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On the regularity of the space of harmonic 2-spheres in the 4-sphere

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Abstract. We give a general overview of the regularity of the space of harmonic maps from the 2-sphere to the 4-sphere of a given degree and why we expected the subspace of linearly full maps to not be regular for $d = 6$. It turns out that this space is regular and we sketch the proof of this fact.

1. Introduction and preliminaries

In 2001 Lemaire and Wood [**LW02**] showed that all Jacobi fields of harmonic 2-spheres into $\mathbb{CP}^2$ are integrable, where $\mathbb{CP}^2$ denotes complex projective 2-space equipped with the Fubini-Study metric of constant holomorphic sectional curvature 4. Then they moved into the study of Jacobi fields of harmonic 2-spheres into the unit 4-sphere S^4. Interestingly, in this case they found Jacobi fields that were not integrable [**LW09**].

A Jacobi field along a harmonic map ϕ_0 can be thought of as a 'harmonic' infinetesimal deformation of ϕ_0, in the sense that it is the derivative at 0 of a deformation ϕ_t of ϕ_0 which is *harmonic to first order*, i.e. $d\tau(\phi_t)/dt = 0$ at $t = 0$, where τ is the tension field (see [**LW09**], Proposition 1.2 for details). A Jacobi field is called *integrable* if it is the derivative of a deformation of ϕ_0 *through harmonic maps*, i.e. ϕ_t is harmonic for t near 0.

With this characterisation, it becomes clear that the study of integrability of Jacobi fields of harmonic maps is intimately related to the study of singularities of the space of harmonic maps. A Jacobi field is a vector field in the space of harmonic maps, and it is integrable if it appears as the derivative of a curve in that space. In fact, if all Jacobi fields are integrable, then the space of maps is a manifold [**AS, LW09**]. The converse is not true in general. For example, in [**LW09**], nonintegrable Jacobi fields of harmonic 2-spheres in S^3 were found; however, the space of harmonic maps of a given degree from S^2 to S^3 is a manifold.

Let us briefly recall some general facts about harmonic maps from S^2 to S^m. In 1967 Calabi [**C**] first showed that if a map $\phi : S^2 \to S^m$ is harmonic and

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linearly full—i.e. not lying in a proper totally geodesic subsphere of S^m—then m must be even. Furthermore, given a harmonic map $\phi : S^2 \to S^{2n}$, one can construct a holomorphic map ψ into the *twistor space* $\mathcal{Z}_n$, defined, for example, as the submanifold of the complex Grassmannian consisting of n-planes that are totally isotropic with respect to a complex bilinear inner product.

There is a 'projection' $\pi : \mathcal{Z}_n \to S^{2n}$ as follows: given an orthonormal basis $\{P_1, P_2, \ldots, P_n\}$ of $P \in \mathcal{Z}_n$, $\pi(P)$ is the unique real unit vector such that $\{\pi(P), P_1, \ldots, P_n, \bar{P}_1, \ldots, \bar{P}_n\}$ is a positively oriented orthonormal basis of $\mathbb{C}^{2n+1}$. The map ψ is a *horizontal lift* with respect to π, in the sense that $d\psi$ is perpendicular to the fibres of π and $\phi = \pm\pi \circ \psi$. Conversely, if $\psi : S^2 \to \mathcal{Z}_n$ is holomorphic, horizontal and linearly full (as defined in [**FGKO**]), $\pm\pi \circ \psi$ are harmonic.

The *twistor lift* ψ of a harmonic map $\phi : S^2 \to S^{2n}$ is uniquely defined for any n when ϕ is linearly full; if ϕ is not linearly full, it is still uniquely defined if $n = 2$ (see [**V85**]). Calabi also showed that if $\phi : S^2 \to S^m$ is harmonic, then $\mathrm{Area}(\phi(S^2)) = 2\pi d$, where d is a positive integer; this was improved to $\mathrm{Area}(\phi(S^2)) = 4\pi d$ by Barbosa in [**Ba**]. The integer d is called the *degree* of the harmonic map.

Let $\mathrm{Harm}_d(S^2, S^m)$ denote the space of harmonic maps of degree d from S^2 to S^m. Then

$$\mathrm{Harm}_d(S^2, S^m) = \mathrm{Harm}_d^{\text{full}}(S^2, S^m) \sqcup \mathrm{Harm}_d^{\text{non}}(S^2, S^m),$$

where $\mathrm{Harm}_d^{\text{full}}(S^2, S^m)$ is the subset of linearly full maps and $\mathrm{Harm}_d^{\text{non}}(S^2, S^m)$ is its complement. As stated above, when m is odd, $\mathrm{Harm}_d^{\text{full}}(S^2, S^m)$ is empty. The space $\mathrm{Harm}_d^{\text{full}}(S^2, S^{2n})$ is also empty if $d < n(n+1)/2$, but is nonempty otherwise [**Ba**]; in this latter case it has two disconnected components [**FGKO**]:

$$\mathrm{Harm}_d^{\text{full}}(S^2, S^{2n}) = \mathrm{Harm}_d^{\text{full},+}(S^2, S^{2n}) \sqcup \mathrm{Harm}_d^{\text{full},-}(S^2, S^{2n}).$$

If we let $\mathrm{HH}_d^{\text{full}}(S^2, \mathcal{Z}_n)$ denote the space of holomorphic, horizontal and linearly full maps of degree d from S^2 to $\mathcal{Z}_n$ and $\mathrm{HH}_d^{\text{non}}(S^2, \mathcal{Z}_n)$ the space of non-full maps, then the results explained above imply

$$\begin{array}{rcll} \mathrm{HH}_d^{\text{non}}(S^2, \mathcal{Z}_n) & \to & \mathrm{Harm}_d^{\text{non}}(S^2, S^{2n}) & \\ \psi & \to & \pi \circ \psi & \text{is onto, and} \end{array} \tag{1.1}$$
$$\mathrm{Harm}_d^{\text{non}}(S^2, S^4) \simeq \mathrm{HH}_d^{\text{non}}(S^2, \mathcal{Z}_2), \quad \mathrm{Harm}_d^{\text{full},+}(S^2, S^{2n}) \simeq \mathrm{HH}_d^{\text{full}}(S^2, \mathcal{Z}_n).$$

Here the symbol '$\simeq$' means 'isomorphic' in the appropriate category. For $n = 2$, these sets are isomorphic in the real analytic category [**LW09**]; for $n > 2$ this has not been studied thoroughly, so we take it as an algebraic isomorphism by transferring the structure of $\mathrm{HH}_d^{\text{full}}(S^2, \mathcal{Z}_n)$ to $\mathrm{Harm}_d^{\text{full},+}(S^2, S^{2n})$ via this correspondence.

The structure of $\mathrm{Harm}_d(S^2, S^m)$ is well understood when $m = 4$: it has three irreducible components $\mathrm{Harm}_d^{\text{non}}(S^2, S^4)$, $\mathrm{Harm}_d^{\text{full},+}(S^2, S^4)$ and $\mathrm{Harm}_d^{\text{full},-}(S^2, S^4)$, each of dimension $2d + 4$ [**L, V85, V83, V88**]. Figure 1 gives a picture.

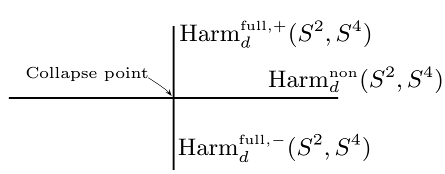

FIGURE 1. Structure of $\mathrm{Harm}_d(S^2, S^4)$

The points of intersection of these components were called *collapse points* by Lemaire and Wood. Their idea to find non-integrable Jacobi fields was to take a one-parameter deformation of harmonic maps along $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^4)$ and another along $\mathrm{Harm}_d^{\mathrm{full},+}(S^2, S^4)$ approaching a collapse point. Then their derivatives with respect to the parameter at the collapse point would give rise to (integrable) Jacobi fields. Now, a linear combination with nonzero coefficients of these two Jacobi fields gives another Jacobi field which is not integrable, represented by a dashed arrow in the picture below (Figure 2).

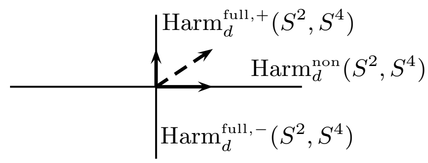

FIGURE 2. Nonintegrable Jacobi fields in $\mathrm{Harm}_d(S^2, S^4)$

The same picture holds for $\mathrm{Harm}_d(S^2, S^{2n})$: since the set $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^{2n})$ is closed, $\mathrm{Harm}_d(S^2, S^{2n})$ is the union of the closed subsets $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^{2n})$ and $\overline{\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^{2n})}$, so it is not irreducible. Therefore, the intersection of these subsets (which is exactly the set of collapse points) consists of non-manifold points (see for example [**Ke**], Lemma 6.2.3), and then Proposition 1.4 of [**LW09**] implies that there are nonintegrable Jacobi fields there.

Now the following question arises: are they also non-manifold points *as a subset of* $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^{2n})$? Note that for $n = 2$ this is not the case since $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^4)$ is a manifold (it is a fibre bundle over the Grassmannian of 3-planes in $\mathbb{R}^5$ with fibre $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^2)$). This problem directly relates to the problem of whether $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^{2n})$ is a manifold. So far no singular points have been found in this variety, and it is known to be regular for any n when $d = n(n+1)/2$ [**Ba**], and, when $n = 2$, for $d = 3, 4, 5$ [**BW**] and $d = 6$ [**BF**]. Concentrating in the case $n = 3$, the relation between $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^6)$ and $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^4)$ can be seen as follows.

A map in $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^6)$ must be linearly full in some 2 or 4-dimensional subsphere of S^6. Fix a 5-dimensional subspace U in $\mathbb{R}^7$ and consider the space $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^6 \cap U) \subset \mathrm{Harm}_d^{\mathrm{non}}(S^2, S^6)$ of maps that are linearly full in $S^6 \cap U$. This set is isomorphic to $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^4)$ (fix an isometric embedding $i : S^4 \to S^6 \cap U$ and use $\phi \in \mathrm{Harm}_d^{\mathrm{full}}(S^2, S^4) \to i \circ \phi \in \mathrm{Harm}_d^{\mathrm{full}}(S^2, S^6 \cap U)$). The set of collapse points in $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^6)$ lying in $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^6 \cap U)$ correspond, via this isomorphism, to *harmonic spheres in* S^4 *with a pair of extra eigenfunctions* [**Ko**] or to *extendable maps* [**F**] described below. Pictorially this can be seen as follows (Figure 3).

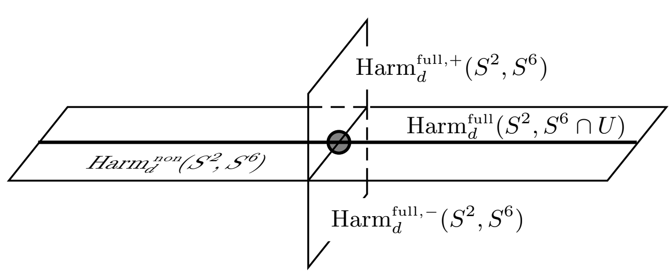

FIGURE 3. Structure of $\mathrm{Harm}_d(S^2, S^6)$

The circled points in Figure 3 correspond to collapse points from the point of view of $\mathrm{Harm}_d(S^2, S^6)$, and to maps with extra eigenvalues or extendable maps from the point of view of $\mathrm{Harm}_d^{\mathrm{full},+}(S^2, S^4) \simeq \mathrm{Harm}_d^{\mathrm{full}}(S^2, S^6 \cap U)$.

2. Algebraic characterization of the space of harmonic maps

It is well known that the twistor space $\mathcal{Z}_2$ is biholomorphic to $\mathbb{CP}^3$, and that a map $\psi : S^2 \to \mathcal{Z}_2$ is horizontal and holomorphic if and only if $\psi = [s : \alpha_1 : \alpha_2 : \tau_{12}]$ satisfies

$$(s\tau_{12}' - s'\tau_{12}) - (\alpha_1'\alpha_2 - \alpha_1\alpha_2') = 0. \tag{2.1}$$

Here $s, \alpha_1, \alpha_2, \tau_{12}$ are coprime polynomials in one complex variable z with maximum degree d (we are thinking of S^2 as $\mathbb{C} \cup \{\infty\}$). We use this choice of notation and ordering of the polynomials (why not, say, $[f_1 : f_2 : f_3 : f_4]$?) for consistency with the higher dimensional case.

For $n = 3$ (and also for general n) one can do a similar treatment by means of a birational map $b : \mathbb{CP}^6 \to \mathcal{Z}_3$. The set of holomorphic and horizontal maps $\psi : S^2 \to \mathcal{Z}_3$ is (essentially—see [**CFW, F**]) isomorphic to the set of maps $\tilde{\psi} = b^{-1} \circ \psi = [s : \alpha_1 : \alpha_2 : \alpha_3 : \tau_{12} : \tau_{23} : \tau_{31}]$ satisfying

$$\alpha_i'\alpha_j - \alpha_i\alpha_j' = s\tau_{ij}' - s'\tau_{ij} \quad 1 \le i, j \le 3 \quad \text{with } \tau_{ij} = -\tau_{ji}. \tag{2.2}$$

One can furthermore assume that s is monic with d distinct real roots and α_1 nonzero at the roots of s. In addition, ψ is linearly full if and only if the polynomials $s, \alpha_1, \alpha_2, \alpha_3$ are linearly independent.

In order to avoid introducing new notation, we will also use $\mathrm{HH}_d^{\mathrm{non}}(S^2, \mathcal{Z}_3)$ and $\mathrm{HH}_d^{\mathrm{full}}(S^2, \mathcal{Z}_3)$ to denote the set of maps $\tilde{\psi} : S^2 \to \mathbb{CP}^6$ satisfying (2.2), and use the isomorphisms (1.1) to translate these maps into harmonic maps into S^6. In this setting, one can easily and explicitly immerse $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^4)$ in $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^6)$ by

$$[s : \alpha_1 : \alpha_2 : \tau_{12}] \longrightarrow [s : \alpha_1 : \alpha_2 : 0 : \tau_{12} : 0 : 0], \tag{2.3}$$

(which amounts to embedding the original map into an equatorial S^4 in S^6) and one can also do explicit deformations of twistor lifts of harmonic maps via

$$\tilde{\psi}_t = [s : \alpha_1 : \alpha_2 : t\alpha_3 : \tau_{12} : t\tau_{23} : t\tau_{31}].$$

Letting $t \to 0$ in this deformation we obtain a collapse point in $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^6)$.

A map $[s : \alpha_1 : \alpha_2 : \tau_{12}]$ will be called *extendable* if there exists a map $[s : \alpha_1 : \alpha_2 : \alpha_3 : \tau_{12} : \tau_{23} : \tau_{31}]$ of the same degree satisfying (2.2) with $\{s, \alpha_1, \alpha_2, \alpha_3\}$ linearly independent. It is clear then that if $[s : \alpha_1 : \alpha_2 : \tau_{12}]$ is extendable then $[s : \alpha_1 : \alpha_2 : 0 : \tau_{12} : 0 : 0]$ corresponds to a collapse point in $\mathrm{Harm}_d^{\mathrm{non}}(S^2, S^6)$. Extendable points were used in [**F**] to prove that $\mathrm{Harm}_d^{\mathrm{full}}(S^2, S^{2n})$ has dimension $2d + n^2$ by showing that the subvariety of extendable points has codimension one and using that the inverse of the map (2.3) (for general n) defines a projection from $\mathrm{HH}_d^{\mathrm{full}}(S^2, \mathcal{Z}_n)$ to the set of extendable points in $\mathrm{HH}_d^{\mathrm{full}}(S^2, \mathcal{Z}_{n-1})$.

In order to look at extendable maps more closely we need a further characterisation of the maps satisfying (2.2). Since s has distinct complex roots, say $z_1, z_2, \ldots, z_d$, we can write

$$s = \prod_{\ell=1}^{d}(z - z_\ell), \quad \alpha_i = a_{i0}s + \sum_{\ell=1}^{d} a_{i\ell}\,\frac{s}{z - z_\ell}, \quad \tau_{jk} = t_{jk0}s + \sum_{\ell=1}^{d} t_{jk\ell}\,\frac{s}{z - z_\ell}, \tag{2.4}$$

for some complex numbers $a_{i\ell}$, $t_{jk\ell}$, with $1 \le i,j,k \le n$ and $0 \le \ell \le d$. It turns out that for general n, the solutions of (2.2) can be characterised by the equation

$$\begin{pmatrix} \lambda_1 & \frac{1}{(z_1-z_2)^2} & \cdots & \frac{1}{(z_1-z_d)^2} \\ \frac{1}{(z_2-z_1)^2} & \lambda_2 & \cdots & \frac{1}{(z_2-z_d)^2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(z_d-z_\ell)^2} & \frac{1}{(z_d-z_2)^2} & \cdots & \lambda_d \end{pmatrix} \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1d} & a_{2d} & \cdots & a_{nd} \end{pmatrix} = 0. \tag{2.5}$$

Linear fullness is characterised by the second matrix of the left hand side having rank n. The numbers $t_{ij\ell}$, $1 \le i,j \le 3$, $\ell \ge 1$ are given by a formula in terms of z_ℓ and $a_{i,\ell}$. The λ_ℓ are auxiliary parameters which are implicitly defined by s and α_1. See [**F**] for details.

Let $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$ denote the matrix on the left hand side of equation (2.5). Note that for a map $[s:\alpha_1:\alpha_2:\tau_{12}]$ to correspond to an element of $\mathrm{Harm}_d^{\mathrm{full}}(S^2,S^4)$, the matrix $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$ must have nullity at least 2, and for a map $[s:\alpha_1:\alpha_2:\alpha_3:\tau_{12}:\tau_{23}:\tau_{31}]$ to correspond to an element of $\mathrm{Harm}_d^{\mathrm{full}}(S^2,S^6)$, the matrix $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$ must have nullity at least 3. It is then clear that extendable maps $[s:\alpha_1:\alpha_2:\tau_{12}]$ are characterised by $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$ having greater nullity 'than needed' (i.e. nullity greater than 2).

For fixed $\boldsymbol{z}$, $\boldsymbol{\lambda}$, equation (2.5) is linear, so it seems sensible to look for singularities of the variety it defines exactly when the matrix $\Sigma_{\boldsymbol{z},\boldsymbol{\lambda}}$ is "more singular than needed". This led the second author to look for singularities of $\mathrm{Harm}_d^{\mathrm{full}}(S^2,S^4)$ in the set of extendable maps. The same was suggested by the approach in the next section.

3. Another general approach

We outline another way to describe the set of solutions of equation (2.2) and hence $\mathrm{Harm}_d^{\mathrm{full}}(S^2,S^{2n})$. We concentrate in the cases $n=2$ and $n=3$, but the construction below is valid for general n.

Consider the map

$$\begin{array}{rcl} W:\mathbb{C}[z]_d\times\mathbb{C}[z]_d & \to & \mathbb{C}[z]_{2d-2} \\ (s,\alpha_1) & \to & W(s,\alpha_1)=s\alpha_1'-s'\alpha_1. \end{array}$$

Note that

$$dW_{(s,\alpha_1)}(-\alpha_2,\tau_{12})=(s\tau_{12}'-s'\tau_{12})-(\alpha_1'\alpha_2-\alpha_1\alpha_2'),$$

and therefore solutions $(s,\alpha_1,\alpha_2,\tau_{12})$ of equation (2.1) can be described as an open subset of

$$\{(\alpha_2,\tau_{12})_{(s,\alpha_1)}\in T(\mathbb{C}[z]_d\times\mathbb{C}[z]_d):dW_{(s,\alpha_1)}(-\alpha_2,\tau_{12})=0\}.$$

Solutions for the case $n=3$ can be described in a similar way. Let $\mathcal{F}_2(\mathbb{C}[z]_d\times\mathbb{C}[z]_d)$ be the space of 2-frames on $\mathbb{C}[z]_d\times\mathbb{C}[z]_d$. Then the solutions of equation (2.2) are given by an open subset of

$$\{((\alpha_2,\tau_{12})_{(s,\alpha_1)},(\alpha_3,\tau_{13})_{(s,\alpha_1)})\in\mathcal{F}_2(\mathbb{C}[z]_d\times\mathbb{C}[z]_d):dW_{(s,\alpha_1)}(-\alpha_j,\tau_{1j})=0\}\times\mathbb{C}.$$

The factor '$\mathbb{C}$' at the end comes from the free choices for τ_{23}: note that (2.2) implies

$$\tau_{23}=c_0s+\int\frac{\alpha_2'\alpha_3-\alpha_2\alpha_3'}{s^2}dz,$$

where c_0 is an arbitrary integration constant. Assuming that s has single complex roots and that α_1 does not vanish at the roots of s one can prove, using equation (2.5), that this formula defines a polynomial τ_{23} of degree at most d.

The map $dW_{(s,\alpha_1)}$ is a generic submersion. Its generic kernel is spanned by $(s,0)$, $(0,\alpha_1)$ and $(-\alpha_1,s) \in T_{(s,\alpha_1)}(\mathbb{C}[z]_d \times \mathbb{C}[z]_d)$. Consider the subvarieties of $\mathbb{C}[z]_d \times \mathbb{C}[z]_d$ given by

$$K^{\geq j} = \{(s,\alpha) \in \mathbb{C}[z]_d \times \mathbb{C}[z]_d : \dim\ker(dW_{(s,\alpha_1)}) \geq 3+j\}.$$

Then

$$\mathbb{C}[z]_d \times \mathbb{C}[z]_d = K^{\geq 0} \supset K^{\geq 1} \supset K^{\geq 2} \supset \dots.$$

Requiring a map to be linearly full is equivalent to requiring the 2-frame and the vectors $(s,0)$, $(0,\alpha_1)$ and $(-\alpha_1,s)$ to be linearly independent. Thus, for a map $[s:\alpha_1:\alpha_2:\tau_{12}]$ to be extendable we need $dF_{(s,\alpha_1)}$ to have two extra dimensions in the kernel. Therefore extendable maps for $n=2$ are a subvariety of $\mathcal{F}_1 K^{\geq 2}$.

The variety of linearly full solutions of (2.1) is therefore a fibration over $K^{\geq 1}$. Notice that, in fact, most of the information about the space of solutions is concentrated in the varieties $K^{\geq j}$, in the sense that $\mathrm{HH}_d^{\mathrm{full}}(S^2,\mathcal{Z}_3)$ can be described from $K^{\geq j}$ via a linear procedure.

Singularities may arise either at non-manifold points of $K^{\geq 1}$ or at points where the fibre changes dimension (which by definition are points lying over $K^{\geq 2}$, i.e. extendable maps). This suggests, again, that extendable points may be singular.

For $d<6$, the set of extendable maps in $\mathrm{Harm}_d^{\mathrm{full}}(S^2,S^4)$ is empty because there are no harmonic linearly full maps from S^2 to S^6 of degree less than 6 [**Ba**] (and $\mathrm{Harm}_d^{\mathrm{full}}(S^2,S^4)$ is a manifold for $d<6$ [**BW**]). So the first interesting case to look for singularities is $\mathrm{Harm}_6^{\mathrm{full}}(S^2,S^4)$. However, after doing an exhaustive search as explained below, it turns out that this variety also has no singularities.

4. $\mathrm{Harm}_6^{\mathrm{full}}(S^2,S^4)$ is a manifold

We only sketch the proof of this fact. Full details appear in [**BF**]. As explained above, twistor lifts of linearly full harmonic 2-spheres into S^4 can be written as linearly full maps $\phi : S^2 \to \mathbb{CP}^3$, $\phi(z) = [s(z):\alpha_1(z):\alpha_2(z):\tau_{12}(z)]$ satisfying (2.1). Let V_d be the subset of $(C[z]_d)^4$ consisting of those quadruplets of coprime polynomials with maximum degree equal to d for which the map $[s(z):\alpha_1(z):\alpha_2(z):\tau_{12}(z)]$ is linearly full in $\mathbb{CP}^3$.

Consider the map $\Phi_d : V_d \to \mathbb{C}[z]_{2d-2}$ given by

$$\Phi_d(s,\alpha_1,\alpha_2,\tau_{12}) = (s\tau_{12}' - s'\tau_{12}) - (\alpha_1'\alpha_2 - \alpha_1\alpha_2').$$

Then $\mathrm{Harm}_d^{\mathrm{full},+}(S^2,S^4)$ is isomorphic to $P(\Phi^{-1}(0))$ (see [**BF**] for details). So if we can show that the zero polynomial is a regular value of Φ_6, it will follow that $\mathrm{Harm}_6^{\mathrm{full}}(S^2,S^4)$ is a manifold.

Our original intention was not to prove regularity but to find singularities of $\mathrm{Harm}_6^{\mathrm{full}}(S^2,S^4)$, so the first step was to rule out all regular points of Φ and find candidates for singularities among the singular points of Φ. The strategy was to write out all the polynomials using the canonical basis, find $d\Phi_6$ in terms of the coefficients of the polynomials, and look for those points where the resulting matrix was singular. However, all the cases were exhausted and no singular points were found. This is the strategy followed in [**BF**].

Proving that an 11×28 matrix depending on 28 variables is regular is certainly a formidable task unless it has a simple form and the number of variables can be reduced. In order to do this, notice that there are two Lie groups that preserve harmonicity: $PSL(2,\mathbb{C})$ acting on the domain S^2 and $PSp(2,\mathbb{C})$ acting on the codomain $\mathbb{CP}^3$. It is not hard to see that these groups preserve the singularities of V_d. It turns out [**BF**] that every element in V_6 can be written, modulo the action of these groups, either as

$$[a_0+a_1z+a_2z^2 : c_1z+c_2z^2+c_4z^4+c_5z^5 : d_3z^3 : b_4z^4+b_5z^5+b_6z^6], \qquad \text{(Form 1)}$$

with $a_0b_6d_3 \neq 0$, or as

$$[a_0+a_1z+a_2z^2 : c_1z+c_2z^2+c_3z^3 : d_3z^3+d_4z^4+d_5z^5 : b_4z^4+b_5z^5+b_6z^6], \ \text{(Form 2)}$$

with $a_0b_6 \neq 0$ and with certain relations between the coefficients coming from the horizontality condition (2.1). In fact, these relations are not needed to treat Form 1 (when regularity follows relatively easily), but they are for Form 2, in which case they are given by

$$2a_0b_4 + c_1d_3 = 0, \tag{4.1}$$

$$5a_0b_5 + 3a_1b_4 + 3c_1d_4 + c_2d_3 = 0, \tag{4.2}$$

$$3a_0b_6 + 2a_1b_5 + a_2b_4 + 2c_1d_5 + c_2d_4 = 0, \tag{4.3}$$

$$5a_1b_6 + 3a_2b_5 + 3c_2d_5 + c_3d_4 = 0, \tag{4.4}$$

$$2a_2b_6 + c_3d_5 = 0. \tag{4.5}$$

To show regularity, we represent $d\Phi_6$ as an 11×28 matrix in the standard basis of $\mathbb{C}[z]_6$. For Form 2 above, this matrix is

$$\begin{bmatrix}
-a_1 & a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2a_2 & 0 & 2a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2c_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -a_2 & a_1 & 3a_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3c_3 & -c_2 & c_1 & 0 & 0 & 0 & 0 & -3d_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2a_1 & 4a_0 & 0 & 0 & -4b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2c_3 & 0 & 2c_1 & 0 & 0 & 0 & -4d_4 & -2d_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_2 & 3a_1 & 5a_0 & 0 & -5b_5 & -3b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -c_3 & c_2 & 3c_1 & 0 & 0 & -5d_5 & -3d_4 & -d_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2a_2 & 4a_1 & 6a_0 & -6b_6 & -4b_5 & -2b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_2 & 4c_1 & 0 & 0 & -4d_5 & -2d_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 3a_2 & 5a_1 & 0 & -5b_6 & -3b_5 & -b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c_3 & 3c_2 & 5c_1 & 0 & 0 & -3d_5 & -d_4 & d_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 4a_2 & 0 & 0 & -4b_6 & -2b_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2c_3 & 4c_2 & 0 & 0 & 0 & -2d_5 & 0 & 2d_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3b_6 & -b_5 & b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3c_3 & 0 & 0 & 0 & 0 & -d_5 & d_4 & 3d_3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2b_6 & 0 & 2b_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2d_4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -b_6 & b_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & d_5
\end{bmatrix}$$

It remains to prove that this matrix has maximal rank. This can be done using (4.1)-(4.5) via a careful, case-by-case study of the minors of the matrix.

It would be of course be very gratifying to find a more geometrical proof, maybe using some general results for the construction in Section 3. Furthermore, the methods used here (which are similar to the methods used in [**BW**] for $d = 4,5$) cannot easily be used for $d \geq 7$ since the size of the calculations increases exponentially with d.

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