REGULARITY OF THE SPACE OF HARMONIC 2-SPHERES IN THE UNIT 4-SPHERE

John Bolton and Luis Fernandez

Abstract

A harmonic map of the Riemann sphere into the unit 4-dimensional sphere has area $4\pi d$ for some positive integer d, and it is well-known that the space of such maps may be given the structure of a complex algebraic variety of dimension $2d+4$. When d less than or equal to 2, the subspace consisting of those maps which are linearly full is empty. The twistor fibration from complex projective 3-space to the 4-sphere has been used to show that, for $d = 3, 4, 5$, this subspace is a complex manifold. These methods are used here to extend this result to $d = 6$.

1 Introduction

Every harmonic map from the Riemann sphere S^2 into the unit 4-sphere S^4 has area $4\pi d$ for some integer d. It has been known for some time [6, 7, 12] that the space $\mathrm{Harm}_d(S^4)$ of such maps may be studied in terms of the twistor lifts of the elements to horizontal holomorphic curves of degree d in complex projective 3-space $\mathbb{C}P^3$. It follows from this that $\mathrm{Harm}_d(S^4)$ may be given the structure of a complex algebraic variety, and a detailed study of this space has been carried out in [13, 17, 18, 19], where, in particular, it is shown that the complex dimension is $2d + 4$. This result is a special case of the recent verification in [10] of the conjecture in [2] that the moduli space of harmonic maps of S^2 into S^{2m} of degree d is a complex algebraic variety of dimension $2d + m^2$.

A natural topology to put on $\mathrm{Harm}_d(S^4)$ is the compact-open topology, and it is shown in [5] that this topology coincides with that coming from the complex algebraic variety structure. Using the twistorial approach mentioned above, it was also shown in [5] that, in

2000 *Mathematics Subject Classification.* Primary 58D10; Secondary 53C43.

Key words and phrases. harmonic maps, 2-sphere, twistor fibration.

This research was partially supported by a grant from the City University of New York PSC-CUNY Research Award Program and by a Scheme 4 grant from the London Mathematical Society.

this topology, the open subset $\mathrm{Harm}_d^{LF}(S^4)$ of $\mathrm{Harm}_d(S^4)$ consisting of linearly full maps has no singular points for $d \leq 5$, so that $\mathrm{Harm}_d^{LF}(S^4)$ is a manifold.

We were encouraged to use the approach of [5] to consider the case $d = 6$, since it seemed possible (and even likely) that $\mathrm{Harm}_6^{LF}(S^4)$ would have singular points. Indeed, non-linearly full elements of $\mathrm{Harm}_d(S^4)$ which are the limit of linearly full harmonic 2-spheres are singular points of $\mathrm{Harm}_d(S^4)$, and, when $d \geq 6$, there are linearly full harmonic 2-spheres in S^4 of degree d which are the limit of linearly full harmonic 2-spheres in higher-dimensional spheres (see [11] for some results on collapsing in higher dimensions, and see [16] for a study of the related problem of when all Jacobi fields along a harmonic 2-sphere are integrable). However, in this paper we show that the moduli space $\mathrm{Harm}_6^{LF}(S^4)$ is a manifold. Specifically, we prove the following theorem.

THEOREM 1.1. *For $3 \leq d \leq 6$, the space $Harm_d^{LF}(S^4)$ equipped with the compact-open topology, is a complex manifold of complex dimension $2d + 4$. For $d \leq 2$, $Harm_d^{LF}(S^4)$ is empty.*

Unfortunately, it seems unlikely that our methods will enable us to draw conclusions for higher values of d. Firstly, it is not clear how to extend Proposition 4.3 to degree greater than 6, and, secondly, the *ad hoc* mathematica calculation of the rank of the map $d\Phi_d$ carried out in Section 5 is likely to be considerably more complicated. It may be, however, that there is a more systematic way of carrying out this calculation using, for example, the notion of discriminant. As remarked above, Theorem 1.1 was proved in [5] for $d \leq 5$.

Similar questions for the space of harmonic maps from S^2 to $\mathbb{C}P^2$ have been studied in [8] and [14]. In fact, the components of this space consist of the $\pm$-holomorphic maps of degree d, together with harmonic maps of degree d and energy $4\pi E$, where $E = 3|d| + 4 + 2r$ for some non-negative integer r. It is shown in [14] (see also [15]) that these components are smooth manifolds of dimension $6|d| + 4$ in the $\pm$-holomorphic case and $2E + 8$ in the other cases.

2 The twistor fibration

We begin by recalling the *twistor fibration* $\pi : \mathbb{C}P^3 \to S^4$. Regarding $\mathbb{H}^2$ as a left quaternionic vector space, this is obtained by composing the Hopf map $\rho : \mathbb{C}P^3 \to \mathbb{H}P^1$ given by

$$\rho([z_1, z_2, z_3, z_4]) = [z_1 + z_2 j, z_3 + z_4 j],$$

with the canonical identification of $\mathbb{H}P^1$ and $S^4 \subset \mathbb{H} \oplus \mathbb{R} = \mathbb{R}^5$ given in the usual way by stereographic projection from $(0, 0, 0, 0, -1)$ onto the equatorial 4-plane $\mathbb{H}$ in $\mathbb{R}^5$ which is included in $\mathbb{H}P^1$ by $[q] \mapsto [q, 1]$. Specifically, this identification is given by

$$[q_1, q_2] \in \mathbb{H}P^1 \leftrightarrow \frac{(2\bar{q}_1 q_2, |q_1|^2 - |q_2|^2)}{|q_1|^2 + |q_2|^2} \in S^4.$$

We recall [4, 6] that π is a Riemannian submersion when $\mathbb{C}P^3$ is given the Fubini-Study metric of constant holomorphic curvature 1.

Now consider the vector space $\mathbb{C}[z]_d$ of polynomials of degree less than or equal to d, and let V_d be the subset of $(\mathbb{C}[z]_d)^4$ consisting of those quadruplets of coprime polynomials with maximum degree equal to d for which the map $z \mapsto [f_1(z), f_2(z), f_3(z), f_4(z)]$ is linearly full in $\mathbb{C}P^3$. Then V_d is a projective subset of $(\mathbb{C}[z]_d)^4 \setminus \{0\}$, and we identify its projectivisation $P(V_d)$ with the space of linearly full holomorphic maps of degree d from S^2 to $\mathbb{C}P^3$ in the usual way via

$$[f_1, f_2, f_3, f_4] \quad \longleftrightarrow \quad z \mapsto [f_1(z), f_2(z), f_3(z), f_4(z)]. \tag{1}$$

Here, and subsequently, we use the complex coordinate z on S^2 defined by sterographic projection from the south pole of S^2 onto the equatorial plane so that, in the usual sense, we may identify S^2 with $\mathbb{C} \cup \{\infty\}$. If $\psi : S^2 \to \mathbb{C}P^3$ is holomorphic then we call the above representation

$$\psi(z) = [f_1(z), f_2(z), f_3(z), f_4(z)]$$

with $(f_1, f_2, f_3, f_4) \in V_d$ a *reduced form* of ψ.

We give $(\mathbb{C}[z]_d)^4$ its natural topology as a vector space, and $P((\mathbb{C}[z]_d)^4)$ the identification topology. Then V_d is an open subset of $(\mathbb{C}[z]_d)^4$, and $P(V_d)$ is an open subset of $P((\mathbb{C}[z]_d)^4)$. Subsets of any of these spaces are then given the induced (subspace) topology.

A holomorphic curve $\psi = [f_1, f_2, f_3, f_4]$ in $\mathbb{C}P^3$ is *horizontal* if it intersects each fibre orthogonally. It is well known [6] that this holds if and only if

$$f_1' f_2 - f_1 f_2' + f_3' f_4 - f_3 f_4' = 0,$$

or, alternatively, if and only if

$$(\mathbf{f}', J\mathbf{f}) = 0, \tag{2}$$

where $\mathbf{f} = (f_1, f_2, f_3, f_4) \in V_d$, $(\ ,\ )$ denotes the complex bilinear extension to $\mathbb{C}^4$ of the standard real inner product on $\mathbb{R}^4$, and

$$J(f_1, f_2, f_3, f_4) = (-f_2, f_1, -f_4, f_3).$$

Thus, if we define $\Phi_d : V_d \to \mathbb{C}[z]_{2d-2}$ by

$$\Phi_d(\mathbf{f}) = (\mathbf{f}', J\mathbf{f}),$$

then $\Phi_d^{-1}\{0\}$ is a projective subset of V_d, and, using the identification (1) above, $P(\Phi_d^{-1}\{0\})$ is identified with the space of linearly full horizontal holomorphic maps of degree d (and hence area $4\pi d$) from S^2 to $\mathbb{C}P^3$.

The space $\mathrm{Harm}_d^{LF}(S^4)$ is the union of two connected components, $\mathrm{Harm}_d^+(S^4)$ and $\mathrm{Harm}_d^-(S^4)$, with post-composition by the antipodal map of S^4 giving a homeomorphism between them. Each element of $\mathrm{Harm}_d^+(S^4)$ has a unique lift to an element of $P(\Phi_d^{-1}\{0\})$, so that post-composition by π gives a bijective correspondence

$$\pi_* : P(\Phi_d^{-1}\{0\}) \to \mathrm{Harm}_d^+(S^4).$$

The following lemma is proved in [5].

LEMMA 2.1. *$Harm_d^+(S^4)$ is a closed subset of $Harm_d^{LF}(S^4)$, and the map $\pi_* : P(\Phi_d^{-1}\{0\}) \to Harm_d^+(S^4)$ is a homeomorphism.*

We note that the final statement of Theorem 1.1 follows immediately from Lemma 2.1, since V_d is empty for $d \leq 2$.

3 Higher singularities and the Plücker formulae

We first recall the definition of *singularity type* of a linearly full holomorphic curve $\psi : S^2 \to \mathbb{C}P^n$ [9, 2, 3]. We may write such a curve in *reduced form* $\psi(z) = [\mathbf{f}(z)] = [f_1(z), \ldots, f_{n+1}(z)]$, where $f_1, \ldots, f_{n+1}$ are polynomials in z with no common zeros. In fact, $\mathbf{f}$ can be written in the following *normal form* about a point z_0,

$$\mathbf{f}(z) = h_0(z)\mathbf{a}_0 + (z - z_0)^{r_0(z_0)+1} h_1(z)\mathbf{a}_1 + \ldots + (z - z_0)^{r_0(z_0)+\ldots+r_{n-1}(z_0)+n} h_n(z)\mathbf{a}_n$$

for some suitable choice of (unitarily orthonormal) basis $\mathbf{a}_0, \ldots, \mathbf{a}_n$ of $\mathbb{C}^{n+1}$, non-negative integers $r_0(z_0), \ldots, r_{n-1}(z_0)$, and polynomials $h_0(z), \ldots, h_n(z)$, each non-zero at z_0. The point z_0 is a *higher singularity* of ψ if $r_k(z_0) \neq 0$ for at least one $k = 0, \ldots, n-1$, and we let $Z(\psi)$ denote the isolated set of higher singularities. The *singularity type* of ψ is then defined to be the set

$$\{(z; r_0(z), \ldots, r_{n-1}(z)) \mid z \in Z(\psi)\}.$$

For each $k = 0, \ldots, n-1$ the *k-th osculating curve (or associated curve)* of ψ is defined for all z in the domain of ψ, and, away from $Z(\psi)$, is given by $[\mathbf{f} \wedge \ldots \wedge \mathbf{f}^{(k)}]$, where $\mathbf{f}^{(j)}$ denotes the j-th derivative of $\mathbf{f}$ with respect to z. If $r_k(z_0) > 0$ then the derivative of the k-th osculating curve has a zero of order $r_k(z_0)$ at z_0.

In the next section we will use group actions to obtain canonical forms for linearly full horizontal holomorphic 2-spheres in $\mathbb{C}P^3$. We now move towards this by considering the natural action on $\mathbb{C}P^3$ of the complexified symplectic group

$$Sp(2; \mathbb{C}) = \{A : \mathbb{C}^4 \to \mathbb{C}^4 \mid (JA\mathbf{v}, A\mathbf{w}) = (J\mathbf{v}, \mathbf{w}) \ \forall \mathbf{v}, \mathbf{w} \in \mathbb{C}^4\}.$$

In fact, the projectivisation $PSp(2;\mathbb{C})$ of this group acts on $\mathbb{C}P^3$ as the group of holomorphic diffeomorphisms which preserve the horizontal distribution, with $PSp(2) = PSp(2;\mathbb{C}) \cap PU(4)$ being the subgroup of holomorphic isometries which preserve the horizontal distribution [4]. This induces a natural action of $Sp(2;\mathbb{C})$ on V_d, and hence on $P(V_d)$ via $(A\mathbf{f})(z) = A(\mathbf{f}(z))$, and this action preserves the set $\Phi_d^{-1}\{0\}$. We also note that if μ is a Möbius transformation, and if $\psi = [\mathbf{f}]$ is a holomorphic curve in $\mathbb{C}P^3$, then $\psi\mu$ is also holomorphic. Moreover, ψ is horizontal if and only if $\psi\mu$ is horizontal.

LEMMA 3.1. *Let $g : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}$ be a non-singular linear map, and $\mu(z)$ a Möbius transformation. If ψ has singularity type $\{(z; r_0(z), \ldots, r_{n-1}(z)) \mid z \in Z(\psi)\}$ then $g\psi\mu$ has singularity type $\{(\mu^{-1}(z); r_0(z), \ldots, r_{n-1}(z)) \mid z \in Z(\psi)\}$.*

The following proposition is proved in [3]. Here, $v(f)$ denotes the order of vanishing of a holomorphic function $f(z)$ at $z = 0$.

PROPOSITION 3.2. *Let $\psi(z)$ be a linearly full horizontal holomorphic curve in $\mathbb{C}P^3$. Then*

$$r_0(z) = r_2(z).$$

Also, if $\psi(z)$ has a higher singularity at $z = 0$ then there exists $A \in Sp(2;\mathbb{C})$ such that $A\psi(0) = [1, 0, 0, 0]$ and if $A\psi = [f_1, f_2, f_3, f_4]$ is in reduced form then $v(f_3) < v(f_4)$. In this case,

$$A\psi(z) = [h_0(z), z^{2k_1+k_2}h_3(z), z^{k_1}h_1(z), z^{k_1+k_2}h_2(z)],$$

where $k_1 = r_0(0) + 1 = r_2(0) + 1$, $k_2 = r_1(0) + 1$ and $h_k(z)$, $k = 0, 1, 2, 3$, are polynomials each of which is non-zero at $z = 0$.

As shown in [3], the Plücker formulae [9, 1] for a linearly full holomorphic curve $\psi : S^2 \to \mathbb{C}P^n$ imply that if $\psi : S^2 \to \mathbb{C}P^3$ is a linearly full horizontal holomorphic curve in $\mathbb{C}P^3$ then

$$2r_0 + r_1 = 2d - 6, \tag{3}$$

where d is the degree of ψ, and

$$r_k = \sum_{p \in Z(\psi)} r_k(p).$$

4 Canonical forms

The method of proof of Theorem 1.1 is to show that, for $3 \leq d \leq 6$, the zero polynomial is a regular value of Φ_d. It will then follow that $\Phi_d^{-1}\{0\}$ is a submanifold of V_d of dimension

$4(d+1)-(2d-1)=2d+5$, from which the theorem will follow by projectivising and using Lemma 2.1.

We begin by writing down the derivative $d\Phi_d|_{\mathbf{f}}$ of Φ_d at a point $\mathbf{f}=(f_1,\ldots,f_4)$ of V_d. The unusual indexing on the left hand side is to facilitate the writing down of the derivative.

$$d\Phi_d|_{\mathbf{f}}(h_2,h_1,h_4,h_3)=\sum_{p=1}^{4}(-1)^p(f_p h_p'-f_p' h_p). \tag{4}$$

As mentioned earlier, Theorem 1.1 has already been proved in [5] for $d\le 5$, so from now on we consider the case $d=6$. In order to simplify the calculations we extend to $d=6$ the results of [3], where natural group actions are used to obtain canonical forms for elements of $P(\Phi_d^{-1}\{0\})$ for $3\le d\le 5$. We recall from the previous section the action of the complexified symplectic group on V_d and on the space of holomorphic horizontal curves in $\mathbb{C}P^3$ via $(A\mathbf{f})(z)=A(\mathbf{f}(z))$. It is shown in [5] (and is easily checked) that

Lemma 4.1. *If $\psi=[\mathbf{f}]$ and if $A\psi\mu=[\tilde{\mathbf{f}}]$, where $A\in Sp(2;\mathbb{C})$ and μ is a Möbius transformation, then the rank of $d\Phi_d|_{\mathbf{f}}$ is equal to the rank of $d\Phi_d|_{\tilde{\mathbf{f}}}$.*

We now produce our canonical form for an element $\psi=[\mathbf{f}]$ in $P(\Phi_6^{-1}\{0\})$. It will be convenient to recall the following definition from [3]. If z_1, $z_2\in S^2$ then we will say that $\{z_1,z_2\}$ is a *ψ-null pair* if $(\psi(z_1),J\psi(z_2))=0$, that is, if $\overline{\psi(z_1)}$ is unitarily orthogonal to $J\psi(z_2)$. We note that if $\{z_1,z_2\}$ is a ψ-null pair, then it is also an $A\psi$-null pair for any $A\in Sp(2;\mathbb{C})$, while $\{\mu^{-1}(z_1),\mu^{-1}(z_2)\}$ is a $\psi\mu$-null pair for any Möbius transformation μ.

Lemma 4.2. *If ψ has degree $d=6$ then there exist a pair of higher singularities that do not form a ψ-null pair.*

Proof. We assume that $z=0$ is a higher singularity of ψ, and that $\psi(0)=[1,0,0,0]$. In this case, it follows from Proposition 3.2 that, modulo the action of $Sp(2,\mathbb{C})$,

$$\psi(z)=[f_1(z),z^4(b_4+b_5z+b_6z^2),f_3(z),f_4(z)].$$

Since $J(a,b,c,d)$ is unitarily orthogonal to $(1,0,0,0)$ if and only if $b=0$, we see that at most two other points of S^2 are ψ-null pairs with $z=0$. In particular, the lemma is proved if ψ has four higher singularities.

If ψ has exactly three higher singularities then the Plücker formulae (3) show that at least one of them, say $z=0$, does not have $r_0(0)=0,r_1(0)=1$. Assume one of the other singularities is at $z=\infty$ and that $\{0,\infty\}$ form a ψ-null pair. Then, writing $f(z)$ in the form of Proposition 3.2, we have $2k_1+k_2\ge 5$, so that

$$\psi(z)=[h_0(z),b_5z^5,z^{k_1}h_1(z),z^{k_1+k_2}h_2(z)]$$

so that the third higher singularity doesn't form a ψ-null pair with $z = 0$. Finally, as obseved in [3], it follows from Theorem 4.1 of that paper that if ψ is 2-point ramified then the two higher singularities do not form a ψ-null pair. □

PROPOSITION 4.3. *Let* $\psi : S^2 \to \mathbb{C}P^3$ *be a linearly full horizontal holomorphic curve of degree* 6*. Then there exists a Möbius transformation* μ *and an element* A *of* $Sp(2,\mathbb{C})$ *such that* **either**

$$A\psi\mu(z) = [a_0 + a_1 z + a_2 z^2, b_4 z^4 + b_5 z^5 + b_6 z^6, c_1 z + c_2 z^2 + c_4 z^4 + c_5 z^5, d_3 z^3], \tag{5}$$

with $a_0 b_6 d_3 \neq 0$ *and with certain relations between the coefficients (which we will not need) given by the horizontalilty condition (2),*
or

$$A\psi\mu(z) = [a_0 + a_1 z + a_2 z^2, b_4 z^4 + b_5 z^5 + b_6 z^6, c_1 z + c_2 z^2 + c_3 z^3, d_3 z^3 + d_4 z^4 + d_5 z^5], \tag{6}$$

with $a_0 b_6 \neq 0$ *and the following horizontality conditions*

$$2a_0 b_4 + c_1 d_3 = 0, \tag{7}$$

$$5a_0 b_5 + 3a_1 b_4 + 3c_1 d_4 + c_2 d_3 = 0, \tag{8}$$

$$3a_0 b_6 + 2a_1 b_5 + a_2 b_4 + 2c_1 d_5 + c_2 d_4 = 0, \tag{9}$$

$$5a_1 b_6 + 3a_2 b_5 + 3c_2 d_5 + c_3 d_4 = 0, \tag{10}$$

$$2a_2 b_6 + c_3 d_5 = 0. \tag{11}$$

PROOF. It follows from Lemma 4.2 and Lemma 4.2 of [3] that there exists a Möbius transformation μ and $\tilde{A} \in Sp(2,\mathbb{C})$ such that

1. $\tilde{A}\psi\mu(0) = [1, 0, 0, 0]$,
2. $z = 0$ is a higher singularity of $\tilde{A}\psi\mu$,
3. $\tilde{A}\psi\mu(\infty) = [0, 1, 0, 0]$,
4. $z = \infty$ is a higher singularity of $\tilde{A}\psi\mu$.

In this case, it follows from Proposition 3.2 that there exists $A \in Sp(2,\mathbb{C})$ such that

$$A\psi\mu(z) = [a_0 + \ldots + a_5 z^5, b_4 z^4 + b_5 z^5 + b_6 z^6, c_1 z + \ldots + c_5 z^5, d_3 z^3 + d_4 z^4 + d_5 z^5], \quad a_0 b_6 \neq 0.$$

Case 1: $\mathbf{deg(c_1 z + \ldots + c_5 z^5) > deg(d_3 z^3 + d_4 z^4 + d_5 z^5)}$.

Let $\tilde{\phi}(z) = BA\psi\mu(1/z)$, where $B \in Sp(2,\mathbb{C})$ is given by

$$B(f_1, f_2, f_3, f_4) = (-f_2, f_1, f_3, f_4).$$

Then

$$\tilde{\phi}(z) = [-b_6 - b_5 z - b_4 z^2, a_5 z + \ldots + a_0 z^6, c_5 z + \ldots + c_1 z^5, d_5 z + d_4 z^2 + d_3 z^3]$$

has a higher singularity at $z = 0$, and $\tilde{\phi}(0) = [1, 0, 0, 0]$. Moreover, the order of vanishing at $z = 0$ of $c_5 z + \ldots + c_1 z^5$ is less than that of $d_5 z + d_4 z^2 + d_3 z^3$. Hence, by Proposition 3.2, it follows that $a_5 = a_4 = a_3 = 0$, $d_5 = d_4 = 0$. Hence d_3 cannot be zero so we may use a further element of $Sp(2,\mathbb{C})$ to assume that $c_3 = 0$. This give us the canonical form (5).

Case 2: $\mathbf{deg(c_1 z + \ldots + c_5 z^5) \leq deg(d_3 z^3 + d_4 z^4 + d_5 z^5)}$.

Let $\phi(z) = JA\psi\mu(1/z)$. Then

$$\phi(z) = [-b_6 - b_5 z - b_4 z^2, a_5 z + \ldots + a_0 z^6, -d_5 z - d_4 z^2 - d_3 z^3, c_5 z + \ldots + c_1 z^5]$$

has a higher singularity at $z = 0$, and $\phi(0) = [1, 0, 0, 0]$.

Hence, by first applying an element of $Sp(2,\mathbb{C})$ which fixes elements of $\mathbb{C}^4$ of the form $(a, b, c, 0)$, we may assume that $\deg(c_1 z + \ldots + c_5 z^5) < \deg(d_3 z^3 + d_4 z^4 + d_5 z^5)$. It now follows from Proposition 3.2 that $a_5 = a_4 = a_3 = 0$, $c_5 = c_4 = 0$, so that $A\psi\mu$ has the form given in (6).

The horizontality conditions (7), (8), (9), (10), (11), follow immediately from (2). □

5 Proof of Theorem 1.1

As mentioned earlier, we prove the theorem by showing that the zero polynomial is a regular value of $\Phi_6 : V_6 \to \mathbb{C}[z]_{10}$. By Lemma 4.1 and Proposition 4.3, we may show this by showing that, for each $\mathbf{f}$ in $\Phi_6^{-1}\{0\}$ taking one of the two forms (5) or (6), the rank of the derivative $d\Phi_6|_{\mathbf{f}}$, given in (4), is equal to 11. The method we use is a brute-force calculation using *Mathematica*. It would be gratifying to find a more elegant, geometrical proof.

We first fix bases of $(\mathbb{C}[z]_6)^4$ and $\mathbb{C}[z]_{10}$ in the obvious way, using the standard basis $\{1, z, \ldots, z^n\}$ of $\mathbb{C}[z]_n$. For $\mathbf{f} \in V_6$ we then let $D_{\mathbf{f}}$ be 11×28 matrix of $d\Phi_6|_{\mathbf{f}}$ with respect to these bases.

LEMMA 5.1. *If* $\mathbf{f}$ *has the form of* (5), *then* $d\Phi_6|_{\mathbf{f}}$ *has rank 11.*

PROOF. The result here is staightforward. It is easy to identify an 11×11 minor of $D_{\mathbf{f}}$ which may be made lower-triangular by a permutation of the rows and a permutation of the columns, and whose determinant is then easily seen to be a non-zero scalar multiple of ${a_0}^3{b_6}^2{d_3}^6$. □

LEMMA 5.2. *If* $\mathbf{f}$ *has the form of* (6) *with any of* $a_2, b_4, c_1, c_3, d_3, d_5$ *being zero then* $d\Phi_6|_{\mathbf{f}}$ *has rank 11.*

PROOF. By considering the map $\phi(z) = [J\mathbf{f}(1/z)]$, we see that we only need to show that the lemma is true if any of b_4, c_1, d_3 are zero.

Case 1: $\mathbf{b_4 = 0}$. We first note that if b_5 is also zero, then it is easy to pick out an 11×11 minor of $D_{\mathbf{f}}$ which may be made lower-triangular by a permutation of the rows and a permutation of the columns, and whose determinant is then easily seen to be a non-zero scalar multiple of ${a_0}^6{b_6}^5$.

So, we now assume that $b_4 = 0$ but $b_5 \neq 0$. In this case, it follows from Proposition 3.2 that $2k_1 + k_2 = 6$ and so either $k_1 = 1$, and $k_2 = 3$, or $k_1 = 2$, and $k_2 = 1$. If $k_1 = 1$, and $k_2 = 3$ then, by Proposition 3.2 again, $c_1 \neq 0$, $d_3 = 0$ and $d_4 \neq 0$. We may then pick out an 11×11 minor of $D_{\mathbf{f}}$ which may be made lower-triangular by a permutation of the rows and a permutation of the columns, and whose determinant is then easily seen to be a non-zero scalar multiple of ${a_0}^3{b_5}^2{d_4}^6$. On the other hand if $k_1 = 2$, and $k_2 = 1$ then $c_1 = 0$, $c_2 \neq 0$ and $d_3 \neq 0$. We may then pick out an 11×11 minor of $D_{\mathbf{f}}$ which may be made lower-triangular by a permutation of the rows and a permutation of the columns, and whose determinant is then easily seen to be a non-zero scalar multiple of $a_0{b_6}^6{c_2}^2{d_3}^2$.

Case 2: $\mathbf{c_1 = 0}$. Here, we have that $k_1 \geq 2$, so that $2k_1 + k_2 \geq 5$. Hence $b_4 = 0$ and we are back in Case 1.

Case 3: $\mathbf{d_3 = 0}$. Here, we have that $k_1 + k_2 \geq 4$, so that $2k_1 + k_2 \geq 5$. Hence $b_4 = 0$ and we are again back in Case 1. □

So, from now on we assume that $\mathbf{f}$ has the form of (6) with all of $a_0, a_2, b_4, b_6, c_1, c_3, d_3, d_5$ being non-zero. For subsequent cases we will need to consider further 11×11 minors of $D_{\mathbf{f}}$. It turns out that there is a minor whose determinant is a non-zero multiple of ${a_2}^5 b_4 {b_6}^2 c_3(3a_0c_3 + a_2c_1 - a_1c_2)$, so that $D_{\mathbf{f}}$ has rank 11 unless this expression is equal to zero. Consideration of another two suitable minors shows that $D_{\mathbf{f}}$ has rank 11 except possibly when the following three equations hold

$$3a_0c_3 + a_2c_1 - a_1c_2 = 0, \tag{12}$$

$$b_4d_5 - b_5d_4 + 3b_6d_3 = 0, \tag{13}$$

$$(14) \qquad -5c_1d_5 + 2c_2d_4 - 3c_3d_3 = 0.$$

LEMMA 5.3. *If* **f** *has the form of* (6) *with* $a_0, a_2, b_4, b_6, c_1, c_3, d_3, d_5$ *all being non-zero, but* $a_1 = b_5 = 0$*, then* $d\Phi_6|_{\mathbf{f}}$ *has rank 11.*

PROOF. In this case, we can use (7), (11) and (13) to write a_0, a_2 and d_3 in terms of b_4, b_6, c_1, c_3 and d_5. In fact,

$$a_0 = \frac{d_5c_1}{6b_6}, \quad a_2 = -\frac{d_5c_3}{2b_6}, \quad d_3 = -\frac{b_4d_5}{3b_6}.$$

If we substitute these into (9) and (14) we find that

$$c_2d_4 = 0 \quad \text{and} \quad b_4 = \frac{5c_1b_6}{c_3}.$$

By replacing $[\mathbf{f}]$ by the map $\phi(z) = [J\mathbf{f}(1/z)]$ if necessary, we may assume that $d_4 = 0$ in which case (10) shows that $c_2 = 0$ also.

It now follows that the following two matrices

$$\begin{pmatrix} a_0 & -b_4 \\ a_2 & -3b_3 \end{pmatrix}, \quad \begin{pmatrix} 5a_0 & 3c_1 \\ 3a_2 & c_3 \end{pmatrix}$$

have non-zero determinant, these being non-zero scalar multiples of c_1d_5 and $c_1c_3d_5/b_6$ respectively.

Using this, we may then find an 11×11 minor of $D_{\mathbf{f}}$ which, by a permutation of the rows and a permutation of the columns, may be exhibited as the direct sum of a lower-triangular matrix and the above two matrices. The determinant of this minor is then easily seen to be a non-zero scalar multiple of the product of ${a_0}^2{a_2}^2b_4{b_6}^2$ and the determinants of the above two matrices. This shows that $D_{\mathbf{f}}$ has rank 11 if $a_1 = b_5 = 0$. □

Using the usual argument involving ϕ, it remains to prove that $D_{\mathbf{f}}$ has rank 11 when $b_5 \neq 0$. In summary, then, Lemmas 5.2 and 5.3 show that $D_{\mathbf{f}}$ has rank 11 whenever **f** has the form 6 with any of $a_2, b_4, b_5, c_1, c_3, d_3, d_5$ being zero.

LEMMA 5.4. *If* $\mathbf{f} \in V_6$ *has the form of* (6) *with* $a_0, a_2, b_4, b_5, b_6, c_1, c_3, d_3, d_5$ *all being non-zero, then* $d\Phi_6|_{\mathbf{f}}$ *has rank 11.*

PROOF. We first note that by using projective equivalence and by replacing z by λz if necessary, we may assume that $b_5 = c_3 = 1$.

Assuming that $1 - 2b_6c_2$ and $3 - 2b_4b_6 + 10{b_6}^2c_1 - 12b_6c_2 + 12{b_6}^2{c_2}^2$ are both non-zero, we may use (7)-(11), (12)-(14) to write $a_0, a_2, b_4, c_1, c_2, d_3, d_4, d_5$ in terms of a_1 and b_6. We then find that

$$[\mathbf{f}] = [a_1(1+2zb_6)^2, z^4(1+2zb_6)^2, z(1+2zb_6)^2, 2z^3b_6(1+2zb_6)^2],$$

so that $\mathbf{f}$ has a non-trivial common factor and hence is not in V_6.

We now consider what happens when $1-2b_6c_2=0$. Then this, together with (7)-(11), (12)-(14) enables us to write $a_0, a_1, b_4, c_2, d_4, d_5$ in terms of a_2, b_6, c_1, d_3. Mathematica now lets us pull out two 11×11 minors of $D_{\mathbf{f}}$ with determinants non-zero multiples of

$$d_3(1-18{b_6}^2c_1)+60a_2{b_6}^3c_1 \quad \text{and} \quad 3a_2-10a_2{b_6}^2c_1+3b_6d_3.$$

Thus, if either of the above expressions is non-zero then $D_{\mathbf{f}}$ has rank 11. We now consider what happens when

$$d_3(1-18{b_6}^2c_1)+60a_2{b_6}^3c_1=0 \quad \text{and} \quad 3a_2-10a_2{b_6}^2c_1+3b_6d_3=0.$$

Then $1-18{b_6}^2c_1\neq 0$ and and we may solve the above two equations for d_3 and c_1 in terms of a_2 and b_6. We then find that

$$[\mathbf{f}]=[a_2(3+8zb_6)(9+40zb_6), 5b_6z^4(3+8zb_6)(5+8zb_6), \\ 5z(3+8zb_6)(1+8zb_6), -10a_2b_6z^3(3+8zb_6)(9+8zb_6)],$$

so that $\mathbf{f}$ has a non-trivial common factor and hence is not in V_6.

Finally we consider the case when $1-2b_6c_2\neq 0$, but $3-2b_4b_6+10{b_6}^2c_1-12b_6c_2+12{b_6}^2{c_2}^2=0$. In this case we use this equation together with (7)-(11), (12)-(14) to write a_0, b_4, d_3, d_4, d_5 in terms of a_1, a_2, b_6, c_1, c_2. Substituting in (10) then gives $a_1{b_6}^2(1-2b_6c_2)=0$, so that $a_1=0$. At this point, (7) gives $a_2c_1(1-2b_6c_2)=0$, which is a contradiction. □

The lemmas in this section show that the rank of $D_{\mathbf{f}}$ is maximal for every $\mathbf{f}\in\Phi_6^{-1}\{0\}$, that is to say 0 is a regular value of Φ_6. As noted at the beginning of this section, this is enough to complete the proof of Theorem 1.1.

References

[1] J. Bolton, G.R. Jensen, M. Rigoli and L.M. Woodward, On conformal minimal immersions of S^2 into $\mathbb{C}P^n$, *Math. Ann.* 279 (1988), 599–620.

[2] J. Bolton and L. M. Woodward, The space of harmonic maps of S^2 into S^n, *Geometry and Global Analysis* (Sendai, 1993), 143–151, Tohoku Univ., Sendai, 1993.

[3] J. Bolton and L. M. Woodward, Linearly full harmonic 2-spheres in S^4 of area 20π, *Internat. J. Math.* 12 (2001), 535–554.

[4] J. Bolton and L. M. Woodward, Higher singularities and the twistor fibration $\pi:\mathbb{C}P^3\to S^4$, *Geom. Dedicata* 80 (2000), 231–245.

[5] J. Bolton and L. M. Woodward, The space of harmonic two-spheres in the unit four-sphere, *Tohoku Math. J* 58 (2006), 231–236.

[6] R. L. Bryant, Conformal and minimal immersions of compact surfaces into the 4-sphere, *J. Differential Geom.* 17 (1982), 455–473.

[7] S. S. Chern and J. Wolfson, Minimal surfaces by moving frames, *Amer. J. Math.* 105 (1983), 59–83.

[8] T. A. Crawford, The space of harmonic maps from the 2-sphere to the complex projective plane, *Canad. Math. Bull.* 40 (1997), 285–295.

[9] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley, 1978.

[10] L. Fernandez, The dimension of the space of minimal 2-spheres in S^6, *Bull. London Math. Soc.* 38 (2006), 156–162.

[11] M. Kotani, Harmonic 2-spheres with r pairs of extra eigenfunctions, *Proc. Amer. Math. Soc.* 125 (1997), 2083–2092.

[12] H. B. Lawson, Jr., Surfaces minimales et la construction de Calabi-Penrose, *Séminaire Bourbaki* 1983/84, Astérisque 121–122 (1985), 197–211.

[13] B. Loo, The space of harmonic maps of S^2 into S^4, *Trans. Amer. Math. Soc.* 313 (1989), 81–102.

[14] L. Lemaire and J. C. Wood, On the space of harmonic 2-spheres in $\mathbb{C}P^2$, *Internat. J. Math.* 7 (1996), 211–225.

[15] L. Lemaire and J. C. Wood, Jacobi fields along harmonic 2-spheres in $\mathbb{C}P^2$ are integrable, *J. London Math. Soc.(2)* 66 (2002), 468–486.

[16] L. Lemaire and J. C. Wood, Jacobi fields along harmonic 2-spheres in S^3 and S^4 are not all integrable. *To appear.*

[17] J.-L. Verdier, Two dimensional σ-models and harmonic maps from S^2 to S^{2n}, *Lecture Notes in Physics* 180 (1982), 136–141.

[18] J.-L. Verdier, Applications harmoniques de S^2 dans S^4, Geometry Today (Rome, 1984), 267–282, *Progr. Math.* 60, Birkhäuser Boston, Boston, MA, 1985.

[19] J.-L. Verdier, Applications harmoniques de S^2 dans S^4 *II*, Harmonic mappings, twistors, and σ-models (Luminy, 1986), 124–147, *Adv. Ser. Math. Phys.* 4, World Sci. Publishing, Singapore, 1988.

John Bolton, Department of Mathematical Sciences, Durham University, South Road, Durham DH1 3LE, UK.
Luis Fernandez, Department of Mathematics and Computer Science, CUNY (BCC), New York, USA.
e-mail: john.bolton@durham.ac.uk, lmfernand@gmail.com