $\Pi(C) \subset \mathrm{Harm}_d^{(\leq 2n),+}(S^2, S^{2n})$ and $\{y_k\}$ a sequence converging to y (of course, in the C-O topology). We want to prove that $y \in \Pi(C)$.

Let $\{x_k\}$ be a sequence in $\mathrm{HH}_d^{(\leq n)}(S^2, \mathcal{Z}_n)$ such that $\Pi(x_k) = y_k$, $k = 1, 2, \ldots$. Then there is a convergent subsequence, which will also be denoted by $\{x_k\}$, converging to $x \in \overline{C} \subset \mathbb{P}(\mathbb{C}[z]_{2d}^{N_n+1})$. Write

$$x_k = [\boldsymbol{g}_k] \quad x = [\beta\, \boldsymbol{g}] \qquad k = 1, 2, \ldots,$$

where $\boldsymbol{g}, \boldsymbol{g}_k \in \mathbb{C}[z]_{2d}^{N_n+1}$, $\beta \in C[z]_{2d}$, and the components of $\boldsymbol{g}$ and $\boldsymbol{g}_k$ are coprime.

For fixed $z \in \mathbb{C}$ the evaluation map $e_z : \mathbb{C}[z]_{2d}^{N_n+1} \to \mathbb{C}^{N_n+1}$ defined by $e_z(p) = p(z)$ is continuous in the E topology; it is in fact linear, so $e^{-1}(0)$ is a subspace of $\mathbb{C}[z]_{2d}^{N_n+1}$. Therefore the map

$$E_z : \mathbb{P}(\mathbb{C}[z]_{2d}^{N_n+1} \setminus e_z^{-1}(0)) \to \mathbb{CP}^{N_n}$$

is continuous in the E topology.

Let $z \in \mathbb{C}$ such that $\beta(z) \neq 0$. Then x_k and x are in $\mathbb{P}(\mathbb{C}[z]_{2d}^{N_n+1} \setminus e_z^{-1}(0))$, so $\lim_{k\to\infty} x_k(z) = \lim_{k\to\infty} E_z(x_k) = E_z(x) = x(z)$. Similarly, since the evaluation map $e_z : \mathrm{Harm}_d^{(\leq 2n),+}(S^2, S^{2n}) \to S^{2n}$ is continuous in the C-O topology, $y(z) = e_z(y) = \lim_{k\to\infty} e_z(y_k)$. Therefore,

$$\begin{aligned} y(z) &= \lim_{k\to\infty} e_z(y_k) \\ &= \lim_{k\to\infty} e_z(\Pi(x_k)) \\ &= \lim_{k\to\infty} \pi(x_k(z)) \\ &= \pi(x(z)) \\ &= \pi([\beta(z)\boldsymbol{g}(z)]) \\ &= \pi([\boldsymbol{g}(z)]). \end{aligned}$$

Since $(z \to \pi([\boldsymbol{g}(z)]))$ is defined and continuous for every $z \in \mathbb{C}$, and since the functions y and $(z \to \pi([\boldsymbol{g}(z)]))$ are equal for all but a finite number of points of S^2, continuity implies that these two functions must be the same. Thus, $(z \to [\boldsymbol{g}(z)])$ is a twistor lift of y, and since y has area $4\pi d$, $(z \to [\boldsymbol{g}(z)])$ must have degree $2d$, so β must be a constant. Therefore $x = [\boldsymbol{g}] \in \overline{C}$ has degree $2d$ and its components are coprime. On the other hand its image lies in $\mathcal{Z}_n$ and it must be horizontal since these conditions are closed. Thus $x \in \overline{C} \cap \mathrm{HH}_d^{(\leq n)}(S^2, \mathcal{Z}_n) = C$, so $y = \Pi(x) \in \Pi(C)$, as desired.

□

Proposition 3. *The map*

$$\begin{aligned} \rho_k : \mathrm{Harm}_d^{(2k)}(S^2, S^m) &\to \mathrm{Gr}(2k+1, \mathbb{R}^{m+1}) \\ \varphi &\to (2k+1)\text{-subspace where } \varphi(S^2) \text{ lies.} \end{aligned}$$

is continuous, and $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$ *is a fiber bundle over* $\mathrm{Gr}(2k+1, \mathbb{R}^{m+1})$ *with fiber* $\mathrm{Harm}_d^{f,+}(S^2, S^{2k})$.

Proof. For each $(2k+1)$-tuple $\boldsymbol{z} = (z_0, \ldots, z_{2k})$ of distinct complex numbers the map $L_{\boldsymbol{z}}$ that takes $\phi \in \mathrm{Harm}_d^{(2k)}(S^2, S^m)$ to $\phi(z_0) \wedge \cdots \wedge \phi(z_{2k}) \in \Lambda^{2k+1}\mathbb{R}^{m+1}$ is continuous in the C-O topology of $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$ since it is the composition of evaluation and wedging. Therefore $D_{\boldsymbol{z}} := L_{\boldsymbol{z}}^{-1}(0)$ is closed in $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$. Thinking of $\mathrm{Gr}(2k+1, \mathbb{R}^{m+1})$ as a subvariety of $\mathbb{P}(\Lambda^{2k+1}\mathbb{R}^{m+1})$ via the Plücker embedding, note that for all $\phi \in (D_{\boldsymbol{z}})^c$, $\rho_k(\phi) = [L(\phi)]$, so ρ_k is continuous in $(D_{\boldsymbol{z}})^c$. Now, $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$ can be covered by open subsets of the form $(D_{\boldsymbol{z}})^c$, so ρ_k is continuous.

Now we prove that $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$ is a fiber bundle over $\mathrm{Gr}(2k+1, \mathbb{R}^{m+1})$ with fiber $\mathrm{Harm}_d^{f,+}(S^2, S^{2k})$. The group $SO(m+1, \mathbb{R})$ acts on $\mathrm{Gr}(2k+1, \mathbb{R}^{m+1})$ and S^m; the action of $A \in SO(m+1, \mathbb{R})$ on $x \in \mathrm{Gr}(2k+1, \mathbb{R}^{m+1})$ or S^m will be denoted by Ax. Further, since $SO(m+1, \mathbb{R})$ acts by isometries on S^m, it induces an action of $SO(m+1, \mathbb{R})$ on $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$; the action of $A \in SO(m+1, \mathbb{R})$ on $\phi \in \mathrm{Harm}_d^{(2k)}(S^2, S^m)$ will be denoted by $A_*\phi$.

Let $P_0 \in \mathrm{Gr}(2k+1, \mathbb{R}^{m+1})$ be the subspace of $\mathbb{R}^{m+1}$ generated by the first $(2k+1)$ coordinate vectors of the standard basis. Consider the map $\tau : SO(m+1, \mathbb{R}) \to \mathrm{Gr}(2k+1, \mathbb{R}^{m+1})$ defined by $\tau(A) = AP_0$. This gives $SO(m+1, \mathbb{R})$ the structure of a fiber bundle over $\mathrm{Gr}(2k+1, \mathbb{R}^{m+1})$. Given $P \in \mathrm{Gr}(2k+1, \mathbb{R}^{m+1})$ let $U \ni P$ be open and $\sigma : U \to SO(m+1, \mathbb{R})$ be a local section of this bundle, so $\tau(\sigma(Q)) = A\sigma(Q) = Q$ for all $Q \in U$.

Since ρ_k is continuous, $\rho_k^{-1}(U)$ is open in $\mathrm{Harm}_d^{(2k)}(S^2, S^m)$. Let $i : S^{2k} \to S^{2n}$ be the inclusion of S^{2k} in the first $2k+1$ coordinates of S^{2n}. This induces a continuous map

$$i_* : \mathrm{Harm}_d^{f,+}(S^2, S^{2k}) \to \mathrm{Harm}_d^{(2k)}(S^2, S^m).$$

Define the map

$$\begin{aligned} r : U \times \mathrm{Harm}_d^{f,+}(S^2, S^{2k}) &\to \mathrm{Harm}_d^{(2k)}(S^2, S^m) \\ (P, \phi) \to r(P, \phi) = \sigma(P)_* \circ i_*(\phi) & \end{aligned}$$

Since it is a composition of continuous functions, r is continuous. Also, since $i_*(\phi)$ lies in P_0, $\sigma(P)_* \circ i_*(\phi)$ lies in P, so $\sigma_k(r(P, \phi)) = P$.

The inverse of r can be written as

$$r^{-1}(\gamma) = (\rho_k(\gamma), [\sigma(\rho_k(\gamma))]_*^{-1}\gamma),$$

which is also a composition of continuous functions. Therefore r is a homeomorphism, proving the claim. □

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Luis Fernández
Department of Mathematics
CUNY – BCC
2155 University Avenue
Bronx, New York 10453, USA
lmfernand@gmail.com
luis.fernandez01@bcc.cuny.edu