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THE DIMENSION OF THE SPACE OF HARMONIC 2-SPHERES IN THE 6-SPHERE

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Abstract

Using the twistorial approach and some previous results, we prove the conjecture that the dimension of the moduli space of harmonic maps of area $4\pi d$ from the 2-sphere to the $2n$-sphere is $2d + n^2$ for the particular case $n = 3$.

1. *Introduction*

Recall ([**3**]) that a harmonic map from S^2 to S^{2n} can be written as the composition of a holomorphic, horizontal map from S^2 to $\mathcal{Z}_n = SO(2n+1)/U(n)$ (the 'twistor lift'), with plus or minus the natural projection from $\mathcal{Z}_n$ to S^{2n}. The main invariant of these harmonic maps is the degree, defined as their area divided by 4π.

We will denote by $\mathrm{Harm}_d^f(S^2, S^{2n})$ the space of harmonic maps from S^2 to S^{2n} of degree d not lying in a lower dimensional subsphere, and by $\mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$ the space of linearly full horizontal holomorphic maps from S^2 to $\mathcal{Z}_n$ of degree d. Using the twistorial approach it follows that $\mathrm{Harm}_d^f(S^2, S^{2n})$ is isomorphic to the disjoint union of two copies of $\mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$ (see [**7**] for details).

While the dimension and structure of $\mathrm{Harm}_d^f(S^2, S^4)$ has been thoroughly studied ([**10**], [**11**], [**12**], [**13**]), the only case that is completely understood for $n \geq 3$ is when $d = n(n+1)/2$ ([**1**]). However, it is known that $\mathrm{Harm}_d^f(S^2, S^{2n})$ is connected ([**9**], [**8**], [**6**]), and the fundamental group of these spaces was calculated in [**7**].

Based on heuristic arguments, in [**2**] it is conjectured that the dimension of $\mathrm{Harm}_d^f(S^2, S^{2n})$ is equal to $2d + n^2$. This figure is correct for all the known cases, but the conjecture has been open since. In this paper we will use the results in [**6**] to prove this conjecture for the particular case $n = 3$.

Our proof makes use of the following. In [**6**] it was shown that $\mathrm{HH}_d^f(S^2, \mathcal{Z}_n)$ is birrationally equivalent to the moduli space of holomorphic maps

$$\psi : S^2 \to \mathbb{CP}^{n(n+1)/2}$$

of degree d, with

$$\psi(z) = [s(z) : \alpha_1(z) : \ldots : \alpha_n(z) : \tau_{12}(z) : \ldots : \tau_{1n}(z) : \tau_{23}(z) : \ldots : \ldots : \tau_{n-1,n}(z)],$$

satisfying

$$\alpha_i' \alpha_j - \alpha_i \alpha_j' = s\tau_{ij}' - s'\tau_{ij}, \tag{1}$$

and the additional condition given by

$$W\left(\left(\frac{\alpha}{s}\right)'\right) \not\equiv 0, \tag{2}$$

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which guarantees that the corresponding harmonic map from S^2 to S^{2n} will be linearly full. Note that, for convenience, we have absorbed the constant factor appearing in equation (2.13) of **[6]** in the functions τ_{jk}. We also take $\tau_{kj} = -\tau_{jk}$.

We use the notation

$$W\Big((p_1, p_2, \ldots, p_k)\Big) := \begin{vmatrix} p_1 & p_2 & \ldots & p_k \\ p_1' & p_2' & \ldots & p_k' \\ \vdots & \vdots & & \vdots \\ p_1^{(k)} & p_2^{(k)} & \ldots & p_k^{(k)} \end{vmatrix}, \tag{3}$$

and where convenient, we drop the argument '(z)' in all the functions involved. The moduli space of maps ψ as above satisfying (1) and (2) will be denoted by $\mathrm{PD}_d^f(S^2, \mathbb{CP}^{n(n+1)/2})$.

We will show that the dimension of $\mathrm{PD}_d^f(S^2, \mathbb{CP}^6)$ is equal to $2d+9$, which implies that $\mathrm{HH}_d^f(S^2, \mathcal{Z}_3)$, and therefore $\mathrm{Harm}_d^f(S^2, S^6)$, also have dimension $2d+9$, as conjectured.

2. *A Lower Bound on the Dimension of the Moduli Space*

We will regard the functions s, α_i, τ_{jk} involved in equation (1) as polynomials of degree less than or equal to d in one complex variable z and without common factors. The vector space of polynomials of degree less than or equal to d will be denoted by $\mathbb{C}[z]_d$.

In this section we show that there is a $2d+10$-dimensional set of polynomials $\{(s(z), \alpha_1(z), \alpha_2(z), \alpha_3(z), \tau_{12}(z), \tau_{23}(z), \tau_{31}(z))\} \subset (\mathbb{C}[z]_d)^7$ in one complex variable z, of degree less than or equal to d with s having degree d, and without a common factor to all of them, that satisfy (1) and (2). After projectivising this proves the inequality $\dim(\mathrm{PD}_d^f(S^2, \mathbb{CP}^6)) \geq 2d+9$.

This was shown in **[6]** for $d > 7$. Essentially the same proof works for $d = 7$; we outline the proof for $d \geq 7$ for the sake of completeness.

Equation (1) is equivalent to

$$\alpha_j \alpha_i' - \alpha_i \alpha_j' = s^2 \left(\frac{\tau_{ij}}{s}\right)', \quad 1 \leq i < j \leq 3, \tag{4}$$

which is equivalent to

$$\frac{\alpha_j \alpha_i' - \alpha_i \alpha_j'}{s^2} \ \text{ has no residues} \tag{5}$$

and

$$\tau_{ij} = s \int \frac{\alpha_j \alpha_i' - \alpha_i \alpha_j'}{s^2}\, dz \ \text{ is a polynomial of degree} \leq d. \tag{6}$$

Let

$$E(\alpha_i, \alpha_j)(s_m) := \lim_{z \to s_m} \left(\frac{(z - s_m)^2 (\alpha_j(z)\alpha_i'(z) - \alpha_i(z)\alpha_j'(z))}{(s(z))^2} \right)'.$$

If s has only simple zeroes at the points $\{s_1, s_2, \ldots, s_d\}$, then equation (5) is equivalent to

$$E(\alpha_i, \alpha_j)(s_m) = 0, \quad 1 \leq i, j \leq 3, \quad m = 1, 2, \ldots, d-1, \tag{7}$$

and (6) is implicitly satisfied. Note that equation (7) implies that $(\alpha_j \alpha_i' - \alpha_i \alpha_j')/s^2$ has no residues at s_m, $1 \leq m \leq d-1$, so this function cannot have residues at all.

Let

$$F(\alpha_i)(s_m) := \lim_{z\to s_m}\left[\left(\frac{(z-s_m)^2}{(s(z))^2}\right)'\alpha_i'(z) + \frac{(z-s_m)^2}{(s(z))^2}\alpha_i''(z)\right],$$

and note that, if $F(\alpha_i)(s_m) = 0$ and $\alpha_i(s_m) = 0$ for some i and some m, then $E(\alpha_i, \alpha_j)(s_m) = 0$ for all j.

LEMMA 1. *Let $d \geq 7$. If $s(z)$ is any polynomial of degree d with simple zeroes at the points $s_1, s_2, \ldots, s_d$, then there exist polynomials $\alpha_1(z), \alpha_2(z), \alpha_3(z)$ of degree less than or equal to d, and not simultaneously vanishing at any of the points s_m, that satisfy conditions (7) and (2).*

Proof. Let $s(z) = \prod_{i=1}^{d}(z - s_i)$. Let $\alpha_1(z)$ be a solution of the system

$$\begin{aligned} \alpha_1(s_m) &= 0, \quad m = 1,2,3 \\ F(\alpha_1)(s_m) &= 0, \quad m = 1,2,3 \end{aligned}$$

which is not a multiple of $s(z)$ (note that $s(z)$ is also a solution of this system) and such that $\alpha_1(s_j) \neq 0$ for $4 \leq j \leq d$.

Now find independent polynomials $\alpha_2(z), \alpha_3(z)$ of degree d satisfying

$$E(\alpha_1, \alpha_2)(s_m) = 0, \quad 4 \leq m \leq d-1 \tag{8}$$

$$E(\alpha_1, \alpha_3)(s_m) = 0, \quad 4 \leq m \leq d-1 \tag{9}$$

$$E(\alpha_2, \alpha_3)(s_m) = 0, \quad m = 2,3, \tag{10}$$

which are independent from $s(z)$ and $\alpha_1(z)$ and do not vanish simultaneously at any of the points s_m, $m = 1,2,3$. Note that equation (10) is an intersection of two quadrics in the 5-dimensional space of solutions of equations (8) or (9); this guarantees the existence of these polynomials.

Using the relation

$$\alpha_1(s_m)\,E(\alpha_2,\alpha_3)(s_m) + \alpha_2(s_m)\,E(\alpha_3,\alpha_1)(s_m) + \alpha_3(s_m)\,E(\alpha_1,\alpha_2)(s_m) = 0, \tag{11}$$

for $1 \leq m \leq d$, it becomes clear that this construction gives a solution of (7).

By construction, the polynomials α_1, α_2 and α_3 do not all vanish simultaneously at any of the points s_m; thus the polynomials $s, \alpha_1, \alpha_2, \alpha_3$ have no common factors. Also by construction, the set $\{s, \alpha_1, \alpha_2, \alpha_3\}$ is a linearly independent set in $\mathbb{C}[z]_d$, so condition (2) is satisfied. $\square$

Now that we know that the set of solutions of (1) and (2) with s having simple zeroes is nonempty for $d \geq 7$, we only have do a simple count to find a lower bound for the dimension of this set.

PROPOSITION 1. *For $d \geq 6$, the dimension of the moduli space of linearly full harmonic maps from S^2 to S^6 of degree d is at least $2d + 9$.*

Proof. For $d = 6$, **[1]** gives $\dim(\mathrm{Harm}_6^f(S^2, S^6)) = 2 \cdot 6 + 9$.

For $d \geq 7$, define the open subset of $\mathbb{C}^{4d+4}$ given by

$$\mathcal{S} = \{(s, \alpha_1, \alpha_2, \alpha_3) \in (\mathbb{C}[z]_d)^4 : s \ \text{ has } d \text{ distinct roots}\}.$$

Let

$$\mathcal{R} = \{(s,\alpha_1,\alpha_2,\alpha_3) \in \mathcal{S} : W\left(\left(\frac{\alpha}{s}\right)'\right) \not\equiv 0 \text{ and } E(\alpha_i,\alpha_j)(s_m) = 0,$$
$$1 \le i < j \le 3\,,\ 1 \le m \le d-1, \text{where } \{s_1, s_2, \dots, s_d\} \text{ are the roots of } s\}.$$

Using relation (11) and the fact that the sum of the residues of a meromorphic function is 0, we have $d+3$ relations between the equations of the system (7), so this system has no more than $2d-3$ independent homogeneous equations and $4d+4$ variables. The set $\mathcal{R}$ is the intersection of the variety given by equations (7) with the open set given by condition (2). By Lemma 1, $\mathcal{R}$ is not empty. Therefore its dimension is at least $2d+7$.

Finally, define

$$\mathcal{A} = \{(s,\alpha_1,\alpha_2,\alpha_3,\tau_{12},\tau_{23},\tau_{31}) \in (\mathbb{C}[z]_d)^7 : (s,\alpha_1,\alpha_2,\alpha_3) \in \mathcal{R}$$
$$\text{and } \ \tau_{ij} = a_{ij}s + s\int \frac{\alpha_j\alpha_i' - \alpha_i\alpha_j'}{s^2}\,dz,\ 1 \le i < j \le 3\},$$

where a_{ij} are arbitrary complex numbers (integration constants).

The elements of $\mathcal{A}$ are solutions of (1) satisfying (2). Its dimension is the dimension of $\mathcal{R}$ plus the three extra degrees of freedom given by the integration constants. Thus, $\dim(\mathcal{A}) \ge 2d+10$, and therefore we have $\mathbb{P}\mathcal{A} \subset \mathrm{PD}^f_d(S^2,\mathbb{CP}^6)$ and $\dim(\mathbb{P}\mathcal{A}) \ge 2d+9$.

Since $\dim(\mathrm{Harm}^f_6(S^2,S^6)) = \dim(\mathrm{HH}^f_d(S^2,\mathcal{Z}_3)) = \dim(\mathrm{PD}^f_d(S^2,\mathbb{CP}^6)) \ge 2d+9$, the result follows. □

Remark 1. In the proof of Proposition 1 we have used that birrational transformations preserve the dimension. However, we do not really need to use this fact since the set $\mathcal{A}$ is contained in the regular part of the birrational transformation given in [**6**].

Of course we understand dimension as the top dimension of the irreducible components, as we do not know whether any of the varieties involved has pure dimension.

3. *An Upper Bound on the Dimension of the Moduli Space*

Using the notation $\alpha = (\alpha_1,\alpha_2,\alpha_3)$ and $t = (\tau_{23},\tau_{31},\tau_{12})$, equation (1) can be written as

$$\alpha \times \alpha' = st' - s't, \tag{12}$$

where $\times$ denotes the cross product in $\mathbb{C}^3$.

Lemma 2. *If $[s:\alpha_1:\alpha_2:\alpha_3:\tau_{12}:\tau_{23}:\tau_{13}]$ is a solution of degree d of (12) satisfying (2), where all the components are polynomials, then*

$$W(s,t) := \begin{vmatrix} s & \tau_{23} & \tau_{31} & \tau_{12} \\ s' & \tau_{23}' & \tau_{31}' & \tau_{12}' \\ s'' & \tau_{23}'' & \tau_{31}'' & \tau_{12}'' \\ s''' & \tau_{23}''' & \tau_{31}''' & \tau_{12}''' \end{vmatrix} = \frac{(\det(\alpha,\alpha',\alpha''))^2}{s^2}$$

(so it is a perfect square) and

$$\alpha = \frac{st' \times t'' - s't \times t'' + s''t \times t'}{\sqrt{W(s,t)}}.$$

Proof. We have

$$\alpha \times \alpha' = st' - s't.$$

Differentiating this equation we obtain

$$\alpha \times \alpha'' = st'' - s''t,$$

and differentiating again,

$$\alpha \times \alpha''' + \alpha' \times \alpha'' = st''' - s'''t + s't'' - s''t'.$$

Taking the cross product of the first two equations and using the formula $(u \times v) \times (u \times w) = \det(u, v, w)u$ we get

$$\det(\alpha, \alpha', \alpha'')\alpha = s(st' \times t'' - s't \times t'' + s''t \times t'),$$

and taking the cross product with the third equation we get

$$(\det(\alpha, \alpha', \alpha''))^2 = s^2 W(s,t).$$

Condition $W((\alpha/s)') \not\equiv 0$ implies that $\det(\alpha, \alpha', \alpha'') \not\equiv 0$, and we obtain the desired formula. $\square$

The previous formula asserts that the elements of $\mathrm{PD}^f_d(S^2, \mathbb{CP}^6)$ are completely characterised by the polynomials s, τ_{12}, τ_{13} and τ_{23}. Consider the map

$$\Xi : \mathrm{PD}^f_d(S^2, \mathbb{CP}^6) \to \mathrm{Gr}(4, \mathbb{C}[z]_d)$$

given by

$$\Xi([s : \alpha_1 : \alpha_2 : \alpha_3 : \tau_{12} : \tau_{13} : \tau_{23}]) = \langle s, \tau_{23}, \tau_{31}, \tau_{12} \rangle,$$

where Gr stands for the Grassmannian and $\langle v_1, v_2, \ldots, v_k \rangle$ denotes, in general, the subspace spanned by the vectors $v_1, v_2, \ldots, v_k$.

Let

$$\Pi : \mathrm{Gr}(4, \mathbb{C}[z]_d) \to \mathbb{P}(\mathbb{C}[z]_{4d-12})$$

be the map given by

$$\Pi(\langle g_1, g_2, g_3, g_4 \rangle) = [W((g_1, g_2, g_3, g_4))],$$

where $W((g_1, g_2, g_3, g_4))$ is the Wronskian of the polynomials g_1, g_2, g_3, g_4, as defined in (3). Lemma 2 asserts that the image of $\Pi \circ \Xi$ lies in the submanifold Q of $\mathbb{P}(\mathbb{C}[z]_{4d-12})$ defined as the projectivisation of the set of non-zero polynomials in $\mathbb{C}[z]_{4d-12}$ that are perfect squares. In other words, the image of Ξ lies in the subvariety $\Pi^{-1}(Q)$ of $\mathrm{Gr}(4, \mathbb{C}[z]_d)$.

Let

$$L : \Lambda^4 \mathbb{C}[z]_d \to \mathbb{C}[z]_{4d-12}$$

be the linear map defined in basic elements $g_1 \wedge g_2 \wedge g_3 \wedge g_4$ by

$$L(g_1 \wedge g_2 \wedge g_3 \wedge g_4) = W((g_1, g_2, g_3, g_4)).$$

Taking the usual basis $\{z^k\}_{k=0}^d$ of $\mathbb{C}[z]_d$ it is a straightforward computation to show that, for $0 \le i < j < k < \ell \le d$,

$$L(z^i \wedge z^j \wedge z^k \wedge z^\ell) = (\ell - k)(\ell - j)(\ell - i)(k - j)(k - i)(j - i)z^{i+k+j+\ell-6}.$$

This shows, in particular, that L is onto.

Let $K \subset \Lambda^4\mathbb{C}[z]_d$ be the kernel of L, and let

$$\widehat{\Pi} : \mathbb{P}(\Lambda^4\mathbb{C}[z]_d - K) \to \mathbb{P}(\mathbb{C}[z]_{4d-12})$$

be the projectivisation of L.

Let $\mathrm{Pl} : \mathrm{Gr}(4, \mathbb{C}[z]_d) \to \mathbb{P}(\Lambda^4\mathbb{C}[z]_d)$ denote the Plücker embedding. Note that $\mathrm{Pl}(\mathrm{Gr}(4, \mathbb{C}[z]_d))$ is disjoint from $\mathbb{P}K$ (the projectivisation of K), and we have $\Pi = \widehat{\Pi} \circ \mathrm{Pl}$, giving the following diagram:

$$\begin{tikzcd}
\mathrm{PD}_d^f(S^2, \mathbb{CP}^6) \arrow[r, "\Xi"] & \Pi^{-1}(Q) \subset \mathrm{Gr}(4, \mathbb{C}[z]_d) \arrow[r, "\Pi"] \arrow[d, "\mathrm{Pl}"'] & \mathbb{P}(\mathbb{C}[z]_{4d-12}) \supset Q \\
& \mathbb{P}(\Lambda^4\mathbb{C}[z]_d - K) \arrow[ur, "\widehat{\Pi}"'] &
\end{tikzcd}$$

The following result appears in [**5**], pp. 127–128, and in the references cited therein, for the case $\mathrm{Gr}(2, \mathbb{C}[z]_d)$. We include a proof for $\mathrm{Gr}(4, \mathbb{C}[z]_d)$; essentially the same argument would work for the corresponding (Wronski) map in $\mathrm{Gr}(n, \mathbb{C}[z]_d)$.

Lemma 3. *For all $q \in \mathbb{P}(\mathbb{C}[z]_{4d-12})$, the set $\Pi^{-1}(q)$ is finite.*

Proof. If $q \in \mathbb{P}(\mathbb{C}[z]_{4d-12})$, then $\Pi^{-1}(q)$ is the set of points in $\mathrm{Gr}(4, \mathbb{C}[z]_d)$ whose image under the Plücker embedding lies in $\widehat{\Pi}^{-1}(q)$. Now, $\widehat{\Pi}^{-1}(q)$ is an affine subspace of $\mathbb{P}(\Lambda^4\mathbb{C}[z]_d)$ whose closure is a projective subspace of codimension $(4d-12)$ of the form $[\mathbb{C}p+K]$. Note that $[\mathbb{C}p+K] = \widehat{\Pi}^{-1}(q) \cup \mathbb{P}K$. Since the dimension of $\mathrm{Gr}(4, \mathbb{C}[z]_d)$ is $4d-12$, and since $\mathrm{Pl}(\mathrm{Gr}(4, \mathbb{C}[z]_d)) \cap \mathbb{P}K$ is empty, $\mathrm{Pl}(\mathrm{Gr}(4, \mathbb{C}[z]_d)) \cap \widehat{\Pi}^{-1}(q)$ must be nonempty for all $q \in \mathbb{P}(\mathbb{C}[z]_{4d-12})$.

If V were an open subvariety of dimension greater than 0 contained in the intersection of $\mathrm{Pl}(\mathrm{Gr}(4, \mathbb{C}[z]_d))$ and $\widehat{\Pi}^{-1}(q)$, then the closure of V would contain points lying in both $\mathrm{Pl}(\mathrm{Gr}(4, \mathbb{C}[z]_d))$ and $\mathbb{P}K$. This is impossible since $\mathrm{Pl}(\mathrm{Gr}(4, \mathbb{C}[z]_d)) \cap \mathbb{P}K = \emptyset$.

Therefore the set

$$\Pi^{-1}(q) = \mathrm{Pl}^{-1}\big(\mathrm{Pl}(\mathrm{Gr}(4, \mathbb{C}[z]_d)) \cap \widehat{\Pi}^{-1}(q)\big)$$

is a subvariety of dimension 0, so it must consist of a finite number of points. ☐

This implies, in particular, that the dimension of the subvariety $\Pi^{-1}(Q)$ is the same as the dimension of Q, which is $2d-6$, and we have the following result.

Proposition 2. *The dimension of $Harm_d(S^2, S^6)$ is less than or equal to $2d+9$.*

Proof. Since the fiber of the map $\Xi : \mathrm{PD}_d^f(S^2, \mathbb{CP}^6) \to \mathrm{Gr}(4, \mathbb{C}[z]_d)$ has dimension less than or equal to 15, and since $\Xi\ (\mathrm{PD}_d^f(S^2, \mathbb{CP}^6))$ lies in a variety of dimension $2d-6$, it follows that $\dim(\mathrm{PD}_d^f(S^2, \mathbb{CP}^6)) \le 2d+9$. Therefore,

$$\dim(\mathrm{Harm}_d^f(S^2, S^6)) = \dim(\mathrm{HH}_d^f(S^2, \mathcal{Z}_3)) = \dim(\mathrm{PD}_d^f(S^2, \mathbb{CP}^6)) \le 2d+9$$

☐

THEOREM 1. *The (top) dimension of the moduli space of harmonic maps of area $4\pi d$ from S^2 to S^6 is $2d+9$.*

Proof. Immediate from Proposition 1 and Proposition 2. □

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