

# COP-WIN GRAPHS: OPTIMAL STRATEGIES AND CORNER RANK

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ABSTRACT. We investigate the game of *cops and robber*, played on a finite graph, between one cop and one robber. If the cop can force a win on a graph, the graph is called *cop-win*. We describe a procedure we call *corner ranking*, performed on a graph, which assigns a positive integer or  $\infty$  to each vertex. We give a characterization of cop-win in terms of corner rank and also show that the well-known characterization of cop-win via dismantling orderings follows from our work. From the corner rank we can determine the capture time of a graph, i.e. the number of turns the cop needs to win. We describe a class of optimal cop strategies we call *Lower Way* strategies, and a class of optimal robber strategies we call *Higher Way* strategies. Roughly speaking, in a Lower Way strategy, the cop pushes the robber down to lower ranked vertices, while in a Higher Way strategy, the robber moves to a highest rank vertex that is “safe.” While interesting in their own right, the strategies are themselves tools in our proofs. We investigate various properties of the Lower Way strategies.

## 1. INTRODUCTION

The game of cops and robber is a perfect-information two-player pursuit-evasion game played on a graph. To begin the game, the cop and robber each choose a vertex to occupy, with the cop choosing first. Play then alternates between the cop and the robber, with the cop moving first. On a turn a player may move to an adjacent vertex or stay still. If the cop and robber ever occupy the same vertex, the robber is caught and the cop wins. If the cop can force a win on a graph, we say the graph is *cop-win*. The game was introduced by Nowakowski and Winkler [6], and Quilliot [8]. A nice introduction to the game and its many variants is found in the book by Bonato and Nowakowski [2].

For a cop-win graph, the cop can guarantee a win, so Bonato et. al. [1] raise the question of how long the game will last. They define the **capture time** of a cop-win graph  $\mathbf{G}$ , denoted  $\text{capt}(\mathbf{G})$ , to mean the fewest number of moves *the cop* needs to guarantee a win (*not counting her initial placement as a move*); for example, on the path with five vertices the capture time is two. For cop-win graphs, we define what we mean for a player’s strategy to be optimal.

**Definition 1.1.** *Consider a cop-win graph with capture time  $\mathfrak{c}$ .*

- *A cop strategy is **optimal** if no matter what the robber does, the cop will win in at most  $\mathfrak{c}$  cop moves.*
- *A robber strategy is **optimal** if no matter what the cop does, the cop must make at least  $\mathfrak{c}$  moves to win.*

Our paper is centered on the development of optimal strategies which are defined in terms of *corner rank*. In Sections 2 and 3, we introduce corner rank and the associated notion of

projections. On any graph, we define a corner rank function which assigns to each vertex a positive integer or  $\infty$ . The corner rank (or simply rank) of a vertex is the value assigned to it by the corner rank function. The corner rank of a graph is equal to the rank of its highest rank vertex. In Section 4 we define a class of cop strategies we call *Lower Way* strategies; roughly, these strategies corner the robber among the vertices of at least a certain rank, forcing the robber to move to successively lower rank vertices until he is caught. In Section 5 we define a class of robber strategies we call *Higher Way*; in such a strategy, the robber moves to a highest corner rank vertex he can while maintaining a condition guaranteeing his safety.

We say a graph with a corner rank that is a positive integer has *finite corner rank*, while a graph with a corner rank of  $\infty$  has *infinite corner rank*. If the corner rank of a graph is finite then we can easily determine a number  $\mathfrak{c}$  such that a Lower Way strategy will win in at most  $\mathfrak{c}$  cop moves. Conversely, the Higher Way strategy forces the cop to make at least  $\mathfrak{c}$  moves. Thus we can conclude that the capture time is  $\mathfrak{c}$  and that both the Lower Way strategies and Higher Way strategies are optimal. While a Higher Way strategy for the robber on a finite corner rank graph delays his inevitable loss as long as possible, on an infinite corner rank graph, the strategy actually wins. Thus corner rank provides a characterization of cop-win graphs: A graph is cop-win if and only if it has finite corner rank.

Finally, in Section 7, we show that in general the Lower Way strategies do not encompass all optimal strategies. However for a particular class of graphs that we call *iteratively twin-free*, if the robber plays a Higher Way strategy, then all optimal strategies for the cop are Lower Way strategies. We raise the question of “characterizing all the optimal strategies.”

Many of the results in this paper, though obtained independently, are explicitly or implicitly contained in a paper of Clarke, Finbow, and MacGillivray [3]. The fundamental difference between our papers is that all our results follow from the properties of the Lower Way and Higher Way strategies, while their results follow from a combination of the following tools: 1) A cop strategy, 2) inductive arguments on the corner rank, and 3) the well-known characterization of cop-win graphs by dismantling orderings. We do not claim that our approach is better. However, our approach does have the advantage of providing explicit optimal strategies for both players, and we prove some interesting properties of these strategies. In addition, in Section 6 we use our tools to give an alternate proof of the characterization of the cop-win graphs as exactly those that have a dismantling ordering [6, 8]. At various points throughout the paper, we will compare our approach with that of [3].

The notion of corner rank provides a useful tool for investigating cop-win graphs. We already see the usefulness in this paper. We also have a submitted paper [7] in which we use corner rank to characterize graphs of a given size with maximal capture time. We expect corner rank to continue to be a useful tool for future questions about cop-win graphs.

## 2. CORNER RANKING

One of the fundamental results about the game of Cops and Robber is a characterization of the cop-win graphs as those graphs which have a *dismantling ordering* (also known as a *cop-win ordering*—see [6] and [8]). In this section we define corner ranking, a procedure that can be thought of as a modification of the cop-win ordering. In [3], the authors describe

*cop-win partitions*, which are equivalent to corner ranking when restricted to cop-win graphs; at the end of this section we compare our approaches.

In this paper all graphs are finite and non-empty, i.e. have at least one vertex; all numbers are integers. We follow a typical Cops and Robber convention: we assume that all graphs are reflexive, that is *all graphs have a loop at every vertex so that a vertex is adjacent to itself*; we will never draw or mention such edges. This assumption simplifies much of the following discussion while leaving the game play unchanged. For a graph  $\mathbf{G}$ ,  $V(\mathbf{G})$  refers to the vertices of  $\mathbf{G}$ . If  $\mathbf{G}$  is a graph and  $X$  is a vertex or set of vertices in  $\mathbf{G}$ , then by  $\mathbf{G} - X$  we mean the subgraph of  $\mathbf{G}$  induced by  $V(\mathbf{G}) \setminus X$ . We say that a vertex  $u$  **dominates** a set of vertices  $X$  if  $u$  is adjacent to every vertex in  $X$ . Given a vertex  $v$  in a graph, by  $\mathbf{N}[v]$ , the **closed neighborhood of  $v$** , we mean the set of vertices adjacent to  $v$ . For a vertex  $v$  in  $\mathbf{G}$  and an induced subgraph  $\mathbf{H}$  of  $\mathbf{G}$ , let  $\mathbf{H} \cup v$  be the subgraph of  $\mathbf{G}$  induced by  $V(\mathbf{H}) \cup \{v\}$ . For distinct vertices  $v$  and  $w$ , if  $\mathbf{N}[v] \subseteq \mathbf{N}[w]$  then we say that  $v$  is a **corner** and that  $w$  **corners  $v$** .

**Convention.** *We will commonly make reference to “the cop” and “the robber”, to refer to the vertices at which they are located; for example, to say that the cop corners the robber, means that the cop is located at some vertex  $c$ , the robber at some vertex  $r$ , and  $c$  corners  $r$ .*

It is well known that for a graph to be cop-win, it must contain a corner, and furthermore, if  $v$  is a corner in a graph  $\mathbf{G}$ , then  $\mathbf{G}$  is cop-win if and only if  $\mathbf{G} - v$  is cop-win. Thus a graph is cop-win if and only if the vertices can be deleted one by one, where each vertex deleted is a corner at the time of deletion. The ordered list of these vertices is called a **cop-win ordering** or **dismantling ordering** [6, 8]; for example, in the graph of Figure 1, the following is a dismantling ordering, where we delete vertices from left to right:

$$a, d, e, g, h, f, c, b.$$

There may be many cop-win orderings for a given graph. To build upon this work, we classify corners into two categories, which we call *strict corners* and *twins*.

**Definition 2.1.** *Suppose  $v$  and  $w$  are distinct vertices where  $w$  corners  $v$ .*

- *If  $\mathbf{N}[v] \subsetneq \mathbf{N}[w]$ , we say that  $v$  is a **strict corner** and that  $w$  **strictly corners  $v$** .*
- *If  $\mathbf{N}[v] = \mathbf{N}[w]$ , we call  $v$  and  $w$  **twins**.*

Note that if  $v$  and  $w$  are twins then  $v$  corners  $w$  and  $w$  corners  $v$ . If a vertex is a corner, it must either be a strict corner or a twin with some other vertex; it can also be both. For example, in Figure 1, vertex  $a$  is a strict corner but not a twin with any vertex. Vertices  $c$ ,  $d$ , and  $e$  are pair-wise twins, but not strict corners. Vertices  $g$  and  $h$  are both strict corners and twins with one another.

**Definition 2.2.** *In a graph, a **path of strict corners** is a path  $(v_1, \dots, v_k)$  containing at least one vertex such that each  $v_i$  is a strict corner and  $v_{i+1}$  strictly corners  $v_i$ .*

If  $v$  is a strict corner, then the trivial path  $(v)$  counts as a path of strict corners. Also note that if  $x$  is strictly cornered by  $y$  and  $y$  is strictly cornered by  $z$ , then  $x$  is strictly cornered by  $z$ ; thus in a path of strict corners  $(v_1, \dots, v_k)$ , if  $i < j$  then  $v_i$  is strictly cornered by  $v_j$ .

**Lemma 2.3.** *In a graph, any strict corner  $v$  is strictly cornered by some vertex  $w$ , where  $w$  is not a strict corner.*

*Proof.* Let  $v$  be a strict corner and let  $(v_1, \dots, v_k)$  be a maximal path of strict corners such that  $v_1 = v$ . As observed above,  $v_k$  strictly corners  $v_i$  if  $i < k$ . Thus for all  $i < k$ ,  $v_k$  has at least one neighbor not adjacent to  $v_i$ , and so  $v_i$  does not corner  $v_k$ . Since  $v_k$  is a strict corner, it must be strictly cornered by some vertex  $w \notin \{v_1, \dots, v_k\}$ . By the maximality of the path,  $w$  is *not* a strict corner. Since  $v$  is strictly cornered by  $v_k$  and  $v_k$  is strictly cornered by  $w$ ,  $v$  is strictly cornered by  $w$ , which is not a strict corner.  $\square$

We now define the cornering ranking procedure, which can be thought of as an alternative to cop-win ordering where all strict corners are deleted at once in each step. Every successive group of strict corners is assigned the next positive integer rank. The procedure stops when it arrives at a graph with no strict corners. If it terminates with a clique, the vertices are assigned the next positive integer; otherwise the remaining vertices are assigned the value  $\infty$ .

**Definition 2.4 (Corner Ranking Procedure).** *For any graph  $\mathbf{G}$ , we define a corresponding **corner rank** function,  $\mathbf{cr}$ , which maps each vertex of  $\mathbf{G}$  to a positive integer or  $\infty$ . We also define a sequence of associated graphs  $\mathbf{G}^{[1]}, \dots, \mathbf{G}^{[\alpha]}$ .*

- (0) Initialize  $\mathbf{G}^{[1]} = \mathbf{G}$ , and  $k = 1$ .
- (1) If  $\mathbf{G}^{[k]}$  is a clique, then:
  - Let  $\mathbf{cr}(x) = k$  for all  $x \in \mathbf{G}^{[k]}$ .
  - Then **stop**.
- (2) Else if  $\mathbf{G}^{[k]}$  is not a clique and has **no** strict corners, then:
  - Let  $\mathbf{cr}(x) = \infty$  for all  $x \in \mathbf{G}^{[k]}$ .
  - Then **stop**.
- (3) Else:
  - Let  $V$  be the set of strict corners in  $\mathbf{G}^{[k]}$ .
  - For all  $x \in V$ , let  $\mathbf{cr}(x) = k$ .
  - Let  $\mathbf{G}^{[k+1]} = \mathbf{G}^{[k]} - V$
  - Increment  $k$  by 1 and return to Step 1.

Define the **corner rank** of  $\mathbf{G}$ , denoted  $\mathbf{cr}(\mathbf{G})$ , to be the same as the vertex of  $\mathbf{G}$  with largest corner rank; we understand  $\infty$  to be larger than all integers.

Lemma 2.3 implies that every graph that contains a strict corner also contains at least one vertex that is *not* a strict corner, so every graph  $\mathbf{G}^{[k]}$  produced in the corner ranking procedure will be nonempty.

As an example, we apply the corner ranking procedure to the graph in Figure 1. The procedure begins by assigning the strict corners  $a$ ,  $g$ , and  $h$  corner rank 1. After these vertices are removed,  $b$  and  $f$  are strict corners in  $\mathbf{G}^{[2]}$ , and are thus assigned corner rank 2. After these are removed, the remaining vertices,  $c$ ,  $d$ , and  $e$ , form a clique and so are assigned corner rank 3; thus the corner rank of the graph is 3. As another example, consider Figure 2. While  $\mathbf{cr}(x) = 1$  and  $\mathbf{cr}(y) = 2$ , once  $x$  and  $y$  have been removed there are no strict corners, and what remains is not a clique, so the other 5 vertices have corner rank  $\infty$ ; thus the graph has corner rank  $\infty$ . We will prove that a graph is cop-win if and only if it



*Proof.* Let  $\mathbf{G}$  be a graph with finite corner rank  $\alpha$ . It suffices to show that if there is some vertex  $x$  of corner rank  $\alpha$  dominating  $V(\mathbf{G}^{[\alpha-1]})$ , then every vertex of rank  $\alpha$  dominates  $V(\mathbf{G}^{[\alpha-1]})$ . Assume for contradiction that  $y$  is a vertex of rank  $\alpha$  that does not dominate  $V(\mathbf{G}^{[\alpha-1]})$ . Then  $x$  strictly corners  $y$  in  $\mathbf{G}^{[\alpha-1]}$ , contradicting the assumption that  $\mathbf{cr}(y) = \alpha$ .  $\square$

In [3], corner ranking is also defined for cop-win graphs; they call it a *cop-win partition*. One difference is that we define vertices with corner rank  $\infty$ . In Section 5, we show that the robber's Higher Way strategy allows him to safely remain among the corner rank  $\infty$  vertices indefinitely. Another difference is that our procedure appears to be simpler. We simply remove all the strict corners at each step. In [3], they essentially carry out two actions in each step: First they collapse any vertices that are currently twins to a single vertex, and then they remove all the corners.

### 3. PROJECTIONS

A key technical point will be to deal with projections of cop or robber locations rather than their actual locations. First, we introduce some definitions that make our treatment of finite and infinite corner rank graphs uniform.

**Definition 3.1.** *Suppose we are considering a graph with corner rank  $\infty$ .*

- *For any finite corner rank  $k$  exhibited by this graph, let  $k < \infty$ .*
- *If there is a vertex of finite corner rank, then there is a largest finite corner rank, say  $\gamma$ . In this case, we let  $\infty - 1$  refer to  $\gamma$  and let  $\gamma + 1$  refer to  $\infty$ .*

For example, if a graph has vertices with corner ranks 1, 2, and  $\infty$ , and nothing else, then  $1, 2 < \infty$ . Furthermore,  $2 + 1 = \infty$  and  $\infty - 1 = 2$ .

Now we define the projection functions, relative to the corner ranking. A similar definition appears in [3], which is in turn a modification of definitions from [4]. We write  $f : \mathbf{H} \rightarrow \mathbf{G}$  to mean that  $f$  is a function whose domain is the **non-empty subsets** of  $V(\mathbf{H})$  and whose codomain is the **non-empty subsets** of  $V(\mathbf{G})$ .

**Definition 3.2.** *Suppose  $\mathbf{G}$  is a graph with corner rank  $\alpha$ . We define the functions  $f_1, \dots, f_{\alpha-1}$  and  $F_1, \dots, F_{\alpha-1}, F_\alpha$ , where  $f_k : \mathbf{G}^{[k]} \rightarrow \mathbf{G}^{[k+1]}$  and  $F_k : \mathbf{G} \rightarrow \mathbf{G}^{[k]}$ .*

- *For a single vertex  $u \in V(\mathbf{G}^{[k]})$ , define:*

$$f_k(\{u\}) = \begin{cases} \{u\} & \text{if } \mathbf{cr}(u) > k \\ \text{the set of vertices in } \mathbf{G}^{[k+1]} \text{ that strictly corner } u \text{ in } \mathbf{G}^{[k]} & \text{otherwise.} \end{cases}$$

- $f_k(\{u_1, \dots, u_t\}) = \bigcup_{1 \leq i \leq t} f_k(u_i)$
- *If  $\mathbf{G}$  has some finite rank vertex, then:*
  - *Let  $F_1 : \mathbf{G} \rightarrow \mathbf{G}$  be the identity function, and*
  - *For  $1 < k \leq \infty$ , let  $F_k = f_{k-1} \circ \dots \circ f_1$ .*
- *If  $\mathbf{G}$  has no finite rank vertices, then let  $F_\infty : \mathbf{G} \rightarrow \mathbf{G}$  be the identity function.*

For a function  $h$  whose domain is sets of vertices, we adopt the usual convention that  $h(u) = h(\{u\})$  for a single vertex  $u$ . We remark that by Lemma 2.3 the functions  $f_k$  are guaranteed

to have *non-empty* sets for values. We say  $v$  is a  $k$ -**projection** (or simply a **projection**) of  $w$  if  $v \in F_k(w)$ .

**Definition 3.3.** *Given two graphs  $\mathbf{H}$  and  $\mathbf{G}$ , and a function  $h : V(\mathbf{H}) \rightarrow V(\mathbf{G})$ , we say that  $h$  is a **homomorphism** if for vertices  $u, v \in V(\mathbf{H})$  and vertices  $u^* \in h(u), v^* \in h(v)$ :*

$$u \text{ is adjacent to } v \text{ implies } u^* \text{ is adjacent to } v^*.$$

**Lemma 3.4.** *Given a graph  $\mathbf{G}$ , its associated functions  $f_k$  and  $F_k$  are homomorphisms.*

*Proof.* Since the identity function is a homomorphism, for a graph with some vertices of finite corner rank,  $F_1$  is a homomorphism, as is  $F_\infty$  on graphs with no vertices of finite corner rank. We show that each  $f_k$  is a homomorphism, which implies all other  $F_k$ 's are homomorphisms because the property is preserved by composition. Suppose  $u, v \in V(\mathbf{G}^{[k]})$  are distinct and adjacent, and let  $u^* \in f_k(u)$  and  $v^* \in f_k(v)$ ; we show that  $u^*$  and  $v^*$  are adjacent. Note that even if  $u^* = v^*$ , the argument works since our graphs are reflexive; in particular, to say that  $w$  corners  $v$ , simply amounts to saying that  $w$  is adjacent to everything  $v$  is adjacent to (this includes  $w$  being adjacent to itself and to  $v$ ).

**Case 1:** Suppose  $u, v \in V(\mathbf{G}^{[k+1]})$ . Then  $f_k(v) = \{v\}$  and  $f_k(u) = \{u\}$ , and so  $u^* = u$  and  $v^* = v$  are adjacent.

**Case 2:** Suppose  $v \in V(\mathbf{G}^{[k+1]})$ ,  $u \notin V(\mathbf{G}^{[k+1]})$ . So  $v^* = v$  and  $u^* \neq u$ . Since  $u^*$  strictly corners  $u$  in  $\mathbf{G}^{[k]}$ ,  $u^*$  is adjacent to  $v$ , and thus  $u^*$  and  $v^*$  are adjacent.

**Case 3:** Suppose  $u, v \notin V(\mathbf{G}^{[k+1]})$ . So  $v^* \neq v$  and  $u^* \neq u$ . Since  $u^*$  strictly corners  $u$  in  $\mathbf{G}^{[k]}$ ,  $u^*$  is adjacent to both  $u$  and  $v$ . Since  $v^*$  strictly corners  $v$  in  $\mathbf{G}^{[k]}$ , and  $v$  is adjacent to  $u^*$ , we have that  $v^*$  is also adjacent to  $u^*$ .  $\square$

We state an important fact that follows from the properties of the projection functions.

**Definition 3.5.** *For a vertex  $v$  in a graph, let  $N_k[v]$  denote the set of vertices in  $N[v]$  of rank at least  $k$ .*

**Lemma 3.6.** *Let  $c$  be a vertex in a graph, and  $c' \in F_k(c)$ . Then  $N_k[c] \subseteq N_k[c']$ .*

*Proof.* Consider a vertex  $w \in N_k[c]$ . Since  $F_k$  is a homomorphism, any vertex in  $F_k(w)$  is adjacent to  $c'$ , but since  $w$  has rank at least  $k$ ,  $F_k(w) = \{w\}$ , so  $w \in N_k[c']$ .  $\square$

#### 4. COP STRATEGY: LOWER WAY

We describe a class of winning strategies for the cop, which allows us to prove that any graph with a finite corner rank is cop-win, and yields an upper bound on the capture time. These strategies will turn out to be optimal for the cop and will be called *Lower Way* since they (roughly) have the property that as the game proceeds, the cop forces the robber to occupy lower-ranked vertices.

**Definition 4.1.** *Suppose  $c$  and  $r$  are vertices in a graph  $\mathbf{G}$ . We say that  $r$  is **0-cornered** by  $c$  if  $c = r$ . For  $k > 0$ , we say that  $r$  is  **$k$ -cornered** by  $c$  if there is some  $r' \in F_k(r)$  such that  $c$  corners  $r'$  in  $\mathbf{G}^{[k]} \cup c$ .*

We do *not* require strict cornering in this definition. Note that 1-cornered means the same thing as cornered.

**Definition 4.2.** Consider a graph with vertices  $c$  and  $r$ , and an integer  $k \geq 1$ . We say that  $c$  has  **$k$ -caught**  $r$  if  $c \in F_k(r)$ .

Notice that 1-caught means the same thing as caught.

When a cop has actually cornered the robber, then whatever the robber does, the cop can catch him on the next move. The next lemma is a version of that fact for projections.

**Lemma 4.3.** For all integers  $k \geq 1$ , if the cop has  $k$ -cornered the robber, then whatever move the robber makes, the cop can  $k$ -catch the robber on her next move.

*Proof.* Given a graph  $\mathbf{G}$ , assume the cop is at some vertex  $c$  and the robber is at some vertex  $r_0$  such that for some  $r'_0 \in F_k(r_0)$ ,  $c$  corners  $r'_0$  in  $\mathbf{G}^{[k]} \cup c$ . Now suppose the robber moves from  $r_0$  to  $r_1$ ; let  $r'_1 \in F_k(r_1)$ . Since  $r_0$  and  $r_1$  are adjacent and  $F_k$  is a homomorphism,  $r'_0$  and  $r'_1$  must also be adjacent. Since  $c$  corners  $r'_0$  in  $\mathbf{G}^{[k]} \cup c$ ,  $c$  must be adjacent to  $r'_1$ , so the cop can move to  $r'_1$  and thus  $k$ -catch the robber.  $\square$

**Lemma 4.4.** For all integers  $k \geq 1$ , if the cop has  $k$ -caught the robber, then the cop has  $(k - 1)$ -cornered the robber.

*Proof.* If  $k = 1$ , then the cop has caught the robber and thus the robber is 0-cornered. Assume  $k > 1$ . Suppose the cop is at  $c$  and the robber is at  $r$ . Since  $c \in F_k(r) = f_{k-1} \circ F_{k-1}(r)$ , either  $c \in F_{k-1}(r)$  or there is an  $r' \in F_{k-1}(r)$  such that  $c$  strictly corners  $r'$  in  $\mathbf{G}^{[k-1]}$ . In either case, the robber is  $(k - 1)$ -cornered.  $\square$

Lemmas 4.3 and 4.4 yield the following corollaries.

**Corollary 4.5.** For all integers  $k \geq 1$ , if the cop has  $k$ -cornered the robber, then whatever move the robber makes, the cop can  $(k - 1)$ -corner the robber on her next move.

**Corollary 4.6.** For all integers  $k \geq 2$ , if the cop has  $k$ -caught the robber, then whatever move the robber makes, the cop can  $(k - 1)$ -catch the robber on her next move.

To define Lower Way strategies, we need to describe the starting positions for the cop.

**Definition 4.7.** Consider an  $r$ -cop-win graph  $\mathbf{G}$  of finite corner rank  $\alpha$ . An initial placement by the cop is called **standard** if that vertex dominates  $V(\mathbf{G}^{[\alpha-r]})$ .

In both 0-cop-win and 1-cop-win graphs, any vertex of rank  $\alpha$  is a standard initial placement, though there may be more. Furthermore, Lemmas 5.3 and 3.6 imply: in a 0-cop-win graph, while the cop can dominate  $V(\mathbf{G}^{[\alpha]})$  (i.e. the top corner rank vertices), the cop cannot dominate  $V(\mathbf{G}^{[\alpha-1]})$  (i.e. the top two corner rank vertices), and in a 1-cop-win graph, while the cop can dominate  $V(\mathbf{G}^{[\alpha-1]})$ , the cop cannot dominate  $V(\mathbf{G}^{[\alpha-2]})$  (i.e. the top three corner rank vertices).

**Definition 4.8 (Lower Way and Catching Strategies).** Consider an  $r$ -cop-win graph with finite corner rank  $\alpha$ .

- A cop strategy is called **Lower Way** if the cop's initial placement is standard, and it satisfies the condition LW.

**LW:** For  $t \geq 1$ , after  $t$  cop moves, the cop  $k$ -corners the robber for some  $k \leq \alpha - r - t$ .



- A cop strategy is called **Catching** if the cop's initial placement is standard, and it satisfies the condition C.  
**C:** For  $t \geq 1$ , after  $t$  cop moves, the cop has  $k$ -caught the robber for some  $k \leq \alpha - r - t + 1$ .

A simple special case of the Lower Way strategy is as above, with condition LW replaced by: *The cop moves to  $k$ -corner the robber for  $k$  minimum.* Similarly, a simple special case of the Catching strategy replaces condition C by: *The cop moves to  $k$ -catch the robber for  $k$  minimum.* The Catching strategies are basically described in [3] (technically the Catching strategies subsume their strategies, since in [3] they simply have the cop start in  $\mathbf{G}^{[\alpha]}$ ).

The next two theorems follow immediately from repeated application of Corollary 4.5 or Corollary 4.6.

**Theorem 4.9.** *If the cop has  $k$ -cornered the robber or  $(k + 1)$ -caught the robber, then the cop needs at most  $k$  more moves to win.*

**Theorem 4.10.**

- If the cop ever satisfies the **LW** condition, then she can continue to satisfy the **LW** condition for the rest of the game.
- If the cop ever satisfies the **C** condition, then she can continue to satisfy the **C** condition for the rest of the game.

**Lemma 4.11.** *Any Catching strategy is also a Lower Way strategy.*

*Proof.* Suppose the cop plays a Catching strategy. Then she will start in a standard position, consistent with a Lower Way strategy. For all  $t \geq 1$ , after  $t$  cop moves, the cop has  $k$ -caught the robber for some  $k \leq \alpha - r - t + 1$ . Thus by Lemma 4.4, for all  $t \geq 1$ , after  $t$  cop moves, the cop has  $k$ -cornered the robber for some  $k \leq \alpha - r - t$ , and thus the strategy is also a Lower Way strategy.  $\square$

We will discuss some of the differences between Lower Way and Catching strategies in Section 7. In particular, in Theorem 7.1 we show that there are Lower Way strategies that are not Catching strategies.

**Theorem 4.12.** *Consider an  $r$ -cop-win graph of finite corner rank  $\alpha$ . There exist Lower Way and Catching strategies. Furthermore, if the cop follows a Lower Way strategy or a Catching strategy, then the cop will win in at most  $\alpha - r$  moves.*

*Proof.* By Lemma 4.11 to show that a Lower Way strategy exists, it suffices to show that a Catching strategy exists. We choose to have the cop start at a vertex of corner rank  $\alpha$ ; this is a standard start. If the graph is 0-cop-win, then since the cop dominates  $V(\mathbf{G}^{[\alpha]})$ , no matter where the robber's initial placement is, on her first move, she can  $\alpha$ -catch him. If the graph is 1-cop-win, then since the cop dominates  $V(\mathbf{G}^{[\alpha-1]})$ , no matter where the robber's initial placement is, on her first move, she can  $(\alpha - 1)$ -catch him. Either way, after one move, the cop satisfies the **C** condition; thus by Theorem 4.10, the cop can continue to satisfy the **C** condition. So Catching strategies exist, and thus so do Lower Way strategies.

The upper bounds follow immediately from the definitions of the strategies. If the cop plays a Lower Way strategy, the cop will 0-corner (and thus catch) the robber after  $\alpha - r$

cop moves. For a Catching strategy, the cop will 1-catch (and thus catch) the robber after  $\alpha - r$  cop moves.  $\square$

Thus we can derive an important upper bound and justify our use of the terminology  $r$ -cop-win as only referring to cop-win graphs.

**Corollary 4.13.** *Suppose  $\mathbf{G}$  is an  $r$ -cop-win graph with finite corner rank  $\alpha$ . Then  $\mathbf{G}$  is cop-win and  $\text{capt}(\mathbf{G}) \leq \alpha - r$ .*

## 5. ROBBER STRATEGY: HIGHER WAY

In this section we will prove lower bounds that match the upper bounds from Corollary 4.13. Just as the upper bound of the last section was a result of an explicit class of cop strategies, the lower bound of this section will follow from an explicit class of robber strategies we call the *Higher Way* strategies. At the end of the section we will contrast our approach to that of [3].

**Definition 5.1.** *Suppose the cop is at vertex  $c$  in a graph, and  $k \geq 1$ .*

- *We say that vertex  $r$  is  $k$ -safe if  $\text{cr}(r) \geq k$  and  $c$  is not adjacent to  $r$ .*
- *We say that vertex  $r$  is  $k$ -proj-safe if  $\text{cr}(r) \geq k$  and there is  $c' \in F_k(c)$  such that  $c'$  is not adjacent to  $r$ .*

Roughly, a  $k$ -proj-safe robber is safe from the  $k$ -projection of the cop, while a  $k$ -safe robber is safe from the cop herself. Lemma 3.6 implies that if  $r$  is  $k$ -proj-safe, then  $r$  is  $k$ -safe; however the converse need not be true.

**Definition 5.2.** *We say that a robber strategy is **Higher Way** if at every turn (including the initial placement), the robber goes to a  $k$ -proj-safe vertex for  $k$  maximum; if there is no such option, he moves arbitrarily to a vertex not occupied by the cop.*

By our convention, any graph has rank at least two, and so has more than 1 vertex. For a graph with more than one vertex it is immediate that Higher Way strategies exist, because the robber can start at an unoccupied vertex and during play can always stay still if there is no other option.

The next lemma will be used by the robber to start safely at a sufficiently high rank vertex.

**Lemma 5.3.** *Suppose a graph  $\mathbf{G}$  has rank  $\alpha$ ; the cop can be anywhere.*

- (1) *If the graph is 0-cop-win, then there is a  $(\alpha - 1)$ -proj-safe vertex.*
- (2) *If the graph is 1-cop-win and  $\alpha > 2$ , then there is a  $(\alpha - 2)$ -proj-safe vertex.*
- (3) *If the graph has rank  $\infty$ , then there is an  $\infty$ -proj-safe vertex.*

*Proof.* Suppose the cop is at vertex  $c$ . It suffices to show that in Case 1 (resp. Case 2, Case 3) there is no vertex in  $\mathbf{G}^{[\alpha-1]}$  (resp.  $\mathbf{G}^{[\alpha-2]}$ ,  $\mathbf{G}^{[\infty]}$ ) that dominates  $V(\mathbf{G}^{[\alpha-1]})$  (resp.  $V(\mathbf{G}^{[\alpha-2]})$ ,  $V(\mathbf{G}^{[\infty]})$ ).

- (1) Suppose  $\mathbf{G}$  is 0-cop-win, and for the sake of contradiction, assume some vertex in  $\mathbf{G}^{[\alpha-1]}$  dominates  $V(\mathbf{G}^{[\alpha-1]})$ . Such a vertex could not be strictly cornered in  $\mathbf{G}^{[\alpha-1]}$  so it would have rank  $\alpha$ , making  $\mathbf{G}$  1-cop-win, a contradiction.

- (2) Suppose  $\mathbf{G}$  is 1-cop-win, and for the sake of contradiction, assume some vertex  $v$  in  $\mathbf{G}^{[\alpha-2]}$  dominates  $V(\mathbf{G}^{[\alpha-2]})$ . Any vertex not strictly cornered by  $v$  in  $\mathbf{G}^{[\alpha-2]}$  must be a twin of  $v$  in  $\mathbf{G}^{[\alpha-2]}$ .

In  $\mathbf{G}^{[\alpha-2]}$ ,  $v$  cannot strictly corner any vertex in  $\mathbf{G}^{[\alpha-1]}$ , so all the vertices of  $\mathbf{G}^{[\alpha-1]}$  are twins with  $v$ , and so they are twins with one another; i.e.  $\mathbf{G}^{[\alpha-1]}$  is a clique. But  $\mathbf{G}^{[\alpha-1]}$  being a clique contradicts the definition of corner rank.

- (3) Suppose  $\mathbf{G}$  has rank  $\infty$ , and for the sake of contradiction, assume some vertex  $v$  in  $\mathbf{G}^{[\infty]}$  dominates  $V(\mathbf{G}^{[\infty]})$ . Since none of the other vertices in  $\mathbf{G}^{[\infty]}$  are strictly cornered by  $v$ , all vertices in  $\mathbf{G}^{[\infty]}$  must be twins. Hence  $\mathbf{G}^{[\infty]}$  is a clique, contradicting the definition of corner rank. □

The next lemma shows how the robber can avoid being adjacent to a projection of the cop (and hence not adjacent to the cop) while moving down at most one corner rank on each turn.

**Lemma 5.4.** *Suppose the cop is at some vertex  $c_0$  and the robber is at a  $k$ -proj-safe vertex for  $k \geq 2$ . Suppose the cop moves from  $c_0$  to  $c_1$ .*

- (1) *The robber has a move to a  $(k-1)$ -proj-safe vertex.*
- (2) *Furthermore, if there is a  $c'_1 \in F_k(c_1)$  such that the robber is not cornered by  $c'$  in  $\mathbf{G}^{[k]}$ , then the robber has a move to a  $k$ -proj-safe vertex.*

*Proof.* For the first part, suppose the robber is at  $r_0$ . Since  $r_0$  is  $k$ -proj-safe from  $c_0$ , there exists  $c'_0 \in F_k(c_0)$  such that  $c'_0$  is not adjacent to  $r_0$ . Then the cop moves to  $c_1$ . Assume for contradiction that from  $r_0$  the robber does not have a move to a  $(k-1)$ -proj-safe vertex. For  $r_0$  not to have such a move, means that for any  $r_1 \in \mathbf{N}_{k-1}[r_0]$ , for all  $c'_1 \in F_{k-1}(c_1)$ ,  $c'_1$  is adjacent to  $r_1$ , i.e. all  $c'_1 \in F_{k-1}(c_1)$  corner  $r_0$  in  $\mathbf{G}^{[k-1]}$ . Consider any such  $c'_1$  which corners  $r_0$  in  $\mathbf{G}^{[k-1]}$ . Since  $r_0$  has rank  $\geq k$ , this cornering cannot be strict, and thus  $c'_1$  and  $r_0$  are twins in  $\mathbf{G}^{[k-1]}$ . Since  $r_0$  has rank  $\geq k$ ,  $c'_1$  must also have rank  $\geq k$ . This implies that  $f_{k-1}(c'_1) = \{c'_1\}$  and so  $c'_1 \in F_k(c_1) = f_{k-1} \circ F_{k-1}(c_1)$ . However since  $F_k$  is a homomorphism and  $c_1$  is adjacent to  $c_0$ ,  $c'_1$  is adjacent to  $c'_0$ . Since  $r_0$  is not adjacent to  $c'_0$ , this contradicts the fact that  $c'_1$  and  $r_0$  are twins in  $\mathbf{G}^{[k-1]}$ .

For the second part, note that there is a vertex of rank at least  $k$  adjacent to  $r$  that is not adjacent to  $c'_1$ , giving the robber a  $k$ -proj-safe move. □

**Theorem 5.5.** *Suppose  $\mathbf{G}$  is a graph of corner rank  $\alpha$  and the robber follows a Higher Way strategy.*

- (1) *If  $\mathbf{G}$  is  $r$ -cop-win, the game will last at least  $\alpha - r$  cop moves.*
- (2) *If  $\alpha = \infty$ , then the cop will never catch the robber.*

*Proof.* From Lemma 5.3 we conclude that in Case 1, if  $r = 0$ , the robber will start  $(\alpha - 1)$ -proj-safe, and if  $r = 1$ , the robber will start  $(\alpha - 2)$ -proj-safe (unless  $\mathbf{G}$  has rank 2, in which case he will simply avoid placing himself at the cop's location). In Case 2, the robber will start  $\infty$ -safe. We now apply Lemma 5.4 repeatedly in all cases.

**Case 1.**

Suppose the robber is  $k$ -proj-safe for some  $k \geq 2$ . Since the robber is  $k$ -proj-safe, the cop is not adjacent to the robber and by Lemma 5.4 whatever the cop does, the robber can respond by moving to a vertex that is at worst  $(k - 1)$ -proj-safe. Lemma 5.4 can be repeatedly applied until the robber is forced to move to a 1-proj-safe vertex. At this point, the cop is not adjacent to the robber, so it takes at least 2 more cop moves to catch the robber.

Now we calculate lower bounds on the number of cop moves. In the 0-cop-win case, the robber starts at rank at least  $\alpha - 1$  and survives for at least  $\alpha - 2$  cop moves before reaching rank 1; the additional 2 cop moves means that at least  $(\alpha - 2) + 2 = \alpha$  cop moves are needed. The 1-cop-win case is similar for  $\alpha > 2$ , but with the robber starting at a vertex of rank at least  $\alpha - 2$ , so we get at least  $(\alpha - 3) + 2 = \alpha - 1$  cop moves.

If  $\alpha = 2$  and the graph is 1-cop-win, then the robber initially goes to a vertex not occupied by the cop and thus the cop will make at least one move, as required.

**Case 2.**

Suppose the cop's initial position is  $c_0$ , and the robber's initial position is at the  $\infty$ -proj-safe vertex  $r_0$ , i.e.  $r_0$  is not adjacent to some  $c'_0 \in F_\infty(c_0)$ . Suppose the cop moves to  $c_1$ . Assume for contradiction that there is no move from  $r_0$  to a  $\infty$ -proj-safe vertex. That means that for any  $r_1 \in \mathbf{N}_\infty[r_0]$ , for all  $c'_1 \in F_\infty(c_1)$ ,  $c'_1$  is adjacent to  $r_1$ , i.e. all  $c'_1 \in F_\infty(c_1)$  corner  $r_0$  in  $\mathbf{G}^{[\infty]}$ .

Since strict cornering is not possible in  $\mathbf{G}^{[\infty]}$ , this means that  $r_0$  is twins with all  $c'_1 \in F_\infty(c_1)$ . But since any  $c'_1$  is adjacent to  $c'_0$  (because  $c_0$  is adjacent to  $c_1$ , and  $F_\infty$  is a homomorphism), the vertex  $r_0$  is adjacent to  $c'_0$ , a contradiction.

□

**Corollary 5.6.** *If an  $r$ -cop-win graph has finite corner rank  $\alpha$ , then  $\text{capt}(\mathbf{G}) \geq \alpha - r$ .*

We compare our approach to that of [3]. In [3], the lower bound of Corollary 5.6 is proven in a natural way: by induction on the corner rank. Our proof relies on the Higher Way strategy, an explicit description of a robber strategy. As a result of our approach we get some benefits: We have an explicit robber strategy, which not only delays the game as long as possible on cop-win graphs, but wins on non cop-win graphs.

## 6. MAIN RESULTS

We can now put together the facts from the previous two sections to prove Theorem 6.1, one of our main theorems (also appears in [3]). We can then deduce that the Lower Way and Higher Way strategies are in fact optimal (in Theorem 6.2). We also give an alternate proof of the standard theorem characterizing cop-win graphs by dismantling orderings [6, 8]. Unlike [3], we do not use the dismantling ordering characterization in our development, so we can in fact derive the result as a corollary of our work.

**Theorem 6.1.** *A graph is cop-win if and only if it has finite corner rank. Furthermore, for an  $r$ -cop-win graph of finite corner rank  $\alpha$ ,  $\text{capt}(\mathbf{G}) = \alpha - r$ .*

*Proof.* For the claim that a graph is cop-win if and only if it has finite corner rank, the backward direction is noted in Corollary 4.13, while the forward direction follows from Theorem 5.5 (Part 3): if the graph does not have finite corner rank, the robber has a winning strategy. Furthermore, when the corner rank is finite, the upper bound on capture time in Corollary 4.13 matches the lower bound in Corollary 5.6, so these corollaries together determine the exact capture time.  $\square$

Since the Lower Way and Higher Way strategies both achieve the capture time in Theorem 6.1, the following is an immediate corollary.

**Theorem 6.2.** *On a graph with finite corner rank, any Lower Way strategy is optimal for the cop, and any Higher Way strategy is optimal for the robber.*

We now develop some ideas that we use to show how the dismantling ordering characterization [6, 8] follows from our work.

**Definition 6.3.** *Suppose  $\mathbf{G}$  is a graph and  $\bar{u} = (u_1, \dots, u_n)$  is an ordering of its vertices.*

- *By  $\mathbf{G}_{\bar{u}}^{(u_i)}$  we mean the graph induced by  $\{u_i, \dots, u_n\}$ .*
- *We write  $u_i \prec u_j$  when  $i < j$ .*
- *The pair  $(\mathbf{G}, \bar{u})$  is **good** for  $u_i$  if there is some  $u_j$  such that  $u_i \prec u_j$  and  $u_j$  corners  $u_i$  in  $\mathbf{G}_{\bar{u}}^{(u_i)}$ .*

Observe that to show that an ordering  $\bar{u}$  is a dismantling ordering for  $\mathbf{G}$  amounts to showing that for each  $i$  ranging from 1 to  $n - 1$ ,  $(\mathbf{G}, \bar{u})$  is good for  $u_i$ . If the ordering  $\bar{u}$  is apparent from context, we may leave off the subscript  $\bar{u}$ . We prove a small fact about dismantling orderings.

**Lemma 6.4.** *Suppose a graph  $\mathbf{G}$  has a dismantling ordering  $(v_1, \dots, v_n)$  such that some  $v_k$  corners  $v_j$  in  $\mathbf{G}^{(v_{j-1})}$ , where  $1 < j < k$ . Then swapping the positions of  $v_j$  and  $v_{j-1}$  in  $(v_1, \dots, v_n)$  and leaving the rest of the vertices in order, is another dismantling ordering of  $\mathbf{G}$ .*

*Proof.* Let  $\bar{\alpha} = (v_1, \dots, v_n)$ , and let  $\bar{\beta}$  refer to the new ordering:

$$(v_1, \dots, v_{j-2}, v_j, v_{j-1}, v_{j+1}, \dots, v_k, \dots, v_n).$$

For all  $i$  except  $j - 1$  and  $j$ ,  $(\mathbf{G}, \bar{\beta})$  is good for  $v_i$  since  $\mathbf{G}_{\bar{\alpha}}^{(v_i)}$  is the same graph as  $\mathbf{G}_{\bar{\beta}}^{(v_i)}$ , and we are given that  $\bar{\alpha}$  is a dismantling ordering.

- To see that  $(\mathbf{G}, \bar{\beta})$  is good for  $v_j$ , notice that  $\mathbf{G}_{\bar{\alpha}}^{(v_{j-1})}$  and  $\mathbf{G}_{\bar{\beta}}^{(v_j)}$  are the same graphs, so since  $v_k$  corners  $v_j$  in the  $\mathbf{G}_{\bar{\alpha}}^{(v_{j-1})}$ ,  $v_k$  still corners  $v_j$  in  $\mathbf{G}_{\bar{\beta}}^{(v_j)}$ .
- To see that  $(\mathbf{G}, \bar{\beta})$  is good for  $v_{j-1}$ , consider cases on what corners  $v_{j-1}$  in  $\mathbf{G}_{\bar{\alpha}}^{(v_{j-1})}$ :
  - *Case:  $v_j$ .*  
Since  $v_k$  corners  $v_j$  in  $\mathbf{G}_{\bar{\alpha}}^{(v_{j-1})}$ , and  $v_j$  corners  $v_{j-1}$  in  $\mathbf{G}_{\bar{\alpha}}^{(v_{j-1})}$ ,  $v_k$  corners  $v_{j-1}$  in  $\mathbf{G}_{\bar{\alpha}}^{(v_{j-1})}$ , and thus  $v_k$  corners  $v_{j-1}$  in  $\mathbf{G}_{\bar{\beta}}^{(v_{j-1})}$ .
  - *Case: Otherwise.*  
If something other than  $v_j$  cornered  $v_{j-1}$  in  $\mathbf{G}_{\bar{\alpha}}^{(v_{j-1})}$ , then it still does in  $\mathbf{G}_{\bar{\beta}}^{(v_{j-1})}$ .

□

**Lemma 6.5.** *Any graph with a dismantling ordering has a dismantling ordering in which all its rank 1 vertices are to the left of any higher ranked vertices.*

*Proof.* Suppose  $\mathbf{G}$  is a graph, and  $(v_1, \dots, v_n)$  is some dismantling ordering. If the rank 1 vertices are already left of any higher ranked vertices we are done, so assume otherwise. Let  $v_j$  be the leftmost vertex of corner rank 1 such that  $v_{j-1}$  has corner rank at least 2. It suffices to show that there is a vertex  $v_i \prec v_j$  such that  $v_i$  has rank at least 2 and swapping  $v_i$  and  $v_j$  yields a new dismantling ordering of  $\mathbf{G}$ ; this suffices because we can repeatedly swap vertices until all the corner rank 1 vertices are on the left. Since  $v_j$  is corner rank 1, by Lemma 2.3 there is a vertex  $w$  of corner rank at least 2 that corners  $v_j$  in  $\mathbf{G}$ . Consider cases:

(1)  $v_j \prec w$ :

Then  $w$  corners  $v_j$  in  $\mathbf{G}^{(v_{j-1})}$ . Thus we apply Lemma 6.4 to swap  $v_j$  and  $v_{j-1}$ .

(2) If  $w \prec v_j$ :

Then let  $y$  be the rightmost vertex in the dismantling ordering with the following properties: 1)  $y$  has corner rank at least 2, 2)  $y \prec v_j$ , and 3)  $y$  corners  $v_j$  in  $\mathbf{G}^{(y)}$ . Such a  $y$  exists because  $w$  is a vertex with these three properties.

If  $y$  and  $v_j$  are twins in  $\mathbf{G}^{(y)}$  then they can be swapped and the result is still a dismantling ordering. Otherwise  $y$  strictly corners  $v_j$  in  $\mathbf{G}^{(y)}$ . Let  $z$  be the vertex in  $\mathbf{G}^{(y)}$  that corners  $y$  in  $\mathbf{G}^{(y)}$ . Thus  $z$  strictly corners  $v_j$  in  $\mathbf{G}^{(y)}$ , and thus also  $z \neq v_j$ . Note that since  $v_j$  was the leftmost rank 1 vertex such that  $v_{j-1}$  had rank at least 2, the vertices between  $y$  and  $v_j$  have rank at least 2. Thus if  $z$  is between  $y$  and  $v_j$  that would contradict the assumption that  $y$  is the rightmost vertex satisfying the three properties above. Since  $z \neq v_j$ , we have that  $v_j \prec z$ . Thus we apply Lemma 6.4 to swap  $v_j$  and  $v_{j-1}$ .

□

**Lemma 6.6.** *A graph  $\mathbf{G}$  has finite corner rank  $\alpha$  if and only if it has a dismantling ordering.*

*Proof.*

**Forward Direction.** If  $\mathbf{G}$  has finite corner rank then we claim that any list of the vertices where their ranks are non decreasing (i.e. the rank 1 vertices come first, in any order, followed by the rank 2 vertices in any order, and so on) is a dismantling ordering. Lemma 2.3 ensures that any vertex  $v$  with rank less than  $\alpha$  is cornered in  $\mathbf{G}$  by a vertex of higher rank, and thus will be cornered by this vertex in  $\mathbf{G}^{(v)}$ . If there is more than one vertex of rank  $\alpha$  then they all mutually corner each other, and thus any vertex  $v$  of rank  $\alpha$  is a corner in the clique  $\mathbf{G}^{(v)}$ . Thus the sequence just described is in fact a dismantling ordering.

**Backward Direction.** If all the vertices have corner rank 1 we are done, so assume otherwise. We proceed by induction on the number of vertices in  $\mathbf{G}$ . By Lemma 6.5, we can rearrange the given dismantling ordering into a dismantling ordering that has all the corner rank 1 vertices first. By the definition of a dismantling ordering, removing any number of initial vertices from the ordering leaves a dismantling ordering of the remaining graph. Removing all the initial corner rank 1 vertices results in a

dismantling ordering of the smaller graph. But this smaller graph is just  $\mathbf{G}^{[2]}$ , the graph induced by all vertices of  $\mathbf{G}$  with corner rank 2 or higher. By the inductive hypothesis,  $\mathbf{G}^{[2]}$  has finite corner rank. Now consider the corner ranking procedure applied to  $\mathbf{G}$ : we first remove the corner rank 1 vertices, leaving us with  $\mathbf{G}^{[2]}$ , and since  $\mathbf{G}^{[2]}$  has finite corner rank, the corner ranking procedure terminates by assigning finite corner rank to  $\mathbf{G}$ . □

From Theorem 6.1 and Lemma 6.6 we conclude the following.

**Theorem 6.7.** [6, 8] *A graph is cop-win if and only if it has a dismantling ordering.*

## 7. COMPARISON OF COP STRATEGIES

This section is about relationships among different classes of optimal cop strategies. In the first part, we show that for cop-win graphs in general, the Catching strategies are a strict subset of the Lower Way strategies, which are in turn, a strict subset of the set of all optimal strategies. However, in the second part, we describe a natural class of graphs on which, if the robber plays a Higher Way strategy, then all optimal strategies are Lower Way strategies. We conclude with some questions raised by these results.

### 7.1. Catching versus Lower Way versus Everything Else.

**Theorem 7.1.** *There are graphs on which there is a Lower Way strategy that is not a Catching strategy.*

*Proof.* We show that in the cop-win graph in Figure 3 there is a Lower Way strategy that is not a Catching Strategy. Suppose the cop starts in a standard position at vertex  $c_1$ . Wherever the robber places himself, the cop can 5-corner him. Suppose the robber starts at  $r_1$ , following a Higher Way strategy. In her first move, suppose the cop moves to  $c_2$ . Since the robber is now 5-cornered, the cop satisfies the **LW** condition of Definition 4.8, so by Theorem 4.10, can continue to play Lower Way. However the cop has not  $k$ -caught the robber for any  $k$ , and so this is not a valid move in a Catching strategy. □

In a Catching strategy, the cop successively catches the  $k$ -projection of the robber for smaller and smaller values of  $k$ , and thus occupies vertices of smaller rank as the game goes on. It might be natural to assume that when the cop plays a Lower Way strategy, the cop starts at a high rank vertex and similarly moves to reduce her rank on each move. But this need not be the case, and in fact there are graphs for which the cop can start at low-rank vertices or can occupy lower-ranked vertices than the robber while still maintaining condition **LW**. For example, consider how the Lower Way strategy might continue in the proof of Theorem 7.1. The robber could move to  $r_2$ , followed by a cop move to  $c_3$ , followed by a robber move to  $r_3$ . At this point, the cop could move to the rank 1 vertex  $c_4$  in order to 3-corner the robber. As another example, note that in Figure 4, a cop playing a Lower Way (or Catching) strategy can start at the rank 1 vertex  $v$ .

We have shown that every Lower Way strategy is optimal. However the converse is not true.

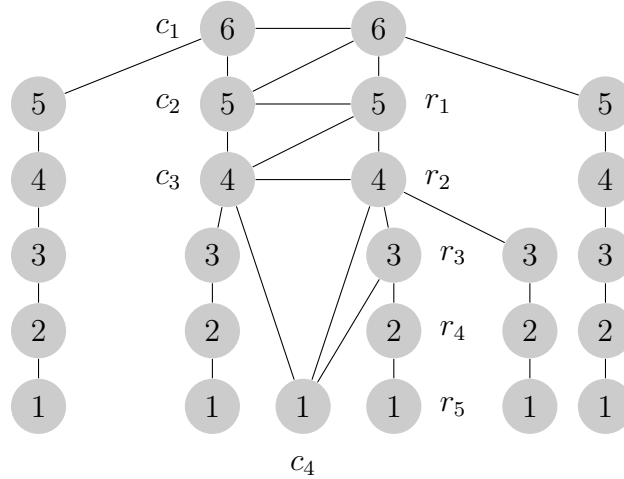


FIGURE 3. A graph with a Lower Way strategy which is not a Catching strategy. In this Lower Way strategy, the cop can move to the rank 1 vertex  $c_4$  while 3-coring the robber.

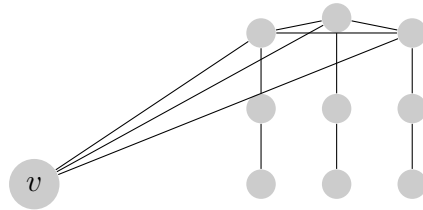


FIGURE 4. A cop-win graph with a standard starting position of rank one (vertex  $v$ ).

**Theorem 7.2.** *There are cop-win graphs with optimal strategies that are not Lower Way.*

*Proof.* Figure 5 shows a 0-cop-win graph of rank 5, which by Theorem 6.1 has capture time 5. In addition to the optimal Lower Way strategies, there is an optimal strategy for this graph where the cop starts at the rank 1 vertex labeled  $x$ . No matter where the robber starts, he will not be  $k$ -cornered for any  $k$ , but we will show that the cop has an optimal strategy nonetheless. If the robber initially chooses  $v_i$  where  $i$  is even, the cop should move to  $v_{14}$  and otherwise to  $v_{13}$ . Note that an optimal robber is still not 4-cornered, (e.g. a robber playing a Higher Way strategy would have started at  $v_3, v_4, v_5,$  or  $v_6$ ) so condition **LW** is not met. On the robber’s next move, he must move to a  $v_i$  for some value of  $i$  greater than 4. If the robber is at a vertex  $v_i$  where  $i$  is even, the cop should move to  $v_4$ , and otherwise the cop should move to  $v_3$ . From there, the cop can force a win within 3 more moves, for a total of five, and thus we have described an optimal strategy that is not Lower Way.  $\square$

**7.2. Iteratively Twin-Free Graphs.** We describe a class of cop-win graphs on which all optimal cop strategies are Lower Way strategies, when the robber plays a Higher Way



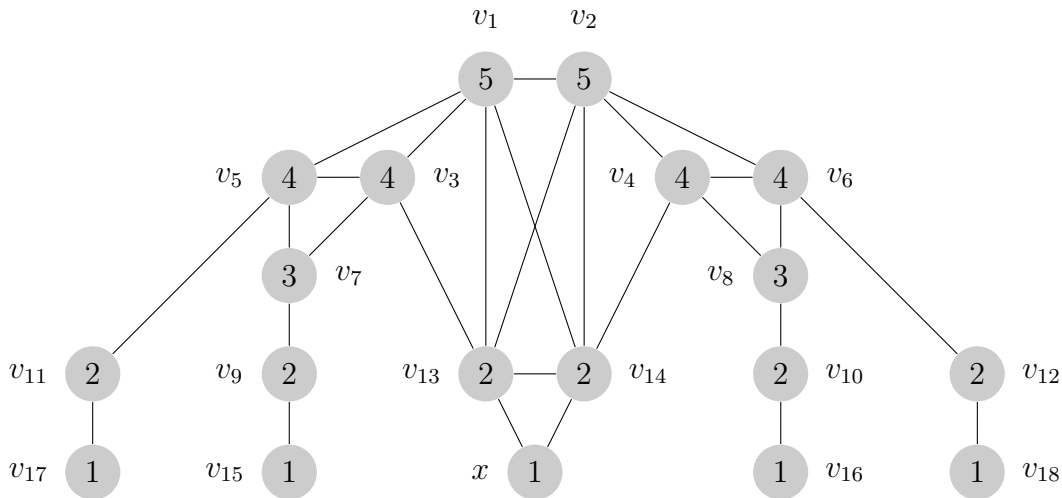


FIGURE 5. A graph with an optimal cop strategy that is not Lower Way.

strategy. Further, on such graphs no matter what strategy the robber plays, an optimal cop must strategy must begin with a standard initial placement.

**Definition 7.3.** Consider  $k \geq 2$  and a graph  $\mathbf{G}$ . We say that two vertices  $x$  and  $y$  are  **$k$ -twins** if they both have rank  $k$  and are twins in  $\mathbf{G}^{[k-1]}$ .

For example, in the graph in Figure 5,  $v_4$  and  $v_6$  are 4-twins, since they both have rank 4 and are twins in  $G_3$ .

**Definition 7.4.** We say that a graph is **iteratively twin-free** if it has finite corner rank  $\alpha$ , and there are no  $k$ -twins for  $2 \leq k \leq \alpha$ .

In other words, a graph is *iteratively twin-free* if each step of the corner ranking procedure (i.e. removing all the current strict corners), leaves a graph with no twins (except possibly at the current lowest rank). For example, the graph in figure 5 is *not* iteratively twin-free because after all the rank 1 and rank 2 vertices are removed,  $v_4$  and  $v_6$  are twins that are not at the lowest rank. Such graphs fit naturally with the definition of corner rank. They include the natural and more restrictive class of graphs:

Those finite rank graphs  $\mathbf{G}$  such that none of the graphs produced in the corner ranking procedure (i.e.  $\mathbf{G}^{[1]}, \dots, \mathbf{G}^{[\alpha]}$ ) have twins, except possibly  $\mathbf{G}^{[\alpha]}$ .

Twins cause complications in some of the development, so avoiding them is interesting. In fact, consider the similar ranking procedure in [3]: In each step, to go from  $\mathbf{G}^{[k]}$  to  $\mathbf{G}^{[k+1]}$ , involves first collapsing twins, and then removing all corners. On this restrictive class of graphs, the procedure of [3] is simplified to just removing all corners (without concern for strict versus non-strict) at each step.

The following lemma mirrors Lemma 5.4. However in this version, the fact that the graph is iteratively twin-free allows the robber to play against the cop rather than the projection of the cop, moving to  $k$ -safe vertices rather than  $k$ -proj-safe vertices.

**Lemma 7.5.** *Suppose  $\mathbf{G}$  is an iteratively twin-free graph, the cop is at some vertex  $c_0$ , and the robber is at some  $k$ -safe vertex where  $k \geq 2$ . Suppose the cop moves from  $c_0$  to  $c_1$ .*

- (1) *The robber has a move to a  $(k - 1)$ -safe vertex.*
- (2) *Furthermore, if the robber is not cornered by  $c_1$  in  $\mathbf{G}^{[k]} \cup c_1$ , then the robber has a move to a  $k$ -safe vertex.*

*Proof.* Since the robber was  $k$ -safe, the cop move to  $c_1$  did not catch him. The second part follows immediately from the definition of not being cornered. Now consider the first part. Suppose the robber is at a vertex  $r$  of rank  $k^* \geq k$ . Assume for contradiction there is no  $(k - 1)$ -safe move for the robber. Then  $c_1$  corners  $r$  in  $\mathbf{G}^{[k-1]} \cup c_1$ . Let  $c'_1 \in F_{k-1}(c_1)$ . Lemma 3.6 implies that  $\mathbf{N}_{k-1}[c_1] \subseteq \mathbf{N}_{k-1}[c'_1]$ , and thus  $c'_1$  also corners  $r$  in  $\mathbf{G}^{[k-1]}$ . Since  $r$  has rank  $k^* \geq k$ , this cornering cannot be strict, so  $c'_1$  and  $r$  must be twins in  $\mathbf{G}^{[k-1]}$ . However since  $r$  has rank  $k^*$ ,  $c'_1$  also has rank  $k^*$ , so  $r$  and  $c'_1$  are both of rank  $k^* \geq k$  and twins in  $\mathbf{G}^{[k^*-1]}$ , i.e.  $r$  and  $c'_1$  are  $k^*$ -twins, contradicting the assumption that  $\mathbf{G}$  is iteratively twin-free.  $\square$

We already know that the standard start positions can always be followed by optimal cop play. For iteratively twin-free graphs we get the converse: to be optimal the cop must start standard.

**Theorem 7.6.** *In an iteratively twin-free cop-win graph the initial placement by the cop is optimal if and only if it is standard.*

*Proof.* The backwards direction follows from the existence of Lower Way strategies, which are optimal and whose initial placement can be anything that is standard. Consider the forward direction. Suppose the cop does not start standard. In the 0-cop-win case, the robber can start  $\alpha$ -safe, and in the 1-cop-win case, the robber can start  $(\alpha - 1)$ -safe. Then by Lemma 7.5, the robber can play to reduce his rank at most 1 per turn; in either case, this prolongs the game at least one cop move more than the optimal capture time.  $\square$

On iteratively twin-free graphs, we classify all optimal cop strategies given the assumption that the robber plays a Higher Way strategy.

**Theorem 7.7.** *Suppose the robber plays a Higher Way strategy on an iteratively twin-free graph. Then a cop strategy is optimal if and only if it is Lower Way.*

*Proof.* The backward direction was noted in Theorem 6.2. Consider the forward direction, just for 0-cop-win graphs (the proof for 1-cop-win graphs is virtually identical). By Theorem 7.6, the cop must start standard, consistent with a Lower Way strategy. Suppose the graph has finite corner rank  $\alpha$ . By Lemma 5.3 (recalling that being  $k$ -proj-safe implies being  $k$ -safe), the robber will start at an  $(\alpha - 1)$ -safe vertex. By Lemma 7.5 (Part 1) the robber will move down at most one rank per a turn. Thus right before the cop makes her  $t^{\text{th}}$  move ( $t \geq 1$ ), the robber will be at some rank  $\geq \alpha - t$ . At this point, if the cop does not follow a Lower Way strategy and corner the robber in  $\mathbf{G}^{[\alpha-t]}$ , then by Lemma 7.5 (Part 2) the robber can safely remain at rank  $\geq \alpha - t$ . But now the cop has made  $t$  moves and by repeated application of Lemma 7.5 (Part 1), after  $(\alpha - t - 1)$  more cop moves, the robber

will be 1-safe, i.e. not adjacent to the cop, thus requiring at least 2 more cop moves for the cop to catch the robber. Thus by *not* following a Lower Way strategy, the cop was forced to make at least  $t + (\alpha - t - 1) + 2 = \alpha + 1$  moves, and the strategy is not optimal.  $\square$

**7.3. Conclusions.** We conclude this paper with a somewhat vague research proposal:

*Find a “nice” characterization of the optimal strategies for the cop and for the robber.*

The Catching strategies are a nice description of a number of optimal cop strategies, but not all of them. The Lower Way strategies are a nice description of a wider class of cop strategies, but still not all of them. So we have nice descriptions which are not comprehensive.

On the other hand, we can give comprehensive descriptions which are not nice. Hahn and MacGillivray [5] define a rank function (right before their Lemma 4) on the collection of all possible game positions (i.e. the pairs consisting of the cop location and the robber location); this rank builds on the “ $\leq_\gamma$ -relation” from [6]. They show that the rank of a position is equal to the number of cop moves required to win from that position. Thus, we could use their rank function to describe a class of cop strategies that is comprehensive: *The cop starts at a location that minimizes the rank over all possible robber responses; then on subsequent moves the cop chooses to reduce the rank.* An analogous approach would work for the robber strategies. However we would not call such a description “nice” because it does not give much insight into the specific game of Cops and Robber. The basic point is that most any game we care about can be described by a similar rank function and a similar class of rank-reducing optimal strategies.

Thus to be considered a nice strategy characterization, we want the description to be tied to the specific structure of the game of Cops and Robber. The notion of corner rank is tied to the specific structure of the game, which is why we find the Catching and Lower Way strategies to be nice. Thus, we can restate our proposal:

*Find a “nice” characterization, in terms of corner rank, of the optimal strategies for the cop and for the robber.*

Our hope would be to extend the definitions of the Lower Way strategies and Higher Way strategies in some nice way, so that they encompassed all the optimal strategies.

## 8. ACKNOWLEDGMENTS

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