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THE SPACE OF ALMOST COMPLEX 2-SPHERES IN THE 6-SPHERE

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Abstract. The complex dimension of the space of linearly full almost complex 2-spheres of area $4\pi d$ in the round 6-sphere is calculated to be $d+8$. Explicit examples of these objects are constructed for every integer value of the degree, $d \geq 6$, $d \neq 7$. Furthermore, it is shown that when $d = 6$ this space is isomorphic to the group $G_2(\mathbb{C})$, and when $d = 7$ this space is empty. We also show that the dimension of the space of non-linearly full almost complex 2-spheres of area $4\pi d$ in the round 6-sphere is $2d + 5$

1. Introduction

Octonionic multiplication in $\mathbb{R}^8$ induces a *cross product* in the vector space, isomorphic to $\mathbb{R}^7$, of imaginary octonions, by defining

$$\boldsymbol{x} \times \boldsymbol{y} = \mathrm{Im}(\boldsymbol{x}\boldsymbol{y})$$

where octonionic multiplication between $\boldsymbol{x}$ and $\boldsymbol{y}$ is understood and Im( ) denotes the octonionic imaginary part. In turn, this defines an *almost complex structure* in $S^6 \subset \mathrm{Im}(\mathbb{O})$: if $\boldsymbol{p} \in S^6$ and $X_{\boldsymbol{p}} \in T_{\boldsymbol{p}}S^6$, define

$$J_{\boldsymbol{p}}(X_{\boldsymbol{p}}) = \boldsymbol{p} \times X_{\boldsymbol{p}}.$$

Then J is an orthogonal almost complex structure in S^6. Furthermore, it is a *nearly Kähler* structure in S^6 in the sense that $(\nabla_X J)X = 0$ for any $X \in TS^6$, where ∇ denotes the Levi-Civita connection in S^6 [15].

A smooth map f from any almost complex manifold (M, J^M) to S^6 is *almost complex* if it is a morphism from (M, J^M) to (S^6, J), i.e.

$$df \circ J^M = J \circ df.$$

The particular case of almost complex maps from $S^2 \cong \mathbb{CP}^1$ to S^6 has been studied by several authors (see for example [8, 7, 10, 15, 22, 23]). In particular, explicit examples of these maps were found in [23], and a Weierstrass-like representation was given in [8].

On the other hand, a map $f : S^2 \to S^6$ is *harmonic* if $\Delta^{S^2} f = \lambda f$ for some function $\lambda : S^2 \to \mathbb{R}$ (see [9] for example). A simple computation shows that almost complex maps from S^2 to S^6 are, in particular, harmonic (see Section 2). This has several implications. The area of a harmonic map $f : S^2 \to S^6$ is graded by the degree: $\mathrm{Area}(f(S^2)) = 4\pi d$, where d is a positive integer [1], and the space of *linearly full* (i.e. whose image does not lie in a proper subsphere of S^6) harmonic

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maps of degree d from S^2 to S^6 can be given the structure of a complex projective variety [10, 16] of dimension $2d+9$ [13, 14]. Therefore, the set of almost complex maps from S^2 to S^6 of a given degree can be furnished with the structure of a projective subvariety of the space of harmonic maps from S^2 to S^6, and the following questions arise naturally: What is its dimension? Are there examples of linearly full almost complex maps from S^2 to S^6 for every value of the degree?

In this paper we use standard techniques in the study of harmonic maps to show that the set of linearly full almost complex maps from S^2 to S^6 is nonempty with dimension $d+8$ for $d \geq 6$, $d \neq 7$, and is empty otherwise. Furthermore, when $d = 6$, this space is isomorphic to $G_2(\mathbb{C})$. In addition, explicit examples of linearly full almost complex maps are found for every value of $d \geq 6$, $d \neq 7$. We also find that the dimension of the space of non-linearly full maps is $2d+5$.

The paper is organized as follows: in Section 2 we give a quick introduction of the tools that will be used in subsequent sections. In Section 3 we find criteria to determine when a harmonic map from S^2 to S^6 is almost complex, and we show that two almost complex maps are $SO(7,\mathbb{C})$-congruent (in the appropriate sense, see for example [1]) if and only if they are $G_2(\mathbb{C})$-congruent. This fact will be used in Section 4 to prove the statements regarding dimension explained above. Finally, in Section 5 we construct explicit examples of linearly full almost complex maps from S^2 to S^6.

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2. Preliminaries

2.1. **The octonions.** Let $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}, \boldsymbol{\epsilon}, \mathbf{i}\boldsymbol{\epsilon}, \mathbf{j}\boldsymbol{\epsilon}, \mathbf{k}\boldsymbol{\epsilon}\}$ be an orthonormal basis of $\mathbb{R}^8$. The (real) octonions, denoted by $\mathbb{O}$, are the (nonassociative, noncommutative) algebra over $\mathbb{R}$ with multiplication table, given in terms of this basis, by

	$\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$\boldsymbol{\epsilon}$	$\mathbf{i}\boldsymbol{\epsilon}$	$\mathbf{j}\boldsymbol{\epsilon}$	$\mathbf{k}\boldsymbol{\epsilon}$
$\mathbf{1}$	$-\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$	$\boldsymbol{\epsilon}$	$\mathbf{i}\boldsymbol{\epsilon}$	$\mathbf{j}\boldsymbol{\epsilon}$	$\mathbf{k}\boldsymbol{\epsilon}$
$\mathbf{i}$	$\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}$	$-\mathbf{j}$	$\mathbf{i}\boldsymbol{\epsilon}$	$-\boldsymbol{\epsilon}$	$-\mathbf{k}\boldsymbol{\epsilon}$	$\mathbf{j}\boldsymbol{\epsilon}$
$\mathbf{j}$	$\mathbf{j}$	$-\mathbf{k}$	$-\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}\boldsymbol{\epsilon}$	$\mathbf{k}\boldsymbol{\epsilon}$	$-\boldsymbol{\epsilon}$	$-\mathbf{i}\boldsymbol{\epsilon}$
$\mathbf{k}$	$\mathbf{k}$	$\mathbf{j}$	$-\mathbf{i}$	$-\mathbf{1}$	$\mathbf{k}\boldsymbol{\epsilon}$	$-\mathbf{j}\boldsymbol{\epsilon}$	$\mathbf{i}\boldsymbol{\epsilon}$	$-\mathbf{k}\boldsymbol{\epsilon}$
$\boldsymbol{\epsilon}$	$\boldsymbol{\epsilon}$	$-\mathbf{i}\boldsymbol{\epsilon}$	$-\mathbf{j}\boldsymbol{\epsilon}$	$-\mathbf{k}\boldsymbol{\epsilon}$	$-\mathbf{1}$	$\mathbf{i}$	$\mathbf{j}$	$\mathbf{k}$
$\mathbf{i}\boldsymbol{\epsilon}$	$\mathbf{i}\boldsymbol{\epsilon}$	$\boldsymbol{\epsilon}$	$-\mathbf{k}\boldsymbol{\epsilon}$	$\mathbf{j}\boldsymbol{\epsilon}$	$-\mathbf{i}$	$-\mathbf{1}$	$-\mathbf{k}$	$\mathbf{j}$
$\mathbf{j}\boldsymbol{\epsilon}$	$\mathbf{j}\boldsymbol{\epsilon}$	$\mathbf{k}\boldsymbol{\epsilon}$	$\boldsymbol{\epsilon}$	$-\mathbf{i}\boldsymbol{\epsilon}$	$-\mathbf{j}$	$\mathbf{k}$	$-\mathbf{1}$	$-\mathbf{i}$
$\mathbf{k}\boldsymbol{\epsilon}$	$\mathbf{k}\boldsymbol{\epsilon}$	$-\mathbf{j}\boldsymbol{\epsilon}$	$\mathbf{i}\boldsymbol{\epsilon}$	$\boldsymbol{\epsilon}$	$-\mathbf{k}$	$-\mathbf{j}$	$\mathbf{i}$	$-\mathbf{1}$

Similarly one defines the *complex octonions* as $\mathbb{O} \otimes \mathbb{C}$ with the multiplication table above. The real part of a real or complex octonion is the term involving $\mathbf{1}$; the imaginary part is the sum of the remaining terms.

Let $\operatorname{Im}(\mathbb{O})$ and $\operatorname{Im}(\mathbb{O}) \otimes \mathbb{C}$ denote the real and complex span, respectively, of $\{\mathbf{i}, \mathbf{j}, \mathbf{k}, \boldsymbol{\epsilon}, \mathbf{i}\boldsymbol{\epsilon}, \mathbf{j}\boldsymbol{\epsilon}, \mathbf{k}\boldsymbol{\epsilon}\}$. Then the formula

$$\boldsymbol{x} \times \boldsymbol{y} = \operatorname{Im}(\boldsymbol{x}\boldsymbol{y})$$

defines a *cross product* in $\operatorname{Im}(\mathbb{O}) \cong \mathbb{R}^7$ or $\operatorname{Im}(\mathbb{O}) \otimes \mathbb{C} \cong \mathbb{C}^7$ with the following properties: for $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in $\mathbb{O}$ or $\mathbb{O} \otimes \mathbb{C}$,

$$\boldsymbol{u} \times \boldsymbol{v} = \frac{1}{2}(\boldsymbol{u}\boldsymbol{v} - \boldsymbol{v}\boldsymbol{u}) \qquad \text{and} \qquad (\boldsymbol{u}, \boldsymbol{v}) = -\frac{1}{2}(\boldsymbol{u}\boldsymbol{v} + \boldsymbol{v}\boldsymbol{u}), \tag{2.1}$$

where $(\ ,\ )$ denotes the standard inner product of $\mathbb{R}^7$ or its bilinear extension to $\mathbb{C}^7$, i.e.

$$\big((u_1,\ldots,u_7),(v_1,\ldots,v_7)\big) = \sum_{i=1}^{7} u_j v_j$$

for u_i, v_i in $\mathbb{R}$ or $\mathbb{C}$, $1 \leq i \leq 7$. We will use $\langle\ ,\ \rangle$ to denote the *hermitian* inner product in $\mathbb{C}^7$, i.e.

$$\langle (u_1,\ldots,u_7),(v_1,\ldots,v_7)\rangle = \sum_{i=1}^{7} u_j \bar{v}_j$$

for $u_i, v_i \in \mathbb{C}$, $1 \leq i \leq 7$. Other important properties of the cross product are

$$\boldsymbol{u}\times(\boldsymbol{v}\times\boldsymbol{w}) + (\boldsymbol{u}\times\boldsymbol{v})\times\boldsymbol{w} = 2(\boldsymbol{u},\boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u},\boldsymbol{v})\boldsymbol{w} - (\boldsymbol{w},\boldsymbol{v})\boldsymbol{u}, \tag{2.2}$$

$$\boldsymbol{u}\times(\boldsymbol{u}\times\boldsymbol{v}) = (\boldsymbol{u},\boldsymbol{v})\boldsymbol{u} - (\boldsymbol{u},\boldsymbol{u})\boldsymbol{v}, \tag{2.3}$$

$$(\boldsymbol{u},\boldsymbol{v}\times\boldsymbol{w}) = (\boldsymbol{v},\boldsymbol{w}\times\boldsymbol{u}) = (\boldsymbol{w},\boldsymbol{u}\times\boldsymbol{v}). \tag{2.4}$$

The group of *automorphisms* of the octonions and complex octonions are G_2 and $G_2(\mathbb{C})$, respectively, i.e. $(g\boldsymbol{u})(g\boldsymbol{v})) = g(\boldsymbol{u}\boldsymbol{v})$ for all g in G_2 and $G_2(\mathbb{C})$ and all $\boldsymbol{u}, \boldsymbol{v}$ in $\mathbb{O}$ and $\mathbb{O}\otimes\mathbb{C}$, respectively. For a very clear and beautiful exposition of the octonions and their properties, see for example [19, 20].

2.2. **Almost complex maps from the 2-sphere to the 6-sphere.** A map $f : S^2 \to S^6$ is almost complex if $J \circ df = df \circ J^{S^2}$, where J denotes the almost complex structure in S^6 defined by

$$J_{\boldsymbol{p}}(X_{\boldsymbol{p}}) = \boldsymbol{p} \times X_{\boldsymbol{p}}$$

for $\boldsymbol{p} \in S^6 \subset \mathrm{Im}(\mathbb{O})$ and $X_{\boldsymbol{p}} \in T_{\boldsymbol{p}}S^6 \subset T_{\boldsymbol{p}}\mathrm{Im}(\mathbb{O})$.

The standard complex structure of $S^2 \cong \mathbb{CP}^1$ can be defined, similarly, using the cross product in $\mathbb{R}^3$:

$$J_q^{S^2}(Y_q) = q \times Y_q,$$

where $q \in S^2$ and $Y_q \in T_qS^2$, and the cross product is given by

$$x \times y = \mathrm{Im}(xy)$$

where in this case *quaternionic* multiplication and imaginary parts are used. In fact, this gives the simplest examples of almost complex maps from S^2 to S^6: if $\hat{f} : S^2 \to S^2 \subset \mathbb{R}^3 \cong \mathrm{Im}(\mathbb{H})$ is any holomorphic map, and if $h : \mathbb{H} \to \mathbb{O}$ is any linear homomorphism of algebras, then $h(\mathrm{Im}(\mathbb{H})) \subset \mathrm{Im}(\mathbb{O})$ and $h(x \times y) = h(x) \times h(y)$ for $x, y \in \mathrm{Im}(\mathbb{H})$, which implies that $f := h \circ \hat{f}$ will be an almost complex map since

$$\begin{aligned} J \circ df &= J \circ dh \circ d\hat{f} \\ &= dh \circ J^{S^2} \circ d\hat{f} \quad \text{because } h \text{ is a linear homomorphism} \\ &= dh \circ d\hat{f} \circ J^{S^2} \quad \text{because } \hat{f} \text{ is holomorphic} \\ &= df \circ J^{S^2}. \end{aligned}$$

We will often identify (S^2, J^{S^2}) with $\mathbb{CP}^1$ (or with $\mathbb{C} \cup \{\infty\}$) via a bi-holomorphic map, for example an appropriate stereographic projection.

In general, let $z = x + iy$ be a local holomorphic coordinate in S^2. Then

$$J^{S^2}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial y} \quad \text{and} \quad J^{S^2}\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial x}.$$

Therefore $f : S^2 \to S^6 \subset \mathbb{R}^7 \cong \mathrm{Im}(\mathbb{O})$ is almost complex if and only if

$$J_{f(z)}\left(df\left(\frac{\partial}{\partial x}\right)\right) = df\left(\frac{\partial}{\partial y}\right) \quad \text{and} \quad J_{f(z)}\left(df\left(\frac{\partial}{\partial y}\right)\right) = -df\left(\frac{\partial}{\partial x}\right)$$

Using subscripts to indicate differentiation, this equation can be written as

$$f \times f_x = f_y \quad \text{and} \quad f \times f_y = -f_x.$$

Differentiating again, and using 2.1,

$$f \times f_{xx} = f_{xy} \quad \text{and} \quad f \times f_{yy} = -f_{yx},$$

and adding these two equations we obtain

$$f \times (f_{xx} + f_{yy}) = 0.$$

Thus $(f_{xx}(z)+f_{yy}(z))$ is orthogonal to $f(z)$ for all $z \in S^2$, and hence f is *harmonic*.

Using $\mathrm{Harm}(S^2, S^6)$ and $\mathrm{Ac}(S^2, S^6)$ to denote the set of harmonic maps and almost complex maps, respectively, from S^2 to S^6, we therefore have

$$\mathrm{Ac}(S^2, S^6) \subset \mathrm{Harm}(S^2, S^6).$$

Recall [1] that the area of the image of a harmonic map from S^2 to S^{2n} is $4\pi d$, where d is a positive integer called the *degree* of the harmonic map. We will use $\mathrm{Harm}_d(S^2, S^6)$ and $\mathrm{Ac}_d(S^2, S^6)$ to denote the subsets of $\mathrm{Harm}(S^2, S^6)$ and $\mathrm{Ac}(S^2, S^6)$, respectively, of maps of degree d. Also, we will use $\mathrm{Harm}_d^f(S^2, S^6)$ and $\mathrm{Ac}_d^f(S^2, S^6)$ to denote the subsets of *linearly full* maps, i.e. whose image is not contained in any proper geodesic subsphere of S^6, and $\mathrm{Harm}_d^{(k)}(S^2, S^6)$ and $\mathrm{Ac}_d^{(k)}(S^2, S^6)$ to denote the subsets of those maps whose image is contained in a k-dimensional subsphere but *not* in a $(k-1)$-dimensional subsphere of S^6. It is known [9] that k has to be an even number. In addition, it is proved in [7, Lemma 4.3] that $\mathrm{Ac}_d^{(4)}(S^2, S^6)$ is empty. Therefore

$$\mathrm{Ac}_d(S^2, S^6) = \mathrm{Ac}_d^{(2)}(S^2, S^6) \sqcup \mathrm{Ac}_d^f(S^2, S^6) \quad \text{(disjoint union)}.$$

The set $\mathrm{Harm}_d(S^2, S^6)$ can be furnished with the structure of an algebraic variety [16]; the dimension of $\mathrm{Harm}_d^f(S^2, S^6)$ is $2d + 9$, and $\dim_{\mathbb{C}}(\mathrm{Harm}_d^{(2k)}(S^2, S^6)) = 2d + 9 - (3 - k)(2 - k)$ for $k = 1, 2$ [13, 14]. Since The set $\mathrm{Ac}_d(S^2, S^6)$ is an algebraic subvariety of $\mathrm{Harm}_d(S^2, S^6)$ [10], to find the dimension of $\mathrm{Ac}_d(S^2, S^6)$ we can use some of the common machinery in the study of harmonic maps into spheres and projective spaces—namely *harmonic sequences* (see for example [24, 6, 7]), *singularity type* (see [17, 4, 1, 6]) and *twistor lifts* (see [9, 1]). We now give a quick introduction to these techniques.

2.3. **Harmonic sequences.** We describe the harmonic sequence of a harmonic map for the specific case of linearly full maps from S^2 to S^6. Details and proofs can be found, for example, in [6, 3, 11], and a more general description appears in [24].

The idea is simple: given a linearly full harmonic map $f : S^2 \to S^6 \subset \mathbb{C}^7$, differentiate it and project the result over the space orthogonal to f to obtain the next element of the sequence. This procedure is independent of the chosen coordinate modulo scalar multiplication, so it produces a sequence of smooth functions from S^2 to $\mathbb{CP}^6$. More precisely, let $f_p : S^2 \to \mathbb{C}^7$ be given inductively by the conditions

$$f_0 = f \tag{2.5}$$

$$f_{p+1} = \frac{\partial f_p}{\partial z} - \frac{1}{|f_p|^2}\left\langle \frac{\partial f_p}{\partial z}, f_p \right\rangle f_p, \quad -3 \le p \le 2 \tag{2.6}$$

$$f_{p-1} = -\frac{|f_{p-1}|^2}{|f_p|^2}\ \frac{\partial f_p}{\partial \bar{z}}, \qquad -2 \le p \le 3, \tag{2.7}$$

where $\langle\ ,\ \rangle$ and $|\ \ |$ denote the hermitian product and associated norm, respectively, in $\mathbb{C}^7$. Since f is assumed to be linearly full, the maps f_p, $-3 \le p \le 3$, are not identically zero, and their definition, away from the points where any of the f_p is zero, is independent of the holomorphic coordinate z chosen, modulo multiplication by scalars. Thus, the maps $\phi_p := [f_p]$, $-3 \le p \le 3$, are well defined in an open subset of S^2; furthermore, their definition can be extended over the points where any of the f_p is zero, giving maps $\phi_p : S^2 \to \mathbb{CP}^6$. It is not hard to check that they are harmonic [11].

The sequence of maps ϕ_p, $-3 \le p \le 3$, is called *the harmonic sequence of* f. Additionally, ϕ_{-3} is holomorphic and ϕ_3 is antiholomorphic. Although the sequence of functions f_p defined above consists only of local representatives of the harmonic sequence ϕ_p, by a slight abuse of language we will also refer to it as 'the harmonic sequence of f'. Note that, although the functions ϕ_p do not depend on the coordinate z used in the definition of the f_p, the functions f_p certainly do.

The maps f_p satisfy the following properties (see, for example, [3]):

$$\bar{f}_p = (-1)^p |f_p|^2 f_{-p} \tag{2.8}$$

$$|f_p||f_{-p}| = 1 \tag{2.9}$$

$$(f_p, f_q) = (-1)^p\, \delta_{-p,q}, \tag{2.10}$$

where δ_{ij} is the Kronecker delta. Together with (2.7), this implies that f_{-3} is holomorphic.

The map ϕ_{-3}, which is usually called the *directrix curve* of f, is characterized by being *totally isotropic*, i.e. for every local representation f_{-3} of ϕ_{-3},

$$\left(\frac{\partial^i f_{-3}}{\partial z^i}, \frac{\partial^j f_{-3}}{\partial z^j} \right) = 0, \quad 0 \le i, j \le 2.$$

Furthermore [1], every holomorphic, linearly full, totally isotropic map $\Xi : S^2 \to \mathbb{CP}^6$ uniquely determines a harmonic map $f : S^2 \to S^6$ (defined using (2.6)) up to composition with the antipodal map of S^6. This implies that much of the study of the set of harmonic maps (or, in particular, of almost complex maps) from S^2 to S^6 can be translated to the study of totally isotropic curves in $\mathbb{CP}^6$. A very useful tool in the study of these curves is the notion of singularity type, which we describe in the next subsection.

2.4. **Singularity type.** We briefly describe the notion of singularity type of holomorphic curves in $\mathbb{CP}^n$. For details, see [4, 17]. Let Σ be a Riemann surface and let

$$G : \Sigma \to \mathbb{CP}^n$$

be a linearly full holomorphic curve. Locally, write $G = [\mathbf{g}(z)] = [g_0(z), \dots, g_n(z)]$, where z is a holomorphic coordinate in Σ and where the g_i, $0 \le i \le n$, do not vanish simultaneously. Then, for $0 \leq k \leq n-1$, the k^{th} *associated curve* of G is the map $\sigma_k : \Sigma \to \mathbb{P}(\Lambda^{k+1}\mathbb{C}^{n+1}) \cong \mathbb{CP}^{\binom{n+1}{k+1}-1}$ locally defined by

$$\mathbf{g} \wedge \frac{\partial \mathbf{g}}{\partial z} \wedge \cdots \wedge \frac{\partial^k \mathbf{g}}{\partial z^k}.$$

A *higher singularity* of G is a point p where the derivative of any of the associated curves of G is zero.

Writing $\sigma_k(z) = [\boldsymbol{\sigma}_k(z)]$ locally, let $r_k(p)$ be the nonnegative integer defined by

$$r_k(p) = \text{Order of vanishing of} \left(\boldsymbol{\sigma}_k \wedge \frac{\partial \boldsymbol{\sigma}_k}{\partial z} \right) \text{ at } z = p.$$

Note that all the $r_i(p)$ are zero except at a finite subset of Σ. The *singularity type* of the original map $G : \Sigma \to \mathbb{CP}^n$ is defined to be the set

$$\{(p; r_0(p), \ldots, r_{n-1}(p)) \mid p \text{ is a higher singularity of } G\}.$$

The *total ramification degree* of σ_k is defined by

$$r_k = \sum_{p \in \Sigma} r_k(p).$$

If δ_k denotes the degree of σ_k, and writing $\delta_{-1} = \delta_n = 0$, we have the *Plücker formulas*

$$\delta_{k-1} - 2\delta_k + \delta_{k+1} = 2g - 2 - r_k, \quad 0 \leq k \leq n-1,$$

where g is the genus of the surface Σ.

When G is the directrix curve of a linearly full almost complex 2-sphere in S^6, the Plücker formulas greatly simplify. Let $f : S^2 \to S^6$ be a linearly full harmonic map, and let Φ_k, $-3 \leq k \leq 3$, be its harmonic sequence. Then $\Phi_{-3} : S^2 \to \mathbb{CP}^6$ is holomorphic and linearly full. Let $\sigma_k : S^2 \to \mathbb{CP}^{\binom{7}{k+1}-1}$, $0 \leq k \leq 5$, be the k^{th} associated curve of Φ_{-3}, let δ_k be the degree of σ_k, and let r_k be the total ramification degree of σ_k. The fact that f_0 is real implies [1]

$$\delta_{5-k} = \delta_k, \quad p = 0, 1, 2,$$

and then the Plücker formulas read

$$\begin{aligned} -2\delta_0 + \delta_1 &= -(2 + r_0) \\ \delta_0 - 2\delta_1 + \delta_2 &= -(2 + r_1) \\ \delta_1 - \delta_2 &= -(2 + r_2). \end{aligned} \tag{2.11}$$

This implies, in particular

$$\delta_2 = 12 + r_0 + 2r_1 + 3r_2. \tag{2.12}$$

On the other hand if f_j, $-3 \leq j \leq 3$, is the harmonic sequence of f, then equations (2.6) and (2.7) imply that

$$\boldsymbol{\sigma}_k = f_{-3} \wedge f_{-2} \wedge \cdots \wedge f_{k-3} \tag{2.13}$$

is a local representation of σ_k. Hence the degree of σ_k can be calculated using the formula

$$\delta_k = \frac{1}{2\pi i} \int_{S^2} \frac{\partial^2}{\partial \bar{z} \partial z} \log |\boldsymbol{\sigma_k}|^2 \, d\bar{z} \wedge dz. \tag{2.14}$$

Using (2.13) and (2.10), we have

$$|\boldsymbol{\sigma_k}|^2 = |f_{-3}|^2 |f_{-2}|^2 \cdots |f_{k-3}|^2,$$

and (2.6) and (2.7) imply, for $0 \le k \le 5$, that

$$\frac{\partial^2}{\partial\bar{z}\partial z}\log(|f_{-3}|^2|f_{-2}|^2\cdots|f_{k-3}|^2) = \frac{|f_{k-2}|^2}{|f_{k-3}|^2},$$

so if we let

$$\gamma_j = |f_{j+1}|^2/|f_j|^2, \quad -3 \le j \le 2,$$

then equation (2.9) implies that $\gamma_j = \gamma_{-j-1}$, and therefore, for $0 \le k \le 2$,

$$\delta_k = \frac{1}{2\pi i}\int_{S^2}\gamma_{k-3}\ d\bar{z}\wedge dz = \frac{1}{2\pi i}\int_{S^2}\gamma_{2-k}\ d\bar{z}\wedge dz$$

(see [6] for details). In the particular case when $f \in \mathrm{Ac}_d^f(S^2, S^6)$, Lemma 5.2 of [7] gives $\gamma_0 = 2\gamma_2$ (which also follows from the equality $|f_1|^2|f_2|^2 = 2|f_3|^2$ obtained in the proof of Proposition 3.1 below). Therefore, if $f \in \mathrm{Ac}_d^f(S^2, S^6)$,

$$2\delta_0 = \delta_2, \tag{2.15}$$

which, using (2.11), gives

$$r_2 = r_0, \tag{2.16}$$

and hence

$$\delta_0 = 6 + 2r_0 + r_1. \tag{2.17}$$

The last three equations have particular importance in what follows. On the one hand, equation (2.15) states that the degree of a linearly full almost complex map from S^2 to S^6 is equal to the degree of its directrix curve, which is peculiar. On the other hand, as we will see in Section 4, a map $f \in \mathrm{Ac}_d^f(S^2, S^6)$ is essentially determined by its singularity type, which is restricted by equation (2.17). This fact will be used to find an upper bound on the dimension of $\mathrm{Ac}_d^f(S^2, S^6)$. A lower bound on the dimension of $\mathrm{Ac}_d^f(S^2, S^6)$ is implicit in [10], where a different approach is used, as described in the next subsection.

2.5. **Twistor lifts.** We give a quick description of the twistor construction started by Calabi in the 60's. For details, see [9, 1]. Given a linearly full harmonic map $f : S^2 \to S^{2n}$, its *twistor lift* is the map $\psi : S^2 \to \mathcal{Z}_n \subset \mathrm{Gr}(n, \mathbb{C}^{2n+1})$ defined by

$$\psi(z) = \mathrm{Span}\left(f, \frac{\partial f}{\partial \bar{z}}, \ldots, \frac{\partial^n f}{\partial \bar{z}^n}\right),$$

where z is a holomorphic coordinate. The set $\mathcal{Z}_n$ is the submanifold of the Grassmannian of n-planes in $\mathbb{C}^{2n+1}$ consisting of isotropic n-planes, i.e. the set of $P \in \mathrm{Gr}(n, \mathbb{C}^{2n+1})$ such that $(\boldsymbol{u}, \boldsymbol{v}) = 0$ for all $\boldsymbol{u}, \boldsymbol{v} \in P$. It is a Kähler manifold isomorphic to the homogeneous space $SO(2n+1, \mathbb{R})/U(n)$ [1].

There is a projection $\pi : \mathcal{Z}_n \to S^{2n}$ that can be defined as follows: given a subspace $P \in \mathcal{Z}_n$, define $\pi(P)$ as the unique real unit vector in $\mathbb{C}^{2n+1}$ such that $\{\pi(P), P_1, P_2, P_3, \bar{P}_1, \bar{P}_2, \bar{P}_3\}$ is a positively oriented basis of $\mathbb{C}^{2n+1}$, where the set $\{P_1, P_2, P_3\}$ is a basis of P. This map is a Riemannian submersion with the metric in $\mathcal{Z}_n$ induced by the standard metric of $\mathrm{Gr}(n, \mathbb{C}^{2n+1})$.

If $f : S^2 \to S^{2n}$ is harmonic and linearly full then its twistor lift is holomorphic, horizontal (i.e. its derivative is perpendicular to the fibers of π) and linearly full (in the sense explained in [16]); conversely, if $\psi : S^2 \to Z_n$ is a holomorphic, horizontal and linearly full map, then $\pm\pi \circ \psi$ are harmonic. The degree of a holomorphic map ψ from S^2 to $\mathcal{Z}_n$ is defined as the image under ψ_* of $\mathbf{1} \in H_2(S^2, \mathbb{Z}) \cong \mathbb{Z}$

in $H_2(\mathcal{Z}_n,\mathbb{Z})\cong\mathbb{Z}$. Note that if ψ is the twistor lift of $f\in \mathrm{Harm}_d^f(S^2,S^{2n})$ then $\deg(\psi)=d$ [1].

If we let

$$\mathrm{HH}_d^f(S^2,\mathcal{Z}_n)=\{\text{Holomorphic, horizontal, full maps } \psi: S^2\to\mathcal{Z}_n \text{ of degree } d\}$$

and

$$\mathrm{Harm}_d^{f,\pm}(S^2,S^{2n})=\{\pm\pi\circ\psi:\psi\in\mathrm{HH}_d^f(S^2,\mathcal{Z}_n)\},$$

then the last paragraph can be summarized by

$$\mathrm{Harm}_d^f(S^2,S^{2n})=\mathrm{Harm}_d^{f,+}(S^2,S^{2n})\sqcup\mathrm{Harm}_d^{f,-}(S^2,S^{2n}).$$

In [10, 12, 14] birational maps $b_E:\mathbb{CP}^{\frac{n(n+1)}{2}}\to\mathcal{Z}_n$ were constructed which translated the problem of finding holomorphic, horizontal, linearly full maps into $\mathcal{Z}_n$ into finding solutions of a differential system in $\mathbb{CP}^{\frac{n(n+1)}{2}}$. More precisely, for the particular case $n=3$, let $E=\{E_0,E_1,E_2,E_3,\bar E_1,\bar E_2,\bar E_3\}$ be a basis of $\mathbb{C}^7$ satisfying

$$(E_0,E_r)=(E_0,\bar E_r)=(E_r,E_s)=(\bar E_r,\bar E_s)=0, \text{ and } (E_r,\bar E_s)=\delta_{rs},\quad r,s=1,2,3.$$

Define the birational map $b_E:\mathbb{CP}^6\to\mathcal{Z}_3$ that takes

$$[s:\alpha_1:\alpha_2:\alpha_3:\tau_{12}:\tau_{23}:\tau_{31}]$$

to the 3-plane in $\mathbb{C}^3$ spanned by the vectors

$$\frac{\alpha_\ell}{s}E_0+E_\ell-\sum_{k=1}^{3}\left(\frac{\alpha_\ell\alpha_k}{2s^2}+\frac{\tau_{\ell k}}{2s}\right)\overline{E}_k,\quad 1\le\ell\le 3,$$

where it is understood that $\tau_{ji}=-\tau_{ij}$.

Under this birational map the horizontality condition translates as follows [18, 10, 13, 14]. A map $\psi:S^2\to\mathcal{Z}_3$ is holomorphic, horizontal and linearly full if and only if the map

$$\tilde\psi:=b_E^{-1}\circ\psi=[s:\alpha_1:\alpha_2:\alpha_3:\tau_{12}:\tau_{23}:\tau_{31}]$$

satisfies

$$\alpha_i'\alpha_j-\alpha_i\alpha_j'=s\tau_{ij}'-s'\tau_{ij},\qquad 1\le i,j\le 3, \tag{2.18}$$

plus the open condition

$$W\left(\left(\frac{\alpha_1}{s}\right)',\left(\frac{\alpha_2}{s}\right)',\left(\frac{\alpha_3}{s}\right)'\right)\not\equiv 0, \tag{2.19}$$

where W denotes the Wronskian, and the dashes denote differentiation with respect to a holomorphic coordinate in S^2. In addition, the image of $\psi\in\mathrm{HH}_d^f(S^2,\mathcal{Z}_3)$ misses the subspace generated by $\{\bar E_1,\bar E_2,\bar E_3\}$ if and only if $\tilde\psi$ has degree exactly d. In other words, if we define $\mathrm{PD}_d^f(S^2,\mathbb{CP}^6)\subset\mathbb{P}(\mathbb{C}[z]_d)^7$ by

$$\mathrm{PD}_d^f(S^2,\mathbb{CP}^6)=\{\tilde\psi:S^2\to\mathbb{CP}^6 \text{ of degree } d \text{ satisfying (2.18) and (2.19)}\},$$

then [10, Theorem 2]

$$\left\{\psi\in\mathrm{HH}_d^f(S^2,\mathcal{Z}_3):\mathrm{span}_{\mathbb{C}}\left\{\bar E_1,\bar E_2,\bar E_3\right\}\notin\psi(S^2)\right\}\cong\mathrm{PD}_d^f(S^2,\mathbb{CP}^6).$$

Now let let $\{\mathbf{i},\mathbf{j},\mathbf{k},\boldsymbol\epsilon,\mathbf{i}\boldsymbol\epsilon,\mathbf{j}\boldsymbol\epsilon,\mathbf{k}\boldsymbol\epsilon\}$ be an orthonormal basis of $\mathbb{R}^7\cong\mathrm{Im}(\mathbb{O})$ satisfying

$$\mathbf{k}=\mathbf{i}\times\mathbf{j}\qquad \mathbf{i}\boldsymbol\epsilon=\mathbf{i}\times\boldsymbol\epsilon\qquad \mathbf{j}\boldsymbol\epsilon=\mathbf{j}\times\boldsymbol\epsilon\qquad \mathbf{k}\boldsymbol\epsilon=\mathbf{k}\times\boldsymbol\epsilon$$

(this is the standard basis of $\mathrm{Im}(\mathbb{O})$) and let $E = \{E_0, E_1, E_2, E_3, \bar{E}_1, \bar{E}_2, \bar{E}_3\}$ be the basis of $\mathrm{Im}(\mathbb{O}) \otimes \mathbb{C}$ defined by

$$\begin{aligned} E_0 = \boldsymbol{\epsilon} \quad E_1 &= \frac{\mathbf{i} + i\mathbf{i}\boldsymbol{\epsilon}}{\sqrt{2}} \quad E_2 = \frac{\mathbf{j} - i\mathbf{j}\boldsymbol{\epsilon}}{\sqrt{2}} \quad E_3 = \frac{\mathbf{k} - i\mathbf{k}\boldsymbol{\epsilon}}{\sqrt{2}} \\ \bar{E}_1 &= \frac{\mathbf{i} - i\mathbf{i}\boldsymbol{\epsilon}}{\sqrt{2}} \quad \bar{E}_2 = \frac{\mathbf{j} + i\mathbf{j}\boldsymbol{\epsilon}}{\sqrt{2}} \quad \bar{E}_3 = \frac{\mathbf{k} + i\mathbf{k}\boldsymbol{\epsilon}}{\sqrt{2}}. \end{aligned} \tag{2.20}$$

Then the basis E satisfies the properties above. If we let

$$\mathrm{HH}^f_d(S^2, \mathcal{Z}_3)_{\mathrm{Ac}} = \{\psi \in \mathrm{HH}^f_d(S^2, \mathcal{Z}_3) : \pi \circ \psi \text{ is almost complex}\},$$

then [10, Proposition 6] (note that the constraint there does not have the factor '$\sqrt{2}$' due to a different choice of the basis (2.20) and the τ_{ij})

$$\begin{aligned} &\left\{\psi \in \mathrm{HH}^f_d(S^2, \mathcal{Z}_3)_{\mathrm{Ac}} : \mathrm{span}_{\mathbb{C}} \left\{\bar{E}_1, \bar{E}_2, \bar{E}_3\right\} \notin \psi(S^2)\right\} \\ &\qquad\qquad \cong \{\tilde{\psi} \in \mathrm{PD}^f_d(S^2, \mathbb{CP}^6) : i\sqrt{2}\alpha_1 = \tau_{23}\}. \end{aligned} \tag{2.21}$$

This last statement immediately gives a lower bound on the dimension of the variety $\mathrm{Ac}^f_d(S^2, S^6)$ which will be used in Lemma 4.1 below.

We need one last observation regarding twistor lifts of maps $f \in \mathrm{Ac}^f_d(S^2, S^6)$. If $\Xi : S^2 \to \mathbb{CP}^6$ is the directrix curve of f, and if $\sigma_2 : S^2 \to \mathbb{CP}^{34}$ denotes the 2nd associated curve of Ξ, then $\sigma_2 = \mathrm{Pl} \circ \psi$, where $\mathrm{Pl} : \mathcal{Z}_3 \subset \mathrm{Gr}(3, \mathbb{C}^7) \to \mathbb{CP}^{34}$ is the Plücker embedding, which has degree 2 [21]. Therefore

$$\delta_2 = \deg(\sigma_2) = 2\deg(\psi).$$

Using (2.15) this implies that if Ξ is the directrix curve of $f \in \mathrm{Ac}^f_d(S^2, S^6)$, then

$$\deg(\Xi) = d. \tag{2.22}$$

3. Cross products and congruence

In this section we state and prove some results that will be needed in the next sections and have an interest of their own. The first proposition gives a convenient criterion, in terms of cross products, to check whether a harmonic map from S^2 to S^6 is $(\pm)$-almost complex (we call a map f '$(-)$-almost complex' if $-f$ is almost complex). As a byproduct we obtain all the cross products of elements in the harmonic sequence of a $(\pm)$-almost complex map. In the second proposition we show that if two almost complex maps are $SO(7, \mathbb{C})$-congruent, then they are $G_2(\mathbb{C})$ congruent. Again, as a byproduct we obtain the cross products of the derivatives of the directrix curve of an almost complex map.

The proofs are computational in nature. We will make extensive use of the properties of the cross product given by (2.2), (2.3) and (2.4).

3.1. **Cross products.** This section is motivated by the following question: What is a simple property that characterizes twistor lifts of almost complex maps? In other words, $\mathrm{HH}^f_d(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$ is the subvariety of $\mathrm{HH}^f_d(S^2, \mathcal{Z}_3)$ of maps that satisfy which condition? Such a condition, namely the vanishing of the torsion 'III', was found in [8, Theorem 4.7] (see also [7, Remark 4.1]). We find a slightly more general criterion here.

Proposition 3.1. *Let $f : S^2 \to S^6$ be a linearly full harmonic map and let z be a holomorphic coordinate in S^2. Then f is $(\pm)$-almost complex (i.e. $f \times f_z = \pm f_z$) if and only if*

$$\operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial f}{\partial z}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^3 f}{\partial z^3} \right\}$$

is closed under $\times$.

Proof. Since f is harmonic, $\left(\frac{\partial^j f}{\partial z^j}, \frac{\partial^k f}{\partial z^k} \right) = 0$, $0 \le j, k \le 2$, so $\operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial f}{\partial z}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^3 f}{\partial z^3} \right\}$ is a totally isotropic subspace, and it has dimension 3 because f is linearly full. Let f_k, $-3 \le k \le 3$, be the harmonic sequence of f. Then $\operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial f}{\partial z}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^3 f}{\partial z^3} \right\} = \operatorname{span}_{\mathbb{C}} \{f_1, f_2, f_3\}$.

If f is $(\pm)$-almost complex, then it is of type (I) in the classification of almost complex curves in [7], so it satisfies (see equations (4.3), (4.4) and (5.1) of [7])

$$\begin{aligned} f \times f_1 &= \pm i f_1 \\ f \times f_2 &= \pm i f_2 \\ f \times f_3 &= \mp i f_3 \end{aligned} \tag{3.1}$$

Differentiating (3.1) and using (2.6) it is easy to obtain

$$f_1 \times f_2 = \pm 2 i f_3, \qquad f_1 \times f_3 = 0, \qquad f_2 \times f_3 = 0,$$

which proves the 'only if' part of the lemma.

The converse is not difficult to prove but it is long. The idea is to use the properties of the cross product and of the harmonic sequence. Suppose that $f_j \times f_k \in \operatorname{span}_{\mathbb{C}} \{f_1, f_2, f_3\}$, $1 \le j, k \le 3$. Note that $f_1 \times f_2 \ne 0$ since otherwise, using (2.7) and (2.3),

$$0 = f_0 \times \frac{\partial (f_1 \times f_2)}{\partial \bar{z}} = -\frac{|f_1|^2}{|f_0|^2} f_0 \times (f_0 \times f_2) = \frac{|f_1|^2}{|f_0|^2} f_2,$$

which is not possible since $f_1, f_2 \ne 0$. Write $f_1 \times f_2 = a_1 f_1 + a_2 f_2 + a_3 f_3$. Then (2.3) implies

$$\begin{aligned} 0 &= f_1 \times (f_1 \times f_2) = a_2 f_1 \times f_2 + a_3 f_1 \times f_3 \\ 0 &= f_2 \times (f_1 \times f_2) = -a_1 f_1 \times f_2 + a_3 f_2 \times f_3 \end{aligned}$$

Since $f_1 \times f_2 \ne 0$, $a_3 \ne 0$. Therefore, writing

$$H := f_1 \times f_2, \qquad d_{13} := -\frac{a_2}{a_3}, \qquad d_{23} := \frac{a_1}{a_3}$$

we have

$$f_1 \times f_3 = d_{13} H \ \text{ and } \ f_2 \times f_3 = d_{23} H. \tag{3.2}$$

Now use (2.7) to find

$$\frac{\partial d_{23}}{\partial \bar{z}} H + d_{23} \frac{\partial H}{\partial \bar{z}} = \frac{\partial (f_2 \times f_3)}{\partial \bar{z}} = -\frac{|f_2|^2}{|f_1|^2} f_1 \times f_3 = -\frac{|f_2|^2}{|f_1|^2} d_{13} H,$$

which implies

$$d_{23} \frac{\partial H}{\partial \bar{z}} = -\left(\frac{\partial d_{23}}{\partial \bar{z}} + \frac{|f_2|^2}{|f_1|^2} d_{13} \right) H \tag{3.3}$$

Similarly,

$$\frac{\partial d_{13}}{\partial \bar{z}} H + d_{13}\frac{\partial H}{\partial \bar{z}} = \frac{\partial (f_1 \times f_3)}{\partial \bar{z}} = -\frac{|f_1|^2}{|f_0|^2} f_0 \times f_3 - \frac{|f_3|^2}{|f_2|^2} H \tag{3.4}$$

and

$$\frac{\partial H}{\partial \bar{z}} = \frac{\partial (f_1 \times f_2)}{\partial \bar{z}} = -\frac{|f_1|^2}{|f_0|^2} f_0 \times f_2. \tag{3.5}$$

Therefore

$$d_{23} f_0 \times f_2 \equiv 0 \pmod{H} \quad \text{and} \quad d_{23} f_0 \times f_3 \equiv 0 \pmod{H},$$

so cross-multiplying these equations by f_0 and using (2.3) we obtain

$$d_{23} f_2 \equiv 0 \pmod{f_0 \times H} \quad \text{and} \quad d_{23} f_3 \equiv 0 \pmod{f_0 \times H},$$

which implies $d_{23} = 0$ since f_2 and f_3 are linearly independent, and then equation (3.3) implies $d_{13} = 0$. Therefore $a_1 = a_2 = 0$, and using (3.2), (3.4) and the fact that $|f_0| = 1$, we have

$$f_1 \times f_2 = a_3 f_3, \qquad f_1 \times f_3 = 0, \qquad f_2 \times f_3 = 0, \qquad f_0 \times f_3 = -\frac{|f_3|^2}{|f_1|^2|f_2|^2} a_3 f_3.$$

Next, use (2.3) to find

$$-f_3 = f_0 \times (f_0 \times f_3) = \left(\frac{|f_3|^2}{|f_1|^2|f_2|^2}\right)^2 a_3^2 f_3,$$

which implies $a_3 = h\frac{|f_1|^2|f_2|^2}{|f_3|^2}$, with $h = \pm i$. Therefore, so far we have

$$f_1 \times f_2 = h\frac{|f_1|^2|f_2|^2}{|f_3|^2} f_3, \qquad f_1 \times f_3 = 0, \qquad f_2 \times f_3 = 0, \qquad f_0 \times f_3 = -h f_3,$$

and using (2.8),

$$f_{-1} \times f_{-2} = -h f_{-3}, \qquad f_{-1} \times f_{-3} = 0, \qquad f_{-2} \times f_{-3} = 0, \qquad f_0 \times f_{-3} = h f_{-3}.$$

Now, (2.2) implies

$$-2f_0 = f_{-3} \times (f_0 \times f_3) + (f_{-3} \times f_0) \times f_3 = -2h f_{-3} \times f_3,$$

and therefore $f_{-3} \times f_3 = -h f_0$. Differentiate and use (2.7) to obtain

$$h\frac{|f_0|^2}{|f_{-1}|^2} f_{-1} = -h\frac{\partial f_0}{\partial \bar{z}} = \frac{\partial (f_{-3} \times f_3)}{\partial \bar{z}} = -\frac{|f_3|^2}{|f_2|^2} f_{-3} \times f_2,$$

then use (2.9) to find $f_{-3} \times f_2 = -h\frac{|f_1|^2|f_2|^2}{|f_3|^2} f_{-1}$, and use (2.8) to find $f_3 \times f_{-2} = h f_1$. Also,

$$0 = f_{-2} \times (f_3 \times f_{-2}) = h f_{-2} \times f_1,$$

and therefore $f_{-2} \times f_1 = f_2 \times f_{-1} = 0$.

Next, use (2.2) to obtain

$$2f_{-3} = f_{-2} \times (f_{-3} \times f_2) + (f_{-2} \times f_{-3}) \times f_2 = h\frac{|f_1|^2|f_2|^2}{|f_3|^2} f_{-1} \times f_{-2} = -h^2\frac{|f_1|^2|f_2|^2}{|f_3|^2} f_{-3}.$$

This implies

$$\frac{|f_1|^2|f_2|^2}{|f_3|^2} = 2, \tag{3.6}$$

and therefore $H := f_1 \times f_2 = 2h f_3$.

Finally, equation (3.5) gives $f_0 \times f_2 = hf_2$, and differentiating,

$$-\frac{|f_0|^2}{|f_{-1}|^2} f_{-1} \times f_2 - \frac{|f_2|^2}{|f_1|^2} f_0 \times f_1 = \frac{\partial (f_0 \times f_2)}{\partial \bar{z}} = -h \frac{|f_2|^2}{|f_1|^2} f_1,$$

which implies $f_0 \times f_1 = hf_1 = \pm i f_1$, as desired.

□

The following condition will be very useful when we compute examples in Section 5.

Corollary 3.2. *Let $\Xi : S^2 \to \mathbb{CP}^6$ be a linearly full holomorphic map and let $\xi(z)$ be a local holomorphic representation of Ξ. Then Ξ is the directrix curve of an almost complex map $f : S^2 \to S^6$ if and only if $\xi \times \xi' = 0$.*

Proof. Suppose that Ξ is the directrix curve of an almost complex map $f : S^2 \to S^6$, and let $\{f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, f_3\}$ be its harmonic sequence. If ξ is any local representation of Ξ, then ξ is a multiple of f_{-3}, and ξ' is a linear combination of f_{-3} and f_{-2}. Therefore $\xi \times \xi'$ is a multiple of $f_{-3} \times f_{-2}$, which was computed to be 0 in the proof of Proposition 3.1.

Conversely, suppose that $\xi \times \xi' = 0$ for a given local holomorphic representation ξ. Since Ξ is holomorphic, the same will be true for *any* local holomorphic representation, so we can assume that $\xi \times \xi' = 0$ for *every* local holomorphic representation ξ. Differentiating we obtain $\xi \times \xi'' = 0$. Then we can use equation (2.3) to obtain

$$\begin{aligned}
0 &= \xi \times (\xi \times \xi') = (\xi, \xi')\xi - (\xi, \xi)\xi' \\
0 &= \xi'' \times (\xi \times \xi'') = -(\xi, \xi'')\xi'' + (\xi'', \xi'')\xi \\
0 &= \xi' \times (\xi \times \xi') = -(\xi, \xi')\xi' + (\xi', \xi')\xi.
\end{aligned}$$

Since ξ is linearly full, ξ, ξ' and ξ'' are linearly independent except at a few points, and therefore $(\xi^{(i)}, \xi^{(i)}) = 0$, for $i = 0, 1, 2$. Differentiating these expressions we find

$$(\xi^{(i)}, \xi^{(j)}) = 0 \quad \text{for } 0 \le i < j \le 3 \tag{3.7}$$

which implies that Ξ is totally isotropic. Therefore Ξ is the directrix curve of some harmonic map $f : S^2 \to S^6$ [1]. Note that, by the definition and properties of the directrix curve,

$$\operatorname{span}_{\mathbb{C}} \{\xi, \xi', \xi''\} = \operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial f}{\partial \bar{z}}, \frac{\partial^2 f}{\partial \bar{z}^2}, \frac{\partial^3 f}{\partial \bar{z}^3} \right\} = \overline{\operatorname{span}_{\mathbb{C}} \left\{ \frac{\partial f}{\partial z}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^3 f}{\partial z^3} \right\}}$$

so in view of Proposition 3.1 it suffices to show that $\operatorname{span}_{\mathbb{C}} \{\xi, \xi', \xi''\}$ is closed under $\times$. Since $\xi \times \xi' = \xi \times \xi'' = 0$, it only remains to show that $\xi' \times \xi'' \in \operatorname{span}_{\mathbb{C}} \{\xi, \xi', \xi''\}$.

To this end, note that equations (2.4), (2.3) and (3.7) imply

$$(\xi, \xi' \times \xi'') = (\xi', \xi' \times \xi'') = (\xi'', \xi' \times \xi'') = 0$$

and

$$(\xi' \times \xi'', \xi' \times \xi'') = -(\xi'', \xi' \times (\xi' \times \xi'')) = 0$$

which implies that $\operatorname{span}_{\mathbb{C}} \{\xi, \xi', \xi'', \xi' \times \xi''\}$ is a totally isotropic subspace of $\mathbb{C}^7$, and as such it must have dimension at most 3. Since $\operatorname{span}_{\mathbb{C}} \{\xi, \xi', \xi''\}$ has dimension 3, it follows that $\xi' \times \xi'' \in \operatorname{span}_{\mathbb{C}} \{\xi, \xi', \xi''\}$.

□

Most of the cross products of elements of the harmonic sequence were computed in Proposition 3.1. For future use, we find the remaining ones in the following lemma.

Lemma 3.3. *Let $\{f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, f_3\}$ be the harmonic sequence of $f \in \mathrm{Ac}_d^f(S^2, S^6)$. Then the table of cross products $(f_i \times f_j)_{ij}$ is given by*

$\times$	f_{-3}	f_{-2}	f_{-1}	f_0	f_1	f_2	f_3
f_{-3}	0	0	0	$-if_{-3}$	$-2if_{-2}$	$-2if_{-1}$	$-if_0$
f_{-2}	0	0	if_{-3}	if_{-2}	0	$-if_0$	$-if_1$
f_{-1}	0	$-if_{-3}$	0	if_{-1}	if_0	0	$-if_2$
f_0	if_{-3}	$-if_{-2}$	$-if_{-1}$	0	if_1	if_2	$-if_3$
f_1	$2if_{-2}$	0	$-if_0$	$-if_1$	0	$2if_3$	0
f_2	$2if_{-1}$	if_0	0	$-if_2$	$-2f_3$	0	0
f_3	if_0	if_1	if_2	if_3	0	0	0

Proof. Many of the cross products were found in the proof of Proposition 3.1. We find the remaining ones here. Use formulas (2.7) and (2.9) to find

$$-\frac{|f_2|^2}{|f_1|^2}\, f_{-3} \times f_1 = \frac{\partial(f_{-3} \times f_2)}{\partial \bar{z}} = -2i\frac{\partial f_{-1}}{\partial \bar{z}} = 2i\frac{|f_{-1}|^2}{|f_{-2}|^2} f_{-2},$$

which gives $f_{-3} \times f_1 = -2if_{-2}$. Then use (2.8) to find

$$|f_{-3}|^2 |f_1|^2 f_3 \times f_{-1} = \overline{f_{-1} \times f_3} = 2i\bar{f}_{-2} = 2i|f_{-2}| f_2.$$

Using (2.9) and (3.6) we obtain $f_{-1} \times f_3 = -if_2$, $f_{-2} \times f_0 = if_{-2}$, $f_{-1} \times f_0 = if_{-1}$.

To find $f_{-1} \times f_1$, differentiate $f_0 \times f_1 = if_1$ and use (2.7) to obtain

$$-\frac{|f_0|^2}{|f_{-1}|^2} f_{-1} \times f_1 = \frac{\partial(f_0 \times f_1)}{\partial \bar{z}} = i\frac{\partial f_1}{\partial \bar{z}} = -i\frac{|f_1|^2}{|f_0|^2} f_0$$

which, using (2.9), gives $f_{-1} \times f_1 = if_0$. Finally, differentiate $f_{-2} \times f_1 = 0$ and use (2.9) to find $f_{-2} \times f_2 = -if_0$.

□

3.2. **Congruence.** The motivation is the following: if the directrix curves of two linearly full harmonic maps from S^2 to S^6 are $SO(7, \mathbb{C})$-congruent, then they certainly have the same singularity type. Moreover, the set of directrix curves of linearly full harmonic maps from S^2 to S^6 with a given singularity type is a finite union of $SO(7, \mathbb{C})$ orbits [4]. Is this also true when we substitute 'harmonic' with 'almost complex' and '$SO(7, \mathbb{C})$' with '$G_2(\mathbb{C})$'?

This fact will be implied by the following: does $SO(7, \mathbb{C})$-congruence of twistor lifts of almost complex maps imply $G_2(\mathbb{C})$-congruence? Intuitively it seems that this should be true. On the one hand its real counterpart is clearly true in view of Proposition 3.1. On the other hand, if $g \in SO(7, \mathbb{C})$ and ψ is the twistor lift of an almost complex map with directrix curve expressed locally by $[\xi]$, then $[g\xi]$ is the directrix of the almost complex map whose twistor lift is $g\psi$, and it is easy to see that, if ξ is suitably normalized,

$$\xi' \times \xi'' = i\xi \quad \text{and} \quad (g\xi') \times (g\xi'') = i(g\xi) = g(\xi' \times \xi'').$$

This implies that g preserves all the cross products of the form $\xi'(p) \times \xi''(p)$, $p \in S^2$, which should include all possible cross products within a basis of $\mathbb{C}^7$.

This heuristic idea is not easy to translate rigorously. Instead, we prove this fact by calculating the cross products of the derivatives, up to order 6, of the directrix curve. The proof is, surprisingly, an easy but lengthy computation using the properties of the cross product.

Lemma 3.4. *Let f_p, $-3 \leq p \leq 3$, be the harmonic sequence of a map $f \in \mathrm{Ac}_d^f(S^2, S^6)$, and let $\xi \equiv f_{-3}$. Then the table of the bilinear products $((\xi^{(i)}, \xi^{(j)}))_{ij}$ of the derivatives of ξ has the form*

(3.8)

$(\, , \,)$	ξ	ξ'	ξ''	ξ'''	$\xi^{(4)}$	$\xi^{(5)}$	$\xi^{(6)}$
ξ	0	0	0	0	0	0	-1
ξ'	0	0	0	0	0	1	0
ξ''	0	0	0	0	-1	0	$2b_{44}$
ξ'''	0	0	0	1	0	$-2b_{44}$	$-3b'_{44}$
$\xi^{(4)}$	0	0	-1	0	$2b_{44}$	b'_{44}	$2b_{46}$
$\xi^{(5)}$	0	1	0	$-2b_{44}$	b'_{44}	$2b_{55}$	b'_{55}
$\xi^{(6)}$	-1	0	$2b_{44}$	$-3b'_{44}$	$2b_{46}$	b'_{55}	$2b_{66}$

where $b_{ij} = (\xi^{(i)}, \xi^{(j)})/2$, and the table of cross products $(\xi^{(i)} \times \xi^{(j)})_{ij}$ has the form

$\times$	ξ	ξ'	ξ''	ξ'''	$\xi^{(4)}$	$\xi^{(5)}$	$\xi^{(6)}$
ξ	0	0	0	$-i\xi$	$-2i\xi'$	C^{05}	C^{06}
ξ'	0	0	$i\xi$	$i\xi'$	$-ib_{44}\xi$	C^{15}	C^{16}
ξ''	0	$-i\xi$	0	C^{23}	C^{24}	C^{25}	C^{26}
ξ'''	$i\xi$	$-i\xi'$	$-C^{23}$	0	C^{34}	C^{35}	C^{36}
$\xi^{(4)}$	$2i\xi'$	$ib_{44}\xi$	$-C^{24}$	$-C^{34}$	0	C^{45}	C^{46}
$\xi^{(5)}$	$-C^{05}$	$-C^{15}$	$-C^{25}$	$-C^{35}$	$-C^{45}$	0	C^{56}
$\xi^{(6)}$	$-C^{06}$	$-C^{16}$	$-C^{26}$	$-C^{36}$	$-C^{46}$	$-C^{56}$	0

where $C^{ij} = \sum_{k=0}^{6} L_k^{ij} \xi^{(k)}$, and the complex functions L_k^{ij} depend only *on the products b_{ij} and their derivatives. (Although we explain below how to find the C^{ij}, we omit their explicit formulas as they are not relevant for the remainder of the paper.)*

Proof. First notice that if $\{f_{-3}, f_{-2}, f_{-1}, f_0, f_1, f_2, f_3\}$ is the harmonic sequence of the almost complex map f, then

$$\xi^{(i)} \equiv f_{i-3} \mod (f_{-3}, f_{-2}, \ldots, f_{i-3}). \tag{3.9}$$

Using (2.10), this implies that $(\xi^{(i)}, \xi^{(j)}) = 0$ if $i + j < 6$ and $(\xi^{(i)}, \xi^{(j)}) = (-1)^{i+1}$ if $i + j = 6$. Then notice that $0 = (\xi^{(3)}, \xi^{(3)})' = 2(\xi^{(3)}, \xi^{(4)})$, and use the formula $(\xi^{(i)}, \xi^{(j)})' = (\xi^{(i+1)}, \xi^{(j)}) + (\xi^{(i)}, \xi^{(j+1)})$ to find the remaining bilinear products.

To find the cross product table, notice that if $\xi^{(i)} \times \xi^{(i+1)}$ is known for $0 \leq i \leq k$, then $\xi^{(i)} \times \xi^{(j)}$ can be easily found, for $0 \leq i + j \leq 2k + 2$, using the formula

$$(\xi^{(i)} \times \xi^{(j)})' = \xi^{(i+1)} \times \xi^{(j)} + \xi^{(i)} \times \xi^{(j+1)}.$$

Hence it suffices to find $\xi^{(i)} \times \xi^{(i+1)}$ for $0 \leq i \leq 5$.

Lemma 3.3 and equation (3.9) imply that $\xi \times \xi' = \xi \times \xi'' = 0$, $\xi' \times \xi'' = i\xi$, $\xi' \times \xi''' = i\xi'$, $\xi \times \xi''' = -i\xi$, and $L_k^{23} = 0$ for $k \geq 3$. Therefore,

$$\xi'' \times \xi''' = L_0^{23}\xi + L_1^{23}\xi' + L_2^{23}\xi''.$$

Use (2.3), (2.2), and (3.8) to find

$$0 = \xi'' \times (\xi'' \times \xi''') = -L_1^{23}\xi' \times \xi'' = -iL_1^{23}\xi$$
$$0 = \xi' \times (\xi'' \times \xi''') + (\xi' \times \xi'') \times \xi''' = L_2^{23}\xi' \times \xi'' + i\xi \times \xi''' = iL_2^{23}\xi + \xi$$

which gives $L_1^{23} = 0$, $L_2^{23} = i$. On the other hand,

$$i\xi'' = (\xi' \times \xi''')' = \xi'' \times \xi''' + \xi' \times \xi^{(4)},$$

and therefore $\xi' \times \xi^{(4)} = -L_0^{23}\xi$. Hence, using (2.2) and (3.8) again, we obtain

$$2b_{44}\xi' = \xi^{(4)} \times (\xi' \times \xi^{(4)}) = L_0^{23}\xi \times \xi^{(4)} = -2iL_0^{23}\xi'$$

since $\xi \times \xi^{(4)} = (\xi \times \xi''')' - \xi' \times \xi''' = -2i\xi'$; this gives $L_0^{23} = ib_{44}$, and therefore

$$\xi'' \times \xi''' = ib_{44}\xi + i\xi''$$

which, differentiating and using (2.14) gives $\xi'' \times \xi^{(4)} = ib'_{44}\xi + ib_{44}\xi' + i\xi'''$ and $\xi' \times \xi^{(4)} = -ib_{44}\xi$. To find $\xi''' \times \xi^{(4)} = \sum_{k=0}^{4} L_k^{34}\xi^{(k)}$, proceed similarly to obtain

$$\begin{aligned}
0 &= \xi \times (\xi''' \times \xi^{(4)}) + (\xi \times \xi''') \times \xi^{(4)} = -iL_3^{34}\xi - 2iL_4^{34}\xi' - 2\xi', \\
0 &= \xi' \times (\xi''' \times \xi^{(4)}) + (\xi' \times \xi''') \times \xi^{(4)} = iL_2^{34}\xi + iL_3^{34}\xi' - ib_{44}L_4^{34}\xi + b_{44}\xi \\
-\xi^{(4)} &= \xi''' \times (\xi''' \times \xi^{(4)}) = iL_0^{34}\xi - iL_1^{34}\xi' - L_2^{34}(ib_{44}\xi + i\xi'') \\
&\qquad + L_4^{34}(L_0^{34}\xi + L_1^{34}\xi' + L_2^{34}\xi'' + L_3^{34}\xi''' + L_4^{34}\xi^{(4)}) \\
2b_{44}\xi''' &= \xi^{(4)} \times (\xi''' \times \xi^{(4)}) = 2iL_0^{34}\xi' + ib_{44}L_1^{34}\xi' - L_2^{34}(ib'_{44}\xi + ib_{44}\xi' + i\xi''') \\
&\qquad - L_3^{34}(L_0^{34}\xi + L_1^{34}\xi' + L_2^{34}\xi'' + L_3^{34}\xi''' + L_4^{34}\xi^{(4)})
\end{aligned}$$

which leads to

$$\xi''' \times \xi^{(4)} = ib_{44}^2\xi + 2ib'_{44}\xi' + 2ib_{44}\xi'' + i\xi^{(4)}.$$

To find $\xi^{(4)} \times \xi^{(5)} = \sum_{k=0}^{6} L_k^{45}\xi^{(k)}$, it is easier to use (2.4) as follows:

$$-L_6^{45} = (\xi, \xi^{(4)} \times \xi^{(5)}) = (\xi^{(5)}, \xi \times \xi^{(4)}) = -2i, \quad \text{so } L_6^{45} = 2i.$$
$$L_5^{45} = (\xi', \xi^{(4)} \times \xi^{(5)}) = (\xi^{(5)}, \xi' \times \xi^{(4)}) = 0, \quad \text{so } L_5^{45} = 0.$$
$$-L_4^{45} + 4b_{44}i = (\xi'', \xi^{(4)} \times \xi^{(5)}) = (\xi^{(5)}, \xi'' \times \xi^{(4)}) = ib_{44} - 2ib_{44}, \quad \text{so } L_4^{45} = 5ib_{44}.$$
$$L_3^{45} - 6b'_{44}i = (\xi''', \xi^{(4)} \times \xi^{(5)}) = (\xi^{(5)}, \xi''' \times \xi^{(4)}) = 2ib'_{44} + ib'_{44}, \quad \text{so } L_3^{45} = 9ib'_{44}.$$
$$-L_2^{45} + 10ib_{44}^2 + 4ib_{46} = (\xi^{(4)}, \xi^{(4)} \times \xi^{(5)}) = 0, \quad \text{so } L_2^{45} = 2i(5b_{44}^2 + 2b_{46}).$$
$$L_1^{45} - 13ib_{44}b'_{44} + 2ib'_{55}4 = (\xi^{(5)}, \xi^{(4)} \times \xi^{(5)}) = 0, \quad \text{so } L_1^{45} = i(13b_{44}b'_{44} - 2b'_{55}).$$

Finding L_0^{45} is trickier: first find $\xi''' \times \xi^{(6)} = (\xi''' \times \xi^{(5)})' - \xi^{(4)} \times \xi^{(5)}$ and then calculate $\xi^{(6)} \times (\xi''' \times \xi^{(6)})$ as above. The result is

$$\begin{aligned}
\xi^{(4)} \times \xi^{(5)} = {} & i(-3b_{44}^3 + 2b_{66} - 7b'^2_{44} + 11b_{44}b''_{44})\xi + i(7b_{44}b'_{44} + 2b'''_{44})\xi' \\
& + 2i(2b_{44}^2 + 3b''_{44})\xi'' + 9ib'_{44}\xi''' + 5ib_{44}\xi^{(4)} + 2i\xi^{(6)}.
\end{aligned}$$

Finally, finding $\xi^{(5)} \times \xi^{(6)}$ is easy: compute $\xi^{(5)} \times (\xi^{(4)} \times \xi^{(5)})$ and solve for $\xi^{(5)} \times \xi^{(6)}$. The long result is

$$\begin{aligned}
\xi^{(5)} \times \xi^{(6)} = {} & -i(11b_{44}^4 - 6b_{44}b_{66} + 19b_{44}b'^2_{44} - 32b_{44}^2b''_{44} - 3b''^2_{44} + 2b'_{44}b'''_{44})\xi \\
& + i(15b_{44}^2b'_{44} + 9b'_{44}b''_{44} + 3b_{44}b'''_{44})\xi' + i(11b_{44}^3 - 2b_{66} + 25b'^2_{44} + b_{44}b''_{44})\xi'' \\
& + i(28b_{44}b'_{44} - b'''_{44})\xi''' + i(11b_{44}^2 + b''_{44})\xi^{(4)} + 5ib'_{44}\xi^{(5)} + 5ib_{44}\xi^{(6)}.
\end{aligned}$$

□

This result—namely that the cross products of the derivatives of the directrix curve are completely determined by their bilinear products—has the following immediate consequence.

Proposition 3.5. *If* $\psi, \chi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$ *and* $\chi = g\psi$*, where* $g \in SO(7, \mathbb{C})$*, then* $g \in G_2(\mathbb{C})$*.*

Proof. Let $\{f_i\}_{i=-3}^3$ and $\{g_i\}_{i=-3}^3$ be the harmonic sequences of $\pi \circ \psi$ and $\pi \circ \chi$, respectively (see Section 2), and let $\xi := f_{-3}$ and $\zeta := g_{-3}$. Then $\zeta = g\xi$. Write $\xi^{(i)} \times \xi^{(j)} = \sum_{k=0}^6 L_k^{ij} \xi^{(k)}$ and $\zeta^{(i)} \times \zeta^{(j)} = \sum_{k=0}^6 M_k^{ij} \zeta^{(k)}$. Then Lemma 3.4 implies that the L_k^{ij} and the M_k^{ij} depend *only* on the products $(\xi^{(i)}, \xi^{(j)})$ and $(\zeta^{(i)}, \zeta^{(j)})$. Since $\zeta^{(i)} = g\xi^{(i)}$, $i \geq 0$, and g is in $SO(7, \mathbb{C})$, $(\xi^{(i)}, \xi^{(j)}) = (\zeta^{(i)}, \zeta^{(j)})$ for all $i, j \geq 0$, and therefore $L_k^{ij} = M_k^{ij}$ for $0 \leq i, j, k \leq 6$. Hence

$$g\xi^{(i)} \times g\xi^{(j)} = \zeta^{(i)} \times \zeta^{(j)} = \sum_{k=0}^{6} M_k^{ij} \zeta^{(k)} = \sum_{k=0}^{6} L_k^{ij} g\xi^{(k)} = g(\xi^{(i)} \times \xi^{(j)}).$$

Since ξ is linearly full, this implies that g preserves all the pairwise cross products of a basis, and therefore is in $G_2(\mathbb{C})$.

□

Corollary 3.6. *If nonempty,* $\mathrm{HH}_6^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$ *is isomorphic to* $G_2(\mathbb{C})$*.*

Proof. Since $G_2(\mathbb{C})$ preserves cross products, Proposition 3.1 implies that $G_2(\mathbb{C})$ acts on $\mathrm{HH}_6^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$. This action is free since $\mathrm{HH}_6^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$ consists of linearly full maps. On the other hand, any two elements of $\mathrm{HH}_6^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$ are $SO(7, \mathbb{C})$-congruent [1], and therefore $G_2(\mathbb{C})$-congruent by Proposition 3.5. Hence $G_2(\mathbb{C})$ acts simply transitively on $\mathrm{HH}_6^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$, and therefore these spaces are isomorphic.

□

4. Dimension

From Section 2 we know that

$$\mathrm{Ac}_d(S^2, S^6) = \mathrm{Ac}_d^{(2)}(S^2, S^6) \sqcup \mathrm{Ac}_d^f(S^2, S^6) \quad \text{(disjoint union)}.$$

Using the tools from the previous sections, we will now find the dimension of each one of the components.

4.1. Linearly full maps.

Recall [16] that $\mathrm{Harm}_d^f(S^2, S^6)$ has two disconnected components, denoted $\mathrm{Harm}_d^{f,+}(S^2, S^6)$ and $\mathrm{Harm}_d^{f,-}(S^2, S^6)$. Since the varieties $\mathrm{Harm}_d^{f,+}(S^2, S^6)$ and $\mathrm{HH}_d^f(S^2, \mathcal{Z}_3)$ are isomorphic as sets [1], we transfer the algebraic structure of $\mathrm{HH}_d^f(S^2, \mathcal{Z}_3)$ to $\mathrm{Harm}_d^{f,+}(S^2, S^6)$ making these two sets algebraically isomorphic. Similarly, since $\mathrm{Ac}_d^f(S^2, S^6) \subset \mathrm{Harm}_d^{f,+}(S^2, S^6)$, we assume throughout that $\mathrm{Ac}_d^f(S^2, S^6)$ is algebraically isomorphic to $\mathrm{HH}_d^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$ by transferring the algebraic structure of the latter to the former via the isomorphism above. Therefore, to find the dimension of $\mathrm{Ac}_d^f(S^2, S^6)$ we only need to find the dimension of $\mathrm{HH}_d^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$. It is very easy to get a lower bound, as follows.

Lemma 4.1. *If nonempty, the dimension of* $\mathrm{Ac}_d^f(S^2, S^6)$ *is at least* $d + 8$*.*

Proof. Use (2.21): The dimension of the variety $\mathrm{PD}_d^f(S^2,\mathbb{CP}^6)$ is $2d+9$ [13, 14]. Since α_1 and τ_{23} are polynomials of degree at most d, the condition $i\sqrt{2}\alpha_1=\tau_{23}$ imposes $d+1$ additional (not necessarily independent) equations Therefore, the left hand side of (2.21), which is an open subset of $\mathrm{HH}_d^f(S^2,\mathcal{Z}_3)_{\mathrm{Ac}}$, has dimension greater than or equal to $d+8$. Hence, if $\mathrm{HH}_d^f(S^2,\mathcal{Z}_3)_{\mathrm{Ac}}$ is not empty, its dimension must be at least $d+8$.

□

To find an upper bound we use the following idea, which appears at the end of [4]. Every harmonic map from S^2 to S^6 is determined, modulo $SO(7,\mathbb{C})$ and a finite number of choices, by its singularity type [2]. Thus, up to the action of $SO(7,\mathbb{C})$ and a finite group, every element of $\mathrm{Harm}_d^{f,+}(S^2,S^6)$ is determined by $r_0+r_1+r_2$ complex numbers, where r_0,r_1,r_2 satisfy $d-12=r_0+2r_1+3r_3$ (see equations (2.12) and (2.22)). The maximum of $r_0+r_1+r_2$ is then achieved when $r_1=r_2=0$, $r_0=2d-12$. Since the dimension of $SO(7,\mathbb{C})$ is 21, the dimension of $\mathrm{Harm}_d^{f,+}(S^2,S^6)$ should therefore be $2d-12+21=2d+9$, which is correct.

The same idea was suggested by Bolton for the almost complex case: if Ξ is the directrix curve of $f\in \mathrm{Ac}_d^f(S^2,S^6)$ then, using (2.22), equation (2.17) reads

$$d-6=2r_0+r_1, \tag{4.1}$$

where r_0 and r_1 are the total ramification degrees of Ξ and the first associated curve of Ξ, respectively. Hence, assuming that every element of $\mathrm{Ac}_d^f(S^2,S^6)$ is determined, modulo $G_2(\mathbb{C})$ and a finite number of choices, by its singularity type, then we have r_0+r_1 complex parameters, where r_0,r_1 satisfy (4.1). The maximum of r_0+r_1 is then attained when $r_0=0$, $r_1=d-6$. Since the dimension of $G_2(\mathbb{C})$ is 14, the dimension of $\mathrm{Ac}_d^f(S^2,S^6)$ should be $d-6+14=d+8$. We will now make this idea more rigorous.

If $\psi\in \mathrm{HH}_d^f(S^2,\mathcal{Z}_3)_{\mathrm{Ac}}$, let $\Xi^\psi: S^2\to\mathbb{CP}^6$ denote the directrix curve of $f=\pi\circ\psi$. Note that Ξ^ψ is the only curve such that $\xi,\xi',\xi''\in\psi$, where ξ is a local representation of Ξ^ψ. This implies that the map that takes ψ to Ξ^ψ is algebraic; it is in fact an isomorphism, but we do not need it here.

Let

$$\begin{aligned}\Sigma_{d_0} &= \{(z_{01},\ldots,z_{0d_0})\in(S^2)^{d_0}: z_{0j}\neq z_{0k}, 1\le j<k\le d_0\}\\ \Sigma_{d_1} &= \{(z_{11},\ldots,z_{1d_1})\in(S^2)^{d_1}: z_{1j}\neq z_{1k}, 1\le j<k\le d_1\}\end{aligned}$$

and let $\Sigma_{d_0,d_1}=\Sigma_{d_0}\times\Sigma_{d_1}$. Let $\boldsymbol{m}_0=(m_{01},\ldots,m_{0d_0})$ and $\boldsymbol{m}_1=(m_{11},\ldots,m_{1d_1})$, where the m_{ij} are positive integers satisfying

$$2(m_{01}+\cdots+m_{0d_0})+m_{11}+\cdots+m_{1d_1}=d-6. \tag{4.2}$$

Consider the subsets of $\Sigma_{d_0,d_1}\times \mathrm{HH}_d^f(S^2,\mathcal{Z}_3)_{\mathrm{Ac}}$ given by

$$\mathcal{H}_{\boldsymbol{m}_0,\boldsymbol{m}_1}=\left\{(z_{01},\ldots,z_{0d_0},z_{11},\ldots,z_{1d_1},\psi)\in\Sigma_{d_0,d_1}\times \mathrm{HH}_d^f(S^2,\mathcal{Z}_3)_{\mathrm{Ac}}:\right.$$
$$\left.\left(\boldsymbol{\sigma}_0^{\boldsymbol{\psi}}\wedge\frac{\partial\boldsymbol{\sigma}_0^{\boldsymbol{\psi}}}{\partial z}\right)_0=\sum_{j=1}^{d_0}m_{0j}z_{0j},\ \left(\boldsymbol{\sigma}_1^{\boldsymbol{\psi}}\wedge\frac{\partial\boldsymbol{\sigma}_1^{\boldsymbol{\psi}}}{\partial z}\right)_0=\sum_{k=1}^{d_1}m_{1k}z_{1k}\right\},$$

for any local representations $\boldsymbol{\sigma}_0^{\boldsymbol{\psi}}$ and $\boldsymbol{\sigma}_1^{\boldsymbol{\psi}}$ of the zeroth and first associated curves σ_0^ψ and σ_1^ψ of Ξ^ψ, respectively, and where the parenthesis $(\ )_0$ denotes the divisor of zeros, and z is any holomorphic coordinate in S^2. Since the maps $\psi \to \Xi^\psi$ and $\Xi^\psi \to \sigma_j^\psi$ are both algebraic, $\mathcal{H}_{\boldsymbol{m}_0,\boldsymbol{m}_1}$ is an algebraic subvariety of $\Sigma_{d_0,d_1} \times \mathrm{HH}_d^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$.

If π_1 and π_2 denote the projections over the 1st and 2nd factors of $\Sigma_{d_0,d_1} \times \mathrm{HH}_d^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$, note that $\pi_2(\mathcal{H}_{\boldsymbol{m}_0,\boldsymbol{m}_1})$ is the variety (actually, it is just a constructible set) of maps $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$ such that the zeroth associated curve of Ξ^ψ has d_0 singularities of orders $m_{01}, \ldots, m_{0d_0}$, and the first associated curve of Ξ^ψ has d_1 singularities of orders $m_{11}, \ldots, m_{0d_1}$. Therefore

$$\mathrm{HH}_d^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}} = \bigcup_{\boldsymbol{m}_0,\boldsymbol{m}_1} \pi_2(\mathcal{H}_{\boldsymbol{m}_0,\boldsymbol{m}_1})$$

where $\boldsymbol{m}_0, \boldsymbol{m}_1$ satisfy (4.2), so the union is finite. Hence the dimension of the variety $\mathrm{HH}_d^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$ is the maximum of the dimensions of the $\pi_2(\mathcal{H}_{\boldsymbol{m}_0,\boldsymbol{m}_1})$.

Theorem 4.2. *When nonempty, the (pure) dimension of* $\mathrm{Ac}_d^f(S^2, S^6)$ *is* $d + 8$.

Proof. In view of Lemma 4.1 and the paragraph before it, we only need to prove that the dimension of $\mathrm{HH}_d^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$ is at most $d + 8$. First we find the dimension of $\mathcal{H}_{\boldsymbol{m}_0,\boldsymbol{m}_1}$. The dimension of $\pi_1(\mathcal{H}_{\boldsymbol{m}_0,\boldsymbol{m}_1}) \subset \Sigma_{d_0,d_1}$ is at most $d_0 + d_1$. Each fiber of π_1 is isomorphic to the set of maps $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_3)_{\mathrm{Ac}}$ such that Ξ^ψ has a given singularity type. Since the set of maps $\psi \in \mathrm{HH}_d^f(S^2, \mathcal{Z}_3)$ with a given singularity type is a finite union of $SO(7, \mathbb{C})$ orbits [2], Lemma 3.5 implies that the fiber of π_1 is a finite union of $G_2(\mathbb{C})$-orbits, and therefore has dimension 14. Hence the dimension of $\mathcal{H}_{\boldsymbol{m}_0,\boldsymbol{m}_1}$ is at most $d_0 + d_1 + 14$. On the other hand, the fibre of π_2 consists of all the permutations of the z_{0i} and z_{1i}, so it is finite, and therefore the dimension of $\pi_2(\mathcal{H}_{\boldsymbol{m}_0,\boldsymbol{m}_1})$ is at most $d_0 + d_1 + 14$.

The maximum of d_0+d_1+14 subject to the condition (4.2) happens when $d_0 = 0$ and all the m_{1j}, $1 \le j \le d_1$, are 1. In this case, $d_1 = d - 6$, so $d_0 + d_1 + 14 = d + 8$, as desired.

□

4.2. **Non-linearly full maps.** As explained in Section 2, if $\tilde{f} \in \mathrm{Hol}_d(S^2, S^2)$ and $h : \mathbb{H} \to \mathbb{O}$ is a homomorphism of algebras, then $h \circ \tilde{f} \in \mathrm{Ac}_d^{(2)}(S^2, S^6)$, where $\mathrm{Hol}_d(S^2, S^2)$ denotes the variety of holomorphic maps of degree d from $S^2 \subset \mathrm{Im}(\mathbb{H})$ to itself. It is easy to see that all the elements of $\mathrm{Ac}_d^{(2)}(S^2, S^6)$ have this form: let $f \in \mathrm{Ac}_d^{(2)}(S^2, S^6)$, and let V_f be the smallest subspace of $\mathbb{R}^7$ containing the image of f. Then, if $z = x + iy$ is a holomorphic coordinate in S^2, using subscripts to denote derivatives, we have

$$V_f = \mathrm{span}_{\mathbb{R}} \{f, f_x, f_y\}.$$

Since

$$f \times f_x = f_y, \quad f \times f_y = -f_x, \text{ and } f_x \times f_y = f_x \times (f \times f_x) = (f_x, f_x)\, f - (f_x, f)\, f_x,$$

the subspace $V_f \subset \mathrm{Im}(\mathbb{O})$ is closed under $\times$ and therefore there is an isomorphism of algebras $h : \mathbb{H} \to \mathbb{R} \cdot \mathbf{1} \oplus V_f \subset \mathbb{O}$. Then $\tilde{f} = h^{-1} \circ f \in \mathrm{Hol}_d(S^2, S^2)$ and $f = h \circ \tilde{f}$. In particular, the set $\{V_f : f \in \mathrm{Ac}_d^{(2)}(S^2, S^6)\}$ is isomorphic to the space of subalgebras of $\mathbb{O}$ that are isomorphic to $\mathbb{H}$. This is the homogeneous space $G_2/SO(4)$, which has real dimension 8. Although we do not know whether the space $\mathrm{Ac}_d^{(2)}(S^2, S^6)$ is a complex variety, we will use complex dimension instead of real in order to have a more compact statement.

Theorem 4.3. *The dimension of* $\mathrm{Ac}_d^{(2)}(S^2, S^6)$ *is* $2d + 5$.

Proof. By the observations above, the map

$$\rho : \mathrm{Ac}_d^{(2)}(S^2, S^6) \to G_2/SO(4) \subset \mathrm{Gr}(3, \mathbb{R}^7)$$

defined by

$$\rho(f) = \text{3-dimensional subspace where } f(S^2) \text{ lies}$$

gives a fiber bundle with fiber $\mathrm{Hol}_d(S^2, S^2)$ (see also [14]). Therefore

$$\dim_{\mathbb{C}}(\mathrm{Ac}_d^{(2)}(S^2, S^6)) = \dim_{\mathbb{C}}(\mathrm{Hol}_d(S^2, S^2)) + \dim_{\mathbb{C}}(G_2/SO(4)) = 2d+1+4 = 2d+5$$

□

It is worth noting the following curious fact: as opposed to the harmonic map case (see [14]), the space of nonlinearly full almost complex maps has greater dimension than the space of linearly full almost complex maps.

5. Existence and examples

In this section we construct examples of linealy full almost complex maps from S^2 to S^6 of any degree $d \geq 6$, with $d \neq 7$. This is done by giving explicit formulas for their directrix curves as in [1]. There cannot be linearly full, almost complex maps of degree 7 because if $d = 7$, formula (4.1) gives $r_0 = 0$, $r_1 = 1$, so the map would be one-point ramified, which is impossible by [5].

Let $\{E_0, E_1, E_2, E_3, \bar{E}_1, \bar{E}_2, \bar{E}_3\}$ be the basis described in (2.20). For reference, we give the cross product table in this basis.

$\times$	E_0	E_1	E_2	E_3	$\bar{E}_1$	$\bar{E}_2$	$\bar{E}_3$
E_0	0	iE_1	$-iE_2$	$-iE_3$	$-i\bar{E}_1$	$i\bar{E}_2$	$i\bar{E}_3$
E_1	$-iE_1$	0	0	0	iE_0	$\sqrt{2}E_3$	$-\sqrt{2}E_2$
E_2	iE_2	0	0	$\sqrt{2}E_1$	$-\sqrt{2}\bar{E}_3$	$-iE_0$	0
E_3	iE_3	0	$-\sqrt{2}E_1$	0	$\sqrt{2}\bar{E}_2$	0	$-iE_0$
$\bar{E}_1$	$i\bar{E}_1$	$-iE_0$	$\sqrt{2}\bar{E}_3$	$-\sqrt{2}\bar{E}_2$	0	0	0
$\bar{E}_2$	$-i\bar{E}_2$	$-\sqrt{2}E_3$	iE_0	0	0	0	$\sqrt{2}\bar{E}_1$
$\bar{E}_3$	$-i\bar{E}_3$	$\sqrt{2}E_2$	0	iE_0	0	$-\sqrt{2}\bar{E}_1$	0

In [1], Barbosa finds examples of totally isotropic curves of the form

$$\xi = a_0 E_1 + a_{\ell-2} z^{\ell-2} E_2 + a_{\ell-1} z^{\ell-1} E_3 + a_\ell z^\ell E_0 + a_{\ell+1} z^{\ell+1} \bar{E}_3 + a_{\ell+2} z^{\ell+2} \bar{E}_2 + a_{2\ell} z^{2\ell} \bar{E}_1,$$

where z is a holomorphic coordinate in S^2. Note that all of these examples have higher singularities only at 0 and ∞, and at these points $r_0 = \ell - 3, r_1 = r_2 = 0$, so they cannot be almost complex because they do not satisfy (2.16). In fact, equation (4.1) says that the *generic* almost complex curve has $r_0 = 0$ at every point. This suggests that we try solutions of the form

$$\xi = a_0 E_1 + a_1 z E_2 + a_{\ell-1} z^{\ell-1} E_3 + a_\ell z^\ell E_0 + a_{\ell+1} z^{\ell+1} \bar{E}_3 + a_{2\ell-1} z^{2\ell-1} \bar{E}_2 + a_{2\ell} z^{2\ell} \bar{E}_1.$$

This in fact works and gives solutions for even $d = 2\ell$. By Corollary 3.2, it suffices to solve the equation $\xi \times \xi' = 0$, which gives an underdetermined system of 7 equations. One can actually take $a_{\ell+1} = a_{2\ell-1} = a_{2\ell} = 1$ and solve for the other a_i to obtain

$$a_0 = \frac{(\ell-2)^2(\ell-1)}{\ell^2(2\ell-1)(\ell+1)}, \; a_1 = -\frac{\ell-2}{\ell(2\ell-1)}, \; a_{\ell-1} = \frac{(\ell-2)(\ell-1)}{\ell(\ell+1)}, \; a_\ell = \frac{i\sqrt{2}(\ell-2)}{\ell}.$$

Note that these examples have higher singularities only at 0 and ∞, and $r_0 = 0$, $r_1 = \ell - 3$ at these points.

Finding examples for odd $d = 2\ell + 1$ is trickier. The idea is to keep the singularities at 0 and ∞ and *create* a single one at another point. This is achieved by trying solutions of the form

$$\begin{aligned}\xi(z) = (b_0 + c_0 z)\, E_1 + (b_1 + c_1 z) z\, E_2 + (b_{\ell-1} + c_{\ell-1} z) z^{\ell-1}\, E_3 + (b_\ell + c_\ell z) z^\ell E_0 \\ +(b_{\ell+1} + c_{\ell+1} z) z^{\ell+1} \bar{E}_3 + (b_{2\ell-1} + c_{2\ell-1} z) z^{2\ell-1} \bar{E}_2 + (b_{2\ell} + c_{2\ell} z) z^{2\ell} \bar{E}_1.\end{aligned}$$

Again, the equation $\xi \times \xi' = 0$ leads to an underdetermined system of equations in the b_i, c_i. One can take $b_0 = b_1 = c_1 = 1$ and solve for the other variables to obtain

$$c_0 = \frac{(\ell-3)(\ell+2)}{(\ell-1)\ell}, \qquad b_{\ell-1} = \frac{\ell(\ell+1)}{\ell-2}, \qquad c_{\ell-1} = \frac{(\ell-3)^2(\ell+1)(\ell+2)}{(\ell-2)(\ell-1)\ell},$$

$$b_\ell = -i\sqrt{2}(\ell+1), \qquad c_\ell = -\frac{i\sqrt{2}(\ell-3)(\ell+1)}{\ell}, \qquad b_{\ell+1} = (\ell-1),$$

$$c_{\ell+1} = \frac{(\ell-1)^2}{\ell+2}, \qquad b_{2\ell-1} = -\frac{\ell(\ell+1)^2}{(\ell-2)(2\ell-1)}, \qquad c_{2\ell-1} = -\frac{(\ell-3)^2(\ell+1)^2}{\ell(\ell-2)(2\ell-1)},$$

$$b_{2\ell} = \frac{(\ell-1)(\ell+1)}{2\ell-1}, \qquad c_{2\ell} = \frac{(\ell-3)(\ell-1)^2(\ell+1)}{\ell(\ell+2)(2\ell-1)}.$$

Note that in the case $d = 7$ (so $\ell = 3$) the coefficient c_6 is 0, and the solution obtained has degree 6.

For d odd, the examples above have higher singularities at 0 and ∞, with $r_0 = 0$, $r_1 = \ell - 3$, and at $\ell/(3-\ell)$, with $r_0 = 0$, $r_1 = 1$.

Theorem 5.1. *For $d \geq 6$, $d \neq 7$, the maps $[\xi] : S^2 \to S^6$ defined above are directrix curves of linearly full almost complex spheres in S^6 of degree d.*

Proof. It is clear that all the curves are linearly full and have degree d, so it only remains to check that they are solutions of the equation $\xi \times \xi' = 0$. The expression for $\xi \times \xi'$ in the even dimensional case is as follows: If

$$\xi = a_0 E_1 + a_1 z^1 E_2 + a_{\ell-1} z^{\ell-1} E_3 + a_\ell z^\ell E_0 + z^{\ell+1} \bar{E}_3 + z^{2\ell-1} \bar{E}_2 + z^{2\ell} \bar{E}_1,$$

then

$$\begin{aligned}\xi \times \xi' &= (2i\ell a_0 - i(2\ell-1)a_1 + i a_1 + i(\ell-1)a_{\ell-1} - i(\ell+1)a_{\ell-1})\, z^{2\ell-1} E_0 \\ &+ \left(\sqrt{2}(\ell-1)a_1 a_{\ell-1} - \sqrt{2}a_1 a_{\ell-1} - i\ell a_0 a_\ell\right) z^{\ell-1} E_1 \\ &+ \left(-\sqrt{2}a_0(\ell+1) + i\ell a_1 a_\ell - i a_1 a_\ell\right) z^\ell E_2 \\ &+ \left(\sqrt{2}(2\ell-1)a_0 - i(\ell-1)a_{\ell-1}a_\ell + i\ell a_{\ell-1} a_\ell\right) z^{2\ell-2} E_3 \\ &+ \left(-i\ell a_\ell + \sqrt{2}(\ell+1) - \sqrt{2}(2\ell-1)\right) z^{3\ell-1} \bar{E}_1 \\ &+ \left(-\sqrt{2}(\ell-1)a_{\ell-1} + 2\sqrt{2}\ell a_{\ell-1} - i\ell a_\ell + i(2\ell-1)a_\ell\right) z^{3\ell-2} \bar{E}_2 \\ &+ \left(-2\sqrt{2}\ell a_1 + \sqrt{2}a_1 - i\ell a_\ell + i(\ell+1)a_\ell\right) z^{2\ell} \bar{E}_3.\end{aligned}$$

It is straightforward to check that the solution does work.

We omit the much-lengthier odd-dimensional case.

□

Corollary 5.2. *The space* $\mathrm{Ac}^f_d(S^2, S^6)$ *is empty if* $d < 6$ *or* $d = 7$*, and nonempty otherwise. Its pure dimension is* $d + 8$.

Proof. For $d < 6$, the set $\mathrm{Harm}^f_d(S^2, S^6)$, and therefore $\mathrm{Ac}^f_d(S^2, S^6)$, is empty [1]. The case $d = 7$ was explained at the beginning of this section. The remaining cases are immediate consequences of Theorem 4.2 and Theorem 5.1

□

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