Bounding the k-rainbow total domination number

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Abstract

Let $[k] = \{1, 2, \ldots, k\}$, where the elements of $[k]$ are called colors. Supposing f is a function which assigns a subset of $[k]$ to each vertex of a graph G, if each vertex which is assigned the empty set has all k colors in its neighborhood, then f is called a k-rainbow dominating function for G. If f satisfies the additional condition that for every $v \in V(G)$ such that $f(v) = \{i\}$ for some $i \in [k]$ there exists $u \in N_G(v)$ such that $i \in f(u)$, then f is called a k-rainbow total dominating function. The corresponding invariant $\gamma_{krt}(G)$, which is the minimum sum of numbers of assigned colors over all vertices of G, is called the k-rainbow total domination number of G. We present bounds on this domination invariant in terms of domination, total domination, and k-rainbow (total) domination. We compute $\gamma_{krt}(K_{a,b})$, and use this result in a number of places. We establish that for $k \geq 3$, the following bounds are tight: $\gamma_{(k-1)rt}(G) \leq \gamma_{krt}(G) \leq \frac{k}{k-1}\gamma_{(k-1)rt}(G)$; we prove similar bounds related to rainbow domination and total domination. For a graph G, its ordinary domination number $\gamma(G)$, and $k \geq 2$, we prove that $\gamma_{krt}(G) \geq \frac{2k}{k+1}\gamma(G)$, which presents a generalization of a result for the case $k = 2$ by Goddard and Henning (2018); we then consider some implications of this result. We conclude the paper with Vizing-like conjectures for both rainbow domination and rainbow total domination, and relate the later one with the original Vizing conjecture.

Keywords: graph theory, domination, total domination, rainbow domination, Vizing's Conjecture

1 Introduction and preliminaries

Domination is a topic in graph theory with extensive research activity. Already in 1998, a classic book [10] by Haynes et al. surveyed over 1200 papers. It is not surprising that this number is increasing rapidly as domination presents one of the most applicable branches of graph theory. For example, graphs can be used to model locations which exchange some resource along its edges. In such an application, ordinary domination is an optimization problem to determine the minimum number of locations necessary in order for each location to contain the resource or be adjacent to a location containing the resource. However, there are situations where certain additional requirements must be fulfilled, thus numerous variations of domination, motivated by real life as well as theoretical applications, developed over the time. In this paper we study the recently introduced concept of k-rainbow total domination, compare it with known domination parameters, and show that it leads to an interesting viewpoint on the famous Vizing conjecture.

We begin with some general definitions and notation for graphs, followed by various definitions of domination in graphs. Graphs considered in this paper are finite, simple and undirected. For a graph G, we let $V(G)$ denote its set of vertices and $E(G)$ denote its set of edges. For a graph G, let $N_G(v)$ be the open neighborhood of vertex v in graph G, that is the vertices in G which are adjacent to v. When G is apparent from context, we may just write $N(v)$. The closed neighborhood $N_G[v]$ is $N_G(v) \cup \{v\}$. For graphs G and H, the Cartesian Product $G \square H$ is the graph with vertex set $V(G) \times V(H)$. Vertices (g, h) and (g', h') are adjacent in $G \square H$ if and only if either $g = g'$ and $hh' \in E(H)$ or $h = h'$ and $gg' \in E(G)$. For an integer k, we let $[k]$ refer to the set $\{1, 2, \ldots, k\}$, and sometimes refer to the elements of $[k]$ as colors. We denote the power set of $[k]$ by $2^{[k]}$.

A dominating set of a graph G is a subset D of $V(G)$ such that every vertex not in D is adjacent to some vertex in D. The *domination number*, $\gamma(G)$, is the minimum cardinality of a dominating set of G. If every vertex of G is adjacent to a vertex in D, then D is called a *total dominating set* of G, and the minimum cardinality of a total dominating set of G is the *total domination number*, $\gamma_t(G)$, [11]

One of the well studied domination invariants is k-rainbow domination, introduced by Brešar, Henning and Rall [4]. For a positive integer k, a *k-rainbow dominating function* (or kRDF, for short) of a graph G is a function $f : V(G) \to 2^{[k]}$, such that for any vertex v with $f(v) = \emptyset$ we have $\bigcup_{u \in N(v)} f(u) = [k]$. Let $||f|| = \sum_{v \in V(G)} |f(v)|$; we refer to $||f||$ as the *weight* of f. The *k-rainbow domination number*, $\gamma_{rk}(G)$, of a graph G is the minimum value of $||f||$ over all k-rainbow dominating functions of G. A kRDF of weight $\gamma_{rk}(G)$ is called a γ_{rk}-function. For example, $\gamma_{r2}(C_4) = 2$ since an optimal choice is to assign $\{1\}$ to one vertex v and assign $\{2\}$ to the vertex not adjacent to v in C_4. It was observed in [4] that for all $k \geq 1$, $\gamma_{rk}(G) = \gamma(G \square K_k)$. Since the seminal paper [4] introduced this invariant, it has been studied extensively, for example: its algorithmic properties [6, 23], relevant graph operations and families [3, 21], and general properties [18, 20, 24].

A *k-rainbow total dominating function* f of a graph G (a kRTDF for short) was introduced in [22] as a k-rainbow dominating function satisfying an additional condition that for every $v \in V(G)$ such that $f(v) = \{i\}$ for some $i \in [k]$, there exists some $u \in N(v)$ such that $i \in f(u)$. The weight of a kRTDF is as in the case of a kRDF, $||f|| = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRTDF is called the *k-rainbow total domination number of G*, $\gamma_{krt}(G)$.

A kRTDF of weight $\gamma_{krt}(G)$ is called a γ_{krt}-*function*. For example, $\gamma_{2rt}(C_4) = 4$ since an optimal choice is to pick two vertices and assign each $\{1,2\}$; another optimal choice simply assigns $\{1\}$ to every vertex. The definition of the k-rainbow total domination number is motivated by wanting to understand total domination in the generalized prism; in [22] it was observed that for all $k \geq 1$,

$$\gamma_{krt}(G) = \gamma_t(G \,\square\, K_k). \tag{1}$$

The main point of this paper is to compare the k-rainbow total domination number to other notions of domination, in particular to usual domination (i.e γ), to total domination (i.e. γ_t), and to k-rainbow domination (i.e. γ_{rk}). In Section 2 we calculate some domination numbers of the complete bipartite graph $K_{a,b}$ and a variant of $K_{a,b}$. The reason for these calculations is mostly for showing the tightness of bounds in subsequent sections. In Section 3 we bound the k-rainbow total domination number in terms of the other kinds of domination numbers. In Section 4 we obtain a lower bound on the k-rainbow total domination number in terms of the usual domination number, a generalization of the Goddard and Henning [12] result which applied to the case of $k = 2$. We also investigate the tightness of this lower bound and consider what happens when the graph is bipartite. In Section 5 we consider Vizing's 1968 conjecture that for any graphs G and H,

$$\gamma(G) \cdot \gamma(H) \leq \gamma(G \,\square\, H). \tag{2}$$

Variations on the Vizing conjecture have been considered by other authors, for example, Ho [15] in the case of total domination, and Brešar et al. [2] and Pilipczuk et al. [18] in the case of k-rainbow domination. We investigate what happens in the case of k-rainbow total domination, proving a simple Vizing-like result, and discussing a stronger Vizing-like conjecture, which in the case of bipartite graphs coincides with the original one.

2 Complete bipartite graphs

In this section we compute some domination invariants for $K_{a,b}$ and a related graph we call $K_{a,b}^+$. Some of the results of this section are interesting in their own right, however, the results of this section will be used in later sections, mostly to demonstrate that certain bounds are tight. In many calculations we use an alternative way to measure the weight of a k-rainbow dominating function f:

$$||f|| = \sum_{S \subseteq [k]} |S|\,|V_S|$$

where $V_S = \{x \in V(G) : f(x) = S\}$. If a vertex v is assigned a set S by the function f, we may refer to S as the *label* of v, and when clear from context we may abbreviate labels by removing the set braces, e.g. we let V_{12} refer to the set $V_{\{1,2\}}$, that is, the vertices labeled by set $\{1,2\}$. A vertex v with $f(v) = \emptyset$ will be referred to as *empty vertex*.

For integers a and b such that $a, b \geq 1$, let $K_{a,b}$ be the complete bipartite graph where one of the partite sets, say A, consists of vertices $x_1, x_2, \ldots, x_a$, and B is the other partite set. We define the graph $K_{a,b}^+$ to be $K_{a,b}$ with new vertices $y_1, y_2, \ldots, y_a$ and new edges $x_i y_i$ for every $i \in [a]$; see Figure 1.
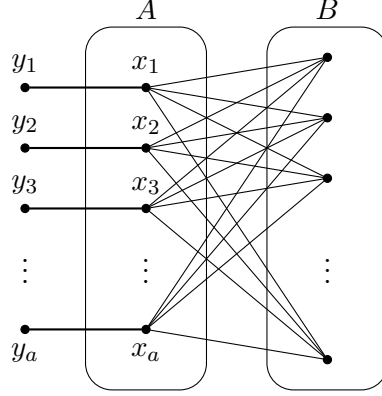
Figure 1: The graph $K_{a,b}^+$.

Lemma 1. $\gamma(K_{a,b}^+) = a$ and $\gamma_{\mathrm{krt}}(K_{a,b}^+) = 2a$ for $a, b \geq 1$ and $2 \leq k \leq a$.

Proof. The set A is clearly a dominating set of $K_{a,b}^+$, thus $\gamma(K_{a,b}^+) \leq a$. On the other hand, every dominating set of $K_{a,b}^+$ must contain either y_i or x_i for every $i \in [a]$, thus $\gamma(K_{a,b}^+) \geq a$, which proves the first equality.

Now let $f : V(K_{a,b}^+) \to 2^{[k]}$ be a function defined by $f(v) = \emptyset$ for every $v \in B$, and for every $i \in [a]$, $f(x_i) = f(y_i) = \{j\}$ for some $j \in [k]$, so that for every $j \in [k]$ there exists i with $f(x_i) = \{j\}$. Clearly f is a kRTDF, thus $\gamma_{\mathrm{krt}}(K_{a,b}^+) \leq 2a$. Finally, note that $\gamma_{\mathrm{krt}}(K_{a,b}^+)$ cannot be less than $2a$, since each pair of vertices x_i, y_i contributes at least 2 to the weight of any kRTDF of $K_{a,b}^+$. $\square$

Lemma 2. *Suppose a, b, and k are positive integers, $k \geq 1$ and $a \leq b$. If $a + b \leq k$, then $\gamma_{\mathrm{rk}}(K_{a,b}) = a + b$. If $a + b > k$, then we have*

$$\gamma_{\mathrm{rk}}(K_{a,b}) = \begin{cases} 2k, & if \ a \geq 2k \\ \max\{a, k\}, & if \ a < 2k. \end{cases}$$

Proof. Let $\{A, B\}$ be the bipartition of $V(K_{a,b})$ with $A = \{x_1, x_2, \ldots, x_a\}$ and $B = \{y_1, y_2, \ldots, y_b\}$, where all the edges are between A and B. We consider the following cases.

- If $a + b \leq k$, the assertion holds by a simple property for a general graph G, that if $|V(G)| \leq k$, then $\gamma_{\mathrm{rk}}(G) = |V(G)|$.

- Let $a + b > k$, $a \geq 2k$ and let $f : V(K_{a,b}) \to 2^{[k]}$ be defined with $f(x_1) = f(y_1) = [k]$ and $f(v) = \emptyset$ for every $v \in V(K_{a,b}) \setminus \{x_1, y_1\}$. Since f is clearly a kRDF we have $\gamma_{\mathrm{rk}}(K_{a,b}) \leq 2k$. Now let g be an arbitrary γ_{rk}-function of $K_{a,b}$. If both A and B contain an empty vertex, or neither of them contains such a vertex, we clearly have $||g|| \geq 2k$. If exactly one of A or B contains an empty vertex, then $||g|| \geq a \geq 2k$, so $\gamma_{\mathrm{rk}}(K_{a,b}) \geq 2k$.

- Let $a + b > k$ and $a < 2k$. Then obviously $\gamma_{\mathrm{rk}}(K_{a,b}) \geq k$. If $a \leq k$, $f : V(K_{a,b}) \to 2^{[k]}$ defined with $f(x_i) = \{i\}$ for $i \in [a - 1]$, $f(x_a) = \{a, a + 1, \ldots, k\}$ and $f(v) = \emptyset$ for

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every $v \in B$, is a kRDF. Therefore in this case we infer $\gamma_{\mathrm{rk}}(K_{a,b}) = k$. If $a > k$, then $f : V(K_{a,b}) \to 2^{[k]}$ defined with $f(x_i) = \{i\}$ for every $i \in [k]$, $f(x_i) = \{1\}$ for every $i \in \{k+1, k+2, \ldots, a\}$, and $f(v) = \emptyset$ for every $v \in B$, is a kRDF, thus $\gamma_{\mathrm{rk}}(K_{a,b}) \le a$. Assume for contradiction that $\gamma_{\mathrm{rk}}(K_{a,b}) < a$, and let g be an arbitrary γ_{rk}-function. Then there is an empty vertex in A and in B since $b \ge a > k$. Thus $||g|| \ge 2k$, contradicting $||g|| < a < 2k$. Therefore $\gamma_{\mathrm{rk}}(K_{a,b}) = a$ if $a > k$.

$\square$

Proposition 3. *Suppose a, b, and k are positive integers, $k \ge 2$ and $a \le b$. If $a + b \le k$, then $\gamma_{krt}(K_{a,b}) = a + b$. If $a + b > k$, then we have*

$$\gamma_{krt}(K_{a,b}) = \begin{cases} k, & \text{if } a \le \lfloor \frac{k}{2} \rfloor \\ a + \lceil \frac{k+1}{2} \rceil, & \text{if } \lfloor \frac{k}{2} \rfloor < a < \lceil \frac{3k-2}{2} \rceil \\ 2k, & \text{if } a \ge \lceil \frac{3k-2}{2} \rceil. \end{cases}$$

Proof. If $a + b \le k$ the assertion is immediate (see Observation 4 in [22]). Now let $a + b > k$ and let $\{A, B\}$ be the bipartition of $V(K_{a,b})$ with $A = \{x_1, x_2, \ldots, x_a\}$ and $B = \{y_1, y_2, \ldots, y_b\}$, where $a \le b$, and all the edges are between A and B.

First we consider the case when $a \le \lfloor \frac{k}{2} \rfloor$. Since $a + b > k$, we clearly have $\gamma_{krt}(K_{a,b}) \ge k$. Let $f : V(K_{a,b}) \to 2^{[k]}$ be defined with $f(x_i) = \{2i - 1, 2i\}$ for every $i \in [a - 1]$, $f(x_a) = \{2a - 1, 2a, \ldots, k\}$, and $f(y_i) = \emptyset$ for every $i \in [b]$. Since f is a kRTDF and $||f|| = k$ we also have $\gamma_{krt}(K_{a,b}) \le k$. Thus $\gamma_{krt}(K_{a,b}) = k$ if $a \le \lfloor \frac{k}{2} \rfloor$.

For the remainder of the proof we assume that $a \ge \lfloor \frac{k}{2} \rfloor + 1 = \lceil \frac{k+1}{2} \rceil$. Before we consider the remaining two cases for the value of a, we prove the following three claims.

Claim 1. $\gamma_{krt}(K_{a,b}) \le 2k$.

Let $f : V(K_{a,b}) \to 2^{[k]}$ be defined with $f(x_1) = f(y_1) = [k]$ and $f(v) = \emptyset$ for every $v \in V(K_{a,b}) \setminus \{x_1, y_1\}$. It is straightforward to see that f is a kRTDF, proving Claim 1.

Claim 2. $\gamma_{krt}(K_{a,b}) \le a + \lceil \frac{k+1}{2} \rceil$.

Let $f : V(K_{a,b}) \to 2^{[k]}$ be defined with $f(x_i) = \{2i - 1, 2i\}$ if $i \in \{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor\}$, $f(x_i) = \{k\}$ if $i \in \{\lfloor \frac{k}{2} \rfloor + 1, \ldots, a\}$, $f(y_1) = \{k\}$ and $f(y_i) = \emptyset$ for $i \in \{2, 3, \ldots, b\}$. It is easy to see that f is a kRTDF. Thus

$$\gamma_{krt}(K_{a,b}) \le ||f|| = 2 \cdot \left\lfloor \frac{k}{2} \right\rfloor + 1 \cdot \left(a - \left\lfloor \frac{k}{2} \right\rfloor \right) + 1 = a + \left\lceil \frac{k+1}{2} \right\rceil,$$

proving Claim 2.

Claim 3. *If f is a kRTDF on $K_{a,b}$ such that all of A consists of non-empty vertices or all of B consists of non-empty vertices, then $||f|| \ge a + \lceil \frac{k+1}{2} \rceil$.*

Let f be a γ_{krt}-function on $K_{a,b}$ such that $|f(x_i)| \ge 1$ for every $i \in [a]$; the argument is similar if all the vertices in B are non-empty. Suppose that in A there are exactly s vertices which are singleton sets (i.e. suppose $|f(x_1)| = \cdots = |f(x_s)| = 1$ and $|f(x_i)| \ge 2$ for $s < i \le a$). Without loss of generality, assume that the colors appearing among the

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singletons are $1, 2, \ldots, r$, for some $r \leq k$. If $s = 0$, then $||f|| \geq 2a \geq a + \left\lceil \frac{k+1}{2} \right\rceil$, so we are done. Thus we assume that $s \geq 1$; also recall that f is a minimum weight kRTDF. In B we must have exactly one occurrence of each of the colors $1, 2, \ldots, r$, and no other colors. Among the labels in A which are not singletons we can assume that collectively they include each color from $\{r + 1, \ldots, k\}$ exactly once, and contain no other colors. To see why we can make that assumption, consider a vertex in A with a non-singleton label L, such that $x, y \in L$ where x and y are distinct and y appears on some other vertex in A. We can remove y from L and add the color x to any label in B to arrive at a new γ_{krt}-function. So we will assume that in A, f consists of the s singletons, and then the rest of A contains each color from $\{r + 1, \ldots, k\}$ exactly once. Thus $||f|| = s + k$, since we have the s singletons in A, the colors $1, 2, \ldots, r$ appearing exactly once in B, and the colors $r + 1, \ldots, k$ appearing exactly once in A. We have at least one singleton, so $r \geq 1$, and thus the non-singletons in A use at most $\left\lfloor \frac{k-1}{2} \right\rfloor$ vertices of A, thus $s \geq a - \left\lfloor \frac{k-1}{2} \right\rfloor$. We can compute as follows

$$||f|| = s + k \geq a - \left\lfloor \frac{k-1}{2} \right\rfloor + k = a + \left\lceil \frac{k+1}{2} \right\rceil,$$

completing the proof of Claim 3.

We turn back to the proof of the proposition, considering the last two cases for the value of a. By Claims 1 and 2, we have the upper bounds on $\gamma_{\mathrm{krt}}(K_{a,b})$ in both cases. For the lower bounds, we first observe the following lower bounds on a kRTDF g depending on how it labels A and B (the first three lower bounds are simple observations, and the fourth is just a statement of Claim 3).

1. If g has empty vertices in both A and B then $||g|| \geq 2k$.

2. If g has empty vertices in all of A or all of B, then $||g|| \geq 2a$.

3. If g has no empty vertices then $||g|| \geq 2a$.

4. If g has no empty vertices in A or no empty vertices in B then $||g|| \geq a + \left\lceil \frac{k+1}{2} \right\rceil$.

Notice that the four conditions above are not mutually exclusive, but do cover all possible labellings of A and B; thus to show a lower bound, it suffices to cover the above cases.

If $\left\lfloor \frac{k}{2} \right\rfloor < a < \left\lceil \frac{3k-2}{2} \right\rceil$, then we refer to the above 4 conditions and note that $2a = a + a \geq a + \left\lceil \frac{k+1}{2} \right\rceil$, thus covering Conditions 2 and 3. Condition 4 already matches this case. To deal with Condition 1, we note its impossibility for a γ_{krt}-function g, because (using Claim 2) $||g|| \leq a + \left\lceil \frac{k+1}{2} \right\rceil < \left\lceil \frac{3k-2}{2} \right\rceil + \left\lceil \frac{k+1}{2} \right\rceil = 2k$.

If $a \geq \left\lceil \frac{3k-2}{2} \right\rceil$, then we refer to the above 4 conditions and note that the following calculations suffice in order to cover all the conditions (recall that $k \geq 2$):

$$2a \geq 2 \cdot \left\lceil \frac{3k-2}{2} \right\rceil \geq 2k \quad \text{and} \quad a + \left\lceil \frac{k+1}{2} \right\rceil \geq \left\lceil \frac{3k-2}{2} \right\rceil + \left\lceil \frac{k+1}{2} \right\rceil = 2k.$$

$\square$

3 Bounds relating to k-rainbow total domination

In this section we study bounds on the k-rainbow total domination number in terms of other domination numbers. Shao et al. [19] proved that $\gamma_{rk'}(G) \leq \gamma_{rk}(G) + (k' - k)\lfloor \frac{\gamma_{rk}(G)}{k} \rfloor$ and consequently $\gamma_{rk'}(G) \leq \frac{k'}{k}\gamma_{rk}(G)$, for a connected graph G and positive integers k and k' such that $k' \geq k$ (see Theorem 1 and Corollary 1 in [19]). Using the same proof we can prove the same relations hold for k-rainbow total domination. For completeness, we briefly recall their proof. Let f be a γ_{krt}-function of G and a_i the number of vertices u for which $i \in f(u)$. We may without loss of generality assume that $a_1 \geq a_2 \geq \cdots \geq a_k$. Since $\gamma_{krt}(G) = a_1 + a_2 + \cdots + a_k$ we conclude that $ka_k \leq \gamma_{krt}(G)$. Now, by adding the colors $k+1, k+1, \ldots, k'$ to the label of any vertex u such that $k \in f(u)$, we clearly get a k'RTDF whose weight is $\gamma_{krt}(G) + (k'-k)a_k$, therefore $\gamma_{k'rt}(G) \leq \gamma_{krt}(G) + (k'-k)\frac{\gamma_{krt}(G)}{k} = \frac{k'}{k}\gamma_{krt}(G)$. Summarizing this result and adding an observation we conclude the following.

Proposition 4. *Let k and k' be integers such that $1 \leq k \leq k'$. Then the following bounds hold:*

$$\gamma_{k'rt}(G) \leq \frac{k'}{k}\gamma_{krt}(G) \quad and \quad \gamma_{rk'}(G) \leq \frac{k'}{k}\gamma_{rk}(G).$$

Furthermore, the bounds are tight when G is $K_{a,b}$ with $a, b \geq 2k'$.

Proof. The upper bounds follow from the Shao et al. proof sketched above. To prove the tightness of the bounds we provide a family of graphs for which the equalities are attained. In both cases, we take the graphs G to be $K_{a,b}$, where $a, b \geq 2k'$. The tightness follows from Lemma 2 and Proposition 3, which imply $\gamma_{rk}(G) = \gamma_{krt}(G) = 2k$ and $\gamma_{rk'}(G) = \gamma_{k'rt}(G) = 2k'$. $\square$

Bringing the above results together with another observation, we can give a full description of how the rainbow domination numbers compare when the parameter is increased by 1.

Theorem 5. *For any graph G, and integer k such that $k \geq 2$, we have the following:*

1. *$\gamma_{(k-1)rt}(G) \leq \gamma_{krt}(G) \leq \frac{k}{k-1}\gamma_{(k-1)rt}(G)$,*

2. *$\gamma_{r(k-1)}(G) \leq \gamma_{rk}(G) \leq \frac{k}{k-1}\gamma_{r(k-1)}(G)$.*

Moreover, both upper bounds are tight for $K_{a,b}$ where $a, b \geq 2k$, and the first lower bound is tight for $k \geq 3$, when the graph is $K_{k,b}^{+}$ and $b \geq 1$.

Proof. The upper bounds and their tightness follow from Proposition 4. Now we discuss the lower bounds.

We prove that $\gamma_{krt}(G) \geq \gamma_{(k-1)rt}(G)$. Let f be a γ_{krt}-function on G. We use f to define $g : V(G) \to 2^{[k-1]}$ by letting $g(v) = f(v)$ as long as $k \notin f(v)$; if $k \in f(v)$ then

$$g(v) = \begin{cases} (f(v) \setminus \{k\}) \cup \{1\}, & \text{if } k - 1 \in f(v) \\ (f(v) \setminus \{k\}) \cup \{k - 1\}, & \text{otherwise.} \end{cases}$$

If $g(v) = \emptyset$, clearly all the colors from $[k-1]$ appear in the neighborhood of v. If $g(v) = \{i\}$ and $i \in [k-2]$, then by the definition of g there exists a neighbor u of v such that $i \in g(u)$. If $g(v) = \{k-1\}$, then we have one of the following situations: either $f(v) = \{k-1\}$, so there exists $u \in N_G(v)$ such that $k-1 \in f(u)$ and therefore also $k-1 \in g(u)$, or $f(v) = \{k\}$ so there exists $u \in N_G(v)$ such that $k \in f(u)$, and thus $k-1 \in g(u)$ again. Thus g is a $(k-1)$RTDF and we have $\gamma_{(k-1)\mathrm{rt}}(G) \leq ||g|| \leq ||f|| = \gamma_{k\mathrm{rt}}(G)$, which shows the first upper bound. To prove the second upper bound, that $\gamma_{\mathrm{r}(k-1)}(G) \leq \gamma_{\mathrm{r}k}(G)$, is even simpler, since by taking a $\gamma_{\mathrm{r}k}$-function f on G we can turn it to a $(k-1)$RDF simply by replacing every occurrence of the element k by $k-1$.

Now we discuss the tightness of the lower bounds. For the first inequality, when $k \geq 3$, we obtain $\gamma_{(k-1)\mathrm{rt}}(G) = \gamma_{k\mathrm{rt}}(G)$ when G is $K_{k,b}^+$ (recall Lemma 1). When $k = 2$ the equality states that $\gamma_t(G) = \gamma_{2\mathrm{rt}}(G)$; there exist graphs satisfying the equality, which were in fact characterized in Theorem 3 of [16]. For the second inequality, when $k \geq 3$ we take integers a, b such that $k < a < 2k \leq b$, so by Lemma 2 $\gamma_{\mathrm{r}(k-1)}(K_{a,b}) = \gamma_{\mathrm{r}k}(K_{a,b}) = a$. When $k = 2$, we are interested in graphs G that satisfy $\gamma(G) = \gamma_{r2}(G)$. Such graphs exist and were characterized by Hartnell and Rall, see Theorem 4 in [14]. $\qquad\square$

The following simple corollary will lead to an interesting question.

Corollary 6. *For any graph G without isolated vertices, $\gamma_t(G) \leq \gamma_{krt}(G) \leq k\gamma_t(G)$, and the upper bound is tight.*

Proof. Recall that if G does not contain isolated vertices, then $\gamma_{1rt}(G) = \gamma_t(G)$. Therefore Theorem 5 implies the lower bound. It is easy to observe the upper bound by assigning the entire set $[k]$ to each vertex in a total dominating set of G. The upper bound is tight when G is $K_{a,b}$ $(a, b \geq 2k)$, since $\gamma_t(G) = 2$ is immediate, while $\gamma_{krt}(G) = 2k$ follows from Proposition 3. $\qquad\square$

Notice that in the last corollary there is no claim about the tightness of the lower bound. As we have already mentioned in the proof of Theorem 5, the lower bound in the case when $k = 2$ can be attained and all such graphs were characterized in [16]. For $k \geq 3$, the lower bound of Corollary 6 does not appear to be tight, which leads to the following question (where we conjecture that $b(k) > 1$).

Question 1. *Find a function $b(k)$ such that for $k \geq 3$, the following bound is true and tight for connected graphs G:*

$$b(k) \cdot \gamma_t(G) \leq \gamma_{krt}(G).$$

We observe that it must be the case that $1 \leq b(k) \leq \frac{3}{2}$. In [9] it was proved that if G is a connected graph of order $n \geq 3$, then $\gamma_t(G) \geq \frac{2n}{3}$, and graphs attaining equality were characterized in [5]. Therefore, since $\gamma_{krt}(G) \leq n$, if G is a graph with maximum total domination number, then $b(k) \leq \frac{3}{2}$.

In the next proposition and question we consider how the k-rainbow domination number compares to the k-rainbow total domination number.

Proposition 7. *For any graph G and any integer k such that $k \geq 2$ the following holds:*

$$\gamma_{\mathrm{r}k}(G) \leq \gamma_{k\mathrm{rt}}(G) \leq 2\gamma_{\mathrm{r}k}(G).$$

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Furthermore, the lower bound is tight for every k, and the upper bound is tight only for $k = 2$, that is, for $k \geq 3$, $\gamma_{\mathrm{krt}}(G) < 2\gamma_{\mathrm{rk}}(G)$.

Proof. As we have already explained, the lower bound follows by definitions of both invariants. To see that the upper bound holds, let f be a γ_{rk}-function on G. Then a k-rainbow total dominating function g can be constructed from f by adding an element to every singleton label of f, so g has no singleton labels. Then $\gamma_{\mathrm{krt}}(G) \leq ||g|| \leq 2||f|| = 2\gamma_{\mathrm{rk}}(G)$.

The tightness of the lower bound holds when G is $K_{a,b}$ for $b \geq a \geq 2k$, since then we have $\gamma_{\mathrm{rk}}(G) = \gamma_{\mathrm{krt}}(G) = 2k$, using Lemma 2 and Proposition 3. The tightness of the upper bound, when $k = 2$, follows by taking G to be $K_{2,b}$, where $b \geq 2$, since by Proposition 3 and Lemma 2 we get $\gamma_{2rt}(G) = 4 = 2\gamma_{r2}(G)$.

Now we show that the upper bound is strict, when $k \geq 3$, i.e. $\gamma_{krt}(G) < 2\gamma_{\mathrm{rk}}(G)$ if $k \geq 3$. Let f be a γ_{rk}-function on G, i.e. $\gamma_{\mathrm{rk}}(G) = ||f|| = |V_1| + \cdots + |V_k| + \Sigma_{S \subseteq [k], |S| \geq 2} |S| \cdot |V_S|$. As pointed out above, from f we can construct a kRTDF g by adding one label for each singleton, so the weight of g is at most $2|V_1| + \cdots + 2|V_k| + \Sigma_{S \subseteq [k], |S| \geq 2} |S| \cdot |V_S|$. If there exists at least one vertex v with $|f(v)| \geq 2$, we derive

$$2\gamma_{\mathrm{rk}}(G) = 2||f|| > 2|V_1| + \cdots + 2|V_k| + \Sigma_{S \subseteq [k], |S| \geq 2} |S| \cdot |V_S| \geq ||g|| \geq \gamma_{\mathrm{krt}}(G),$$

and we are done. Thus now assume that f assigns only singletons and empty sets to vertices of G. We can assume that f has at least one empty vertex, otherwise we could change all the labels to $\{1\}$ to achieve a kRTDF of the same weight, so we would be done.

Let $f_1 : V(G) \to 2^{[k]}$ be defined from f as follows:

$$f_1(v) = \begin{cases} \{1\}, & \text{if } f(v) = \{k-1\} \\ \{k-1, k\}, & \text{if } f(v) = \{k\} \\ f(v), & \text{otherwise} \end{cases}$$

Now construct f_2 from f_1 according to the following:

- let v be a vertex such that $f(v) = f_1(v) = \emptyset$, and define $f_2(v) = \{1\}$. Note that since f is a kRDF there exist $x \in V_1$ and $y \in V_{k-1}$ such that v is adjacent to both of them. Recall that $f_1(x) = f_1(y) = f_2(v) = \{1\}$,

- for all non-empty vertices except x, y, and v, add an element to it, so that it is no longer a singleton.

It is easy to see that f_2 is a kRTDF, thus we derive

$$\gamma_{\mathrm{krt}}(G) \leq ||f_2|| \leq 2|V_1| + 2|V_2| + \cdots + 2|V_k| - 2 + 1 = 2||f|| - 1 < 2\gamma_{\mathrm{rk}}(G),$$

which concludes the proof. $\qquad\square$

The strict inequality in Proposition 7 leads to the next question.

Question 2. *Find a function $a(k)$ such that for every k we have the tight bound*

$$\gamma_{krt}(G) \leq a(k) \cdot \gamma_{rk}(G).$$

From Proposition 7 we know $a(k) < 2$ for $k \geq 3$. By taking G to be $K_{k,b}$, for $b \geq k$, we get $\gamma_{rk}(G) = k$ and $\gamma_{krt}(G) > (3/2)k$, by Lemma 2, and Proposition 3, respectively; thus $a(k) > 3/2$. In summary, we know that for $k \geq 3$, we must have $3/2 < a(k) < 2$.

4 Lower bounding k-rainbow total domination

The main point of this section is to lower bound $\gamma_{krt}(G)$ in terms of $\gamma(G)$. In [22] it was observed that for a graph G of order n where $n > k > 1$, it is always the case that $\max\{k, \gamma(G)\} \leq \gamma_{krt}(G)$. While the lower bound of k can be achieved, we will show that the lower bound of $\gamma(G)$ cannot. Goddard and Henning [12] show that for any graph G, $\frac{4}{3}\gamma(G)$ is a tight lower bound for $\gamma_{2rt}(G)$. We generalize their lower bound to all k in the following theorem, where the interesting issue of tightness will be discussed after the theorem.

Theorem 8. *For a graph G and $k \geq 2$ we have*

$$\gamma_{krt}(G) \geq \frac{2k}{k+1}\gamma(G).$$

Proof. We use the notation $G[S]$ for the subgraph of G induced by the set $S \subseteq V(G)$. Let $f : V(G) \to 2^{[k]}$ be a γ_{krt}-function. Let $V_i' \subseteq V_i$ be the set of vertices in V_i having a neighbor in V_i, and let D_i be a minimum dominating set for $G[V_i']$. Since $G[V_i']$ contains no isolated vertices, we have (due to a result of Ore in [17]) that $|D_i| \leq \frac{|V_i'|}{2} \leq \frac{|V_i|}{2}$. For $i \in [k]$ we define a set U_i as follows:

$$U_i = V_i \ \cup \bigcup_{\substack{S \subseteq [k], \\ |S| \geq 2}} V_S \ \cup \ \bigcup_{j \neq i} D_j.$$

In other words, U_i consists of the following set of vertices in G: those that are labeled by just $\{i\}$, those that are labeled by a subset of $[k]$ of size at least 2, and from among those labeled by $\{j\}$, where $j \neq i$, take just the ones in D_j.

Now we show that each U_i is a dominating set of G. First, consider a vertex v with $f(v) = \emptyset$. Since f is a kRTDF, there is a neighbor x of v such that $i \in f(x)$. This means that v is adjacent to a vertex in V_i or a vertex labeled by a set of size 2 or greater. Therefore v is adjacent to a vertex in U_i. Second, consider a vertex $u \in V_j \setminus D_j$ where $j \neq i$. If u is in V_j' then it is adjacent to a vertex from D_j, otherwise, u is not adjacent to a vertex labeled by $\{j\}$, so it must be adjacent to a vertex labeled by S where $j \in S$ and $|S| \geq 2$. Thus u is adjacent to a vertex in U_i. Finally, the rest of the vertices are actually in U_i. Thus U_i is a dominating set.

Observe that each U_i contains all the vertices labeled by a set of size at least 2, all of the vertices in V_i, and at most half of the of vertices from any V_j ($j \neq i$). Letting m be the number of vertices colored by a subset of size at least 2, and s the number of vertices colored by only one color, we can conclude that

$$\sum_{i=1}^{k} |U_i| \leq k \sum_{\substack{S \subseteq [k], \\ |S| \geq 2}} |V_S| + \sum_{i=1}^{k} |V_i| + \frac{k-1}{2} \sum_{i=1}^{k} |V_i| = km + s + \frac{k-1}{2}s.$$

Using the last statement and the fact that $s \leq \gamma_{krt}(G) - 2m$ (since $\gamma_{krt}(G) \geq 2m + s$), we derive

$$k \cdot \gamma(G) \leq \sum_{i=1}^{k} |U_i| \leq km + \frac{k+1}{2}s \leq \frac{k+1}{2}\gamma_{krt}(G) - m \leq \frac{k+1}{2}\gamma_{krt}(G),$$

thus $k \cdot \gamma(G) \leq \frac{k+1}{2}\gamma_{krt}(G)$, which completes the proof. $\qquad \square$

In what follows we consider the tightness of the bound in the above theorem. The tightness for $k = 2$ follows from the mentioned result by Goddard and Henning; see Theorem 3.1 in [12]. To see that the lower bound is tight for $k = 3$, we modify a construction from [12] in order to define a family of graphs $\mathcal{T}_m$, where $m \geq 2$ is an integer parameter. A graph G is in the family $\mathcal{T}_m$ if we can construct it as follows. There are two parts to G: H and I. H consists of the three graphs H_1, H_2, and H_3, where each H_i is m disjoint edges. To construct I, for each 3-tuple of vertices $\bar{x} = (x_1, x_2, x_3)$, where $x_i \in H_i$, add an independent set of size at least m to I, calling it $I_{\bar{x}}$. Attach each x_i to all vertices in $I_{\bar{x}}$. See Figure 2 for an illustration of a graph from $\mathcal{T}_3$; for clarity, only 3 of the 6^3 independent sets are depicted. We calculate $\gamma_{3rt}(G)$ for G in $\mathcal{T}_m$, and then state the immediate and interesting corollary.
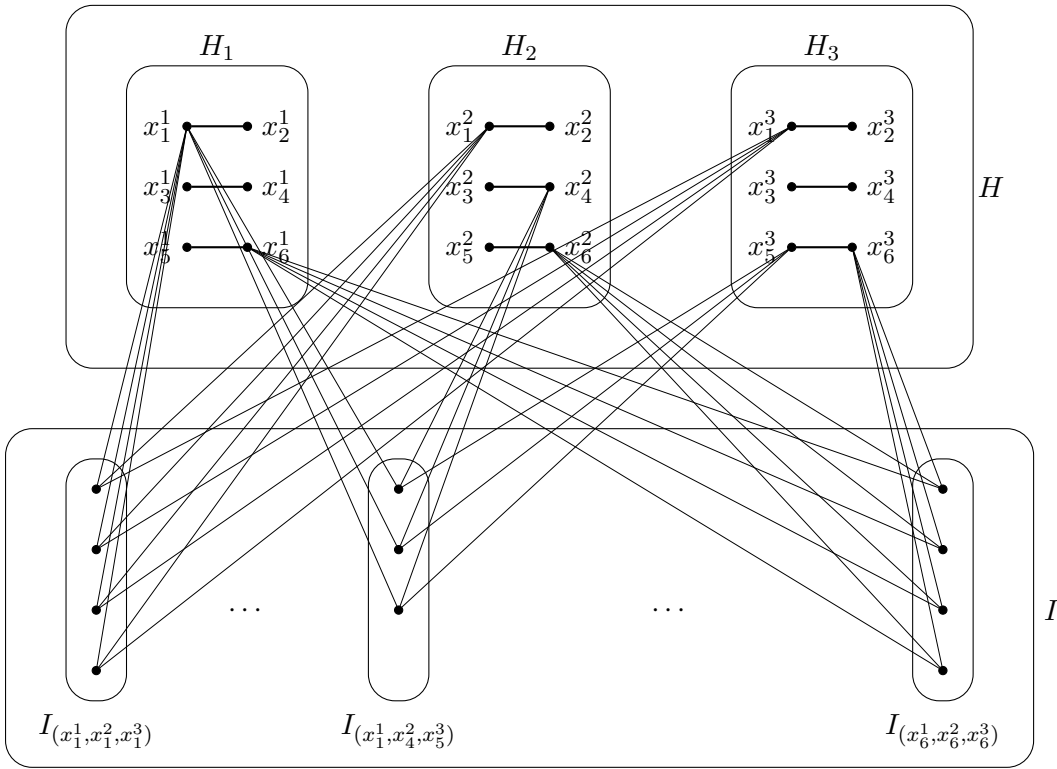

Figure 2: A graph from $\mathcal{T}_3$.

Proposition 9. *Suppose G is in the family $\mathcal{T}_m$. Then*

$$\gamma_{3rt}(G) = \frac{3}{2}\gamma(G).$$

Proof. We obtain $\gamma_{3rt}(G) \geq \dfrac{3}{2}\gamma(G)$ from Theorem 8. The rest of the proof shows that $\gamma_{3rt}(G) \leq \dfrac{3}{2}\gamma(G)$. It suffices to show that $\gamma_{3rt}(G) \leq 6m$ and $\gamma(G) \geq 4m$. For the first inequality, let $f : V(G) \to 2^{[3]}$ be defined by $f(v) = \{i\}$ if $v \in V(H_i)$ and $f(v) = \emptyset$ for every

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vertex v in I. It is easy to see that f is a 3RTDF of G thus $\gamma_{3rt}(G) \leq ||f|| \leq 6m$. The rest of the proof shows that $\gamma(G) \geq 4m$.

Let D be a minimum size dominating set of G, where D is chosen to have the following property: D contains as many vertices from $H_1 \cup H_2 \cup H_3$ as possible. We claim that such a D must contain every vertex of H_i for some $i = 1, 2$, or 3. Assume for contradiction that there is no such H_i; then there must be a tuple (u_1, u_2, u_3) such that $u_i \in H_i$ and no u_i is in D. The structure of G implies that every vertex of $I_{(u_1, u_2, u_3)}$ must be in D. However the closed neighborhood of $I_{(u_1, u_2, u_3)}$ contains the vertices $I_{(u_1, u_2, u_3)}$ along with $\{u_1, u_2, u_3\}$, while the closed neighborhood of $\{u_1, u_2, u_3\}$ contains the same vertices as well as others. Since $I_{(u_1, u_2, u_3)}$ contains at least 3 vertices, we could remove $I_{(u_1, u_2, u_3)}$ from D and replace them by $\{u_1, u_2, u_3\}$ to arrive at a different minimum dominating set that contains more vertices of $H_1 \cup H_2 \cup H_3$, contradicting the property of D having as many vertices from $H_1 \cup H_2 \cup H_3$ as possible.

Thus we assume D contains all of H_1. The vertices of H_1 are not adjacent to the vertices of $H_2 \cup H_3$, so we still must dominate all of them. Note that for any vertex in the graph, it dominates at most two vertices in $H_2 \cup H_3$, thus to dominate the $4m$ vertices of $H_2 \cup H_3$ we need at least $2m$ more vertices in addition to the $2m$ vertices of H_1. Thus $\gamma(G) \geq 4m$.

$\square$

Corollary 10. *For a graph G we have the following tight inequality:*

$$\gamma_{3rt}(G) \geq \frac{3}{2}\gamma(G).$$

The construction of $\mathcal{T}_m$, along with the last corollary does not obviously generalize to $k \geq 4$. Summarizing, we have the following tight lower bounds for the cases of $k = 2$ and $k = 3$: $\gamma_{2rt}(G) \geq \frac{4}{3}\gamma(G)$ and $\gamma_{3rt}(G) \geq \frac{3}{2}\gamma(G)$. We make the following conjecture for the other values of k.

Conjecture 11. *For a graph G and $k \geq 4$, we have the tight bound $\gamma_{krt}(G) \geq 2\gamma(G)$.*

In Conjecture 11 the correct coefficient c in front of $\gamma(G)$ (which we conjecture to be 2) satisfies $\frac{2k}{k+1} \leq c \leq 2$. The first inequality holds by Theorem 8. The second inequality holds because $\gamma_{krt}(K_{k,b}^+) = 2k = 2 \cdot \gamma(K_{k,b}^+)$, by Lemma 1 (for $k \geq 2, b \geq 1$).

In the case of bipartite graphs, when $k = 2$, the bound in Theorem 8 can be improved. Using hypergraphs Azarija et al. showed that for a bipartite graph G, $\gamma_{2rt}(G) = 2\gamma(G)$ (see Theorem 1 in [1]). Goddard and Henning [12] (see Theorem 2.1) presented a shorter proof using paired domination. Using 2-rainbow total domination the proof of the mentioned result is even simpler.

Theorem 12 ([1, 12]). *If G is a bipartite graph then $\gamma_{2rt}(G) = 2\gamma(G)$.*

Proof. Let G be a bipartite graph with bipartition (L, R), and let f be a γ_{2rt}-function for G. We define a set A as the union of the following three subsets of $V(G)$: all vertices of G with label $\{1, 2\}$, all vertices from L with label $\{2\}$, and all vertices from R with label $\{1\}$. Obviously, A is a dominating set in G. By interchanging L and R in the above definition, a set B is defined, which is also a dominating set in G. Therefore

$$2\gamma(G) \leq |A| + |B| = 2|V_{12}| + |V_1| + |V_2| = ||f||.$$

The opposite inequality $||f|| \leq 2\gamma(G)$ is immediate (see Observation 2.3 in [22]). $\qquad \square$

5 Vizing-like conjectures

Vizing's well known conjecture states that for any graphs G and H

$$\gamma(G) \cdot \gamma(H) \leq \gamma(G \,\square\, H). \qquad (3)$$

By a result of Clark and Suen [8] it is known that for any graphs G and H

$$\gamma(G) \cdot \gamma(H) \leq 2\gamma(G \,\square\, H).$$

Ho [15] proved that for any graphs G and H without isolated vertices,

$$\gamma_t(G) \cdot \gamma_t(H) \leq 2 \cdot \gamma_t(G \,\square\, H),$$

where the inequality is tight. We use above results to provide a simple proof of the following Vizing-like inequalities for k-rainbow and k-rainbow total domination.

Proposition 13. *Let G and H be graphs and $k \geq 2$. Then*

$$\gamma_{krt}(G) \cdot \gamma_{krt}(H) \leq 2k \cdot \gamma_{krt}(G \,\square\, H)$$

and

$$\gamma_{rk}(G) \cdot \gamma_{rk}(H) \leq 2k \cdot \gamma_{rk}(G \,\square\, H).$$

Proof. Using Ho's result, the equality in equation (1), and Corollary 6 we derive

$$
\begin{aligned}
\gamma_{krt}(G) \cdot \gamma_{krt}(H) &= \gamma_t(G \,\square\, K_k) \cdot \gamma_t(H \,\square\, K_k) \\
&\leq 2\gamma_t(G \,\square\, K_k \,\square\, H \,\square\, K_k) \\
&= 2\gamma_t(G \,\square\, H \,\square\, K_k \,\square\, K_k) \\
&= 2\gamma_{krt}(G \,\square\, H \,\square\, K_k) \\
&\leq 2k\gamma_t(G \,\square\, H \,\square\, K_k) \\
&= 2k\gamma_{krt}(G \,\square\, H).
\end{aligned}
$$

Following the same lines, using the above result of Clark and Suen, and known bound $\gamma_{rk}(G) \leq k\gamma(G)$ from [14], the Vizing-like inequality for k-rainbow domination can be derived. $\qquad \square$

We believe that the following stronger inequalities might hold.

Conjecture 14. *Let G and H be graphs and $k \geq 2$. Then*

$$\gamma_{rk}(G) \cdot \gamma_{rk}(H) \leq 2 \cdot \gamma_{rk}(G \,\square\, H).$$

Conjecture 15. *Let G and H be graphs and $k \geq 2$. Then*

$$\gamma_{krt}(G) \cdot \gamma_{krt}(H) \leq 2 \cdot \gamma_{krt}(G \,\square\, H). \qquad (4)$$
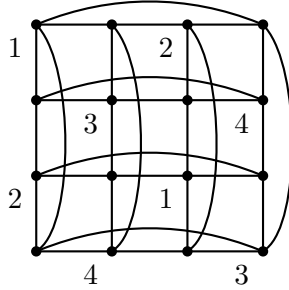
Figure 3: 4-rainbow domination of $C_4 \square C_4$.

For the first conjecture the constant 2 is attained for $G = H = C_4$ and $k = 4$. It is easy to see that $\gamma_{r4}(C_4) = 4$. To see that $\gamma_{r4}(C_4 \square C_4) = 8$, observe that the upper bound $\gamma_{r4}(C_4 \square C_4) \leq 8$ follows from the construction in Figure 3, while the lower bound $\gamma_{r4}(C_4 \square C_4) \geq 8$ follows from Theorem 3 in [19] (that theorem states that for a connected graph G, $\gamma_{rk}(G) \geq \left\lceil \frac{|V(G)|k}{\Delta(G)+k} \right\rceil$, where $\Delta(G)$ stands for the maximum degree of G).

Regarding Conjecture 15, if we exclude graphs containing isolated vertices and take $k = 1$, then inequality (4) is just a restatement of Ho's result. Furthermore, when $k = 1$, Ho observes that the inequality is sharp. When $k = 2$ we obtain an interesting observation relating Vizing's conjecture to Conjecture 15. Due to Theorem 12, if G and H are bipartite graphs, then

$$4\gamma(G)\gamma(H) = \gamma_{2rt}(G)\gamma_{2rt}(H) \quad \text{and} \quad 4\gamma(G \square H) = 2\gamma_{2rt}(G \square H),$$

which implies that for bipartite graphs, inequality (3) holds if and only if for $k = 2$ inequality (4) holds. Furthermore, one inequality is tight if and only if the other is. The next observation summarizes this discussion.

Observation 16. *Restricting to the case $k = 2$ and to the class of bipartite graphs, Conjecture 15 and Vizing's conjecture are equivalent.*

The relevance of the last observation is highlighted by the fact that there are many pairs of graphs for which equality is achieved in Vizing's conjecture (see [13]).

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