Variations of Cops and Robber on the hypercube

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Abstract

We determine the cop number of the hypercube for different versions of the game *Cops and Robber*. Cops and Robber is a two player game played on an undirected graph. One player controls some number of cops; the other player controls a single robber. In the standard game, the cops first choose some vertices to occupy, then the robber chooses a vertex; the players then alternate turns. On a turn, the robber may stay still or move to an adjacent vertex, likewise for the cops. The cop number of a graph is the least number of cops needed to guarantee the robber will be caught. The n-dimensional hypercube (or n-cube) is the graph whose vertices are the length n binary vectors with an edge between vectors that differ at exactly one coordinate. Various authors have investigated the cop number of the n-cube for a few versions of the game. We extend the game rules by considering cops of varying capacities. We refer to a cop who must move as an active cop, and cops who may move or stay still as flexible cops. Three versions of the game have been considered: 1) All cops are flexible, 2) All cops are active, and 3) At least one cop is active. In addition to the active and flexible cops, we introduce a third kind of cop, a passive cop, which must remain still on a turn. By varying the number of flexible, active, and passive cops we consider a whole spectrum of games. We fully classify the tradeoff between active and flexible cops. Introducing passive cops dramatically increases the difficulty and interest of the problem. The most involved proof of the paper achieves only a partial result for the game involving passive cops, and suggests a number of open questions. Finally, we connect Cops and Robber to another vertex pursuit game, *Graph Searching*, and use this relationship to provide a new lower bound for the cop number of the Graph Searching game.

1 Introduction

The game of Cops and Robber was introduced independently by Quilliot [22] and Nowakowski and Winkler [20]. It is a two-player game played on an undirected, loopless, finite graph. One player controls some number of cops and the second player controls a single robber. The standard game begins with the cops choosing some vertices (multiple cops may occupy a single vertex), followed by the robber choosing a vertex. It is the cops' turn first and then play alternates between the two players. On the cops' turn any number of the cops may remain stationary, while each of the other cops moves to an adjacent vertex. On the robber's turn, he may remain stationary or move to an adjacent vertex. The cops win if any one of them "catches" the robber, that is if the cop and the robber ever occupy the same vertex. The **cop number** of a graph, introduced by Aigner and Fromme [1], is the minimum number of cops required to catch the robber on that graph. Most research on the Cops and Robber game has focused on determining the value of the cop number.

Besides providing interesting mathematical questions, the game of Cops and Robber has a number of motivating applications. Many search problems on a network can be formulated as some variant of a Cops and Robber game; for example, searching for a lost person in a network of caves or searching for a virus in a computer network. The cop number of a graph can be thought of as a measure of the ease of searching the graph. For further details on the game see the surveys by Hahn [13] and by Bonato and Nowakowski [9].

The most famous open question related to Cops and Robber is Meyniel's conjecture, made public in a 1987 paper by Frankl [11]. The conjecture states that the cop number of a graph is upper bounded by $O(\sqrt{n})$, where n is the number of vertices in the graph. Though there has been recent progress due to Scott and Sudakov [23] and Lu and Peng [15], the conjecture is still very open. While Meyniel's conjecture concerns the best possible bound for the cop number on general graphs, it is possible to do better on particular classes of graphs, and there are many results in this direction. For example, researchers have studied the cop number of planar graphs [1], random graphs [8, 16, 18, 21], product graphs [25, 19], geometric graphs [6], grid graphs [7], and many other types of graphs [4, 5, 11, 14].

In this paper, we study Cops and Robber on the hypercube, investigating how the cop number varies as the rules of the game are changed. The n-dimensional hypercube, denoted Q_n, is the graph whose vertices are the length n binary vectors, with an edge between every pair of vertices that differ in exactly one coordinate.

The game of Cops and Robber on the hypercube was first studied by Maamoun and Meyniel [17] (where the hypercube is a special case of the Cartesian Product of trees) and later by other researchers [7, 19, 10]. We were surprised (at first) to see that each of these papers claimed different values for the cop number of Q_n. The different cop numbers are explained by the variations in the rules of the game concerning which cops can move on a turn and whether the robber must move on each turn. For the standard version of the game, Maamoun and Meyniel [17] proved the cop number of Q_n is $\lceil \frac{n+1}{2} \rceil$. In [7] the cops are not allowed to choose their own starting positions, and in this case the cop number for Q_n is n, by Theorem 1 and Theorem 2 when their result is specialized from the case of the n-grid to the n-cube. In [19], where the robber must move on each turn, by Theorem 2.4, the cop number of Q_n is $\lceil \frac{n}{2} \rceil$. We introduce the following terminology to describe these and other variations. When it is the cops' turn, each cop may be in one of the three states, **active**, **passive**, or **flexible**, meaning respectively that she must move, must remain still, or has the option of moving or staying still. We may also designate the robber as active, passive, or flexible, though we will always consider an active robber, as explained in Section 2. Ignoring passive cops at first, in Sections 3 and 4 we fully classify the tradeoff between active and flexible cops, showing that having all flexible cops is the best case, with cop number $\lceil n/2 \rceil$, the worst case is having all active cops, with cop number $\lceil 2n/3 \rceil$, and every cop number between $\lceil n/2 \rceil$ and $\lceil 2n/3 \rceil$ can be achieved with an appropriate mixture of active and flexible cops. We obtain results from [7] and [19] as special cases of our Corollary 3.3.

In Section 5, we consider the game in which some fixed number of cops (say k of them) must move when it is the cops' turn (i.e. be active), and the rest must remain stationary (i.e. be passive); call the corresponding cop number $C_{k\mathbf{a}}(n)$. We find that $C_{(\lceil \frac{n}{2} \rceil - 1)\mathbf{a}}(n) = \lceil \frac{n}{2} \rceil$. This is surprising, since it means that on Q_n a single passive cop along with $\lceil \frac{n}{2} \rceil - 1$ active cops is as capable as $\lceil \frac{n}{2} \rceil$ flexible cops. However, as k decreases, $C_{k\mathbf{a}}(n)$ appears to increase rapidly, and in fact in Theorem 5.5 we show for sufficiently large n $C_{1\mathbf{a}}(n) > 2^{\lfloor \sqrt{n}/20 \rfloor}$, while in Theorem 5.4 we show $C_{1\mathbf{a}}(n) \leq O\left(\dfrac{2^n \ln n}{n^{3/2}}\right)$. The lower bound is determined using a complicated strategy for the robber, which is unusual in the literature since most of the work on Cops and Robber deals with upper bounds, and thus cop strategies (an exception is the robber strategy given by Frankl in [11]). Despite the most involved work of the paper going into those bounds, they are far from tight, and leave open the question of the values of $C_{k\mathbf{a}}(n)$ for other values of k.

Introducing passive cops dramatically complicates the game, while also making an interesting connection between Cops and Robber, and another vertex-pursuit game called Graph Searching. These two games are almost disjoint areas of research (see the survey by Alspach [3] for background on Graph Searching). In Graph Searching, the robber is invisible to the cops and can make an arbitrary number of moves on a single turn. The "cop numbers" in the Graph Searching game are often large; in fact, for Q_n, Tosic [24] obtains an upper bound of $2^{n-1}+1$. In Section 5 we show that cop numbers from $C_{(\lceil \frac{n}{2} \rceil -1)\mathbf{a}}(n)$ up to $C_{1\mathbf{a}}(n)$ provide a chain that begins with values typical of the game of Cops and Robber, and ends up with values typical of Graph Searching. In Section 6 we show, in Lemma 6.1, that $C_{1\mathbf{a}}(n)$ is a lower bound on the Graph Searching cop number. We are hopeful that our work is a first step toward bridging a gap between two areas in which there is "a vast gap" (Alspach [3], p. 12).

2 Notation and framework

We now define a framework which will allow us to unify different versions of Cops and Robber in a consistent manner, and generalize previous results. Throughout we will assume that all graphs are connected and the robber is active, except for some comments at the end of this section where we discuss why the case of a flexible robber is not so different. By a **police force** (or just a **force** for short), we mean a collection of cops specified by the tuple $(a_1\mathbf{A} + p_1\mathbf{P} + f_1\mathbf{F} + a_2\mathbf{a} + p_2\mathbf{p} + f_2\mathbf{f})$, where $a_1, p_1, f_1, a_2, p_2, f_2 \geq 0$ are integers, and $\mathbf{A}, \mathbf{P}, \mathbf{F}, \mathbf{a}, \mathbf{p},$ and $\mathbf{f}$ are special symbols. This tuple refers to the following collection of cops:

- There are $m = a_2 + p_2 + f_2$ cops such that every time it is the cops' turn, they choose exactly a_2 of these m cops to move, exactly p_2 of these m to remain still, and exactly f_2 of these m cops which can move or stay still.
- There are exactly a_1 cops that must be active throughout the game (we call these cops **fixed active** cops).
- There are exactly p_1 cops that must be passive throughout the game–they may never move.
- There are exactly f_1 cops that are flexible throughout the game (we call these cops **fixed flexible** cops).

If one of the 6 parameters is not specified, it is assumed to be zero. If a police force $\mathcal{F}$ is sufficient to catch a robber on a graph G, we denote this by $\mathcal{F} \longrightarrow G$. For example, if G is a path, then $(1\mathbf{F}) \longrightarrow G$, or if C_l is a cycle of length l, then $(1\mathbf{a}+1\mathbf{p}) \longrightarrow C_l$. Since the game is monotonic in the sense that adding cops of any type never hurts the cops, we will be interested in the minimal forces sufficient to catch the robber on a given graph. To denote such a minimal force, we use a squiggly arrow to indicate that the force is sufficient, but removing any cop would give the robber a winning strategy. For instance, $(1\mathbf{a}+1\mathbf{p}) \rightsquigarrow C_6$ and $(2\mathbf{f}) \rightsquigarrow C_6$. Finally, we use an arrow with a strike through it to denote a force that is insufficient to guarantee a cop victory, for example $(1\mathbf{A}) \not\longrightarrow C_4$. We can express existing results (basically from [19]) in this notation. For $n \geq 3$:

1. $(\lceil n/2 \rceil \mathbf{f}) \rightsquigarrow Q_n$

2. $(1\mathbf{a} + (\lceil n/2 \rceil - 1)\mathbf{f}) \rightsquigarrow Q_n$

A natural way to expand upon these results is to require more active cops. Using this notation, we determine the exact tradeoff between active and flexible cops in Section 4.

Unless otherwise stated, we will *assume the robber is active*; this assumption simplifies our work, while having little effect on the cop number. Using the idea of a "chaser" (used at various places in the literature, such as Lemma 1.1 of [19]) we observe the following.

Lemma 2.1 *Suppose $\mathcal{F}$ is a force and G a graph, such that $\mathcal{F} \longrightarrow G$. Then the following forces suffice to catch a flexible robber on G: $(\mathcal{F} + 1\mathbf{A})$, $(\mathcal{F} + 1\mathbf{a})$, $(\mathcal{F} + 1\mathbf{F})$, $(\mathcal{F} + 1\mathbf{f})$.*

Proof. We are supposing $\mathcal{F} \longrightarrow G$, meaning that there is a multi-set of vertices, P, from which $\mathcal{F}$ has a strategy to catch an active robber, no matter where that robber starts. It suffices to describe a strategy for $\mathcal{F} + 1\mathbf{A}$ to catch a flexible robber. The cops of $\mathcal{F}$ begin at vertices P and the extra active cop, who we will call the "chaser," starts anywhere. On a turn, the chaser simply chooses any move that brings her closer to the robber. The cops of $\mathcal{F}$ can be "*in retreat*" or "*on the attack*." If the robber stays still on a turn then $\mathcal{F}$ goes into retreat, meaning they begin to return to the original placement P, ignoring the robber until they are back at P. $\mathcal{F}$ can do this by simply reversing their moves. If $\mathcal{F}$ is at P and the robber moves, then $\mathcal{F}$ plays its usual strategy for an active robber. Since $\mathcal{F}$ has a strategy starting at P, irrespective of where the robber is situated, $\mathcal{F}$ can carry out its strategy every time a retreat brings them back to P. $\mathcal{F}$ continues playing this strategy as long as the robber continues to move; we say $\mathcal{F}$ is on the attack. Notice that when $\mathcal{F}$ is on the attack, they are on track to catch the robber as long as they are given enough consecutive moves in which the robber is active. The robber can only stay still a fixed number of times without being caught by the chaser. Thus assuming the chaser does not catch the robber, $\mathcal{F}$ will only go into retreat a fixed number of times, so eventually $\mathcal{F}$ will be on the attack long enough to catch the robber. ■

Remark. In the previous proof, if all of the cops in $\mathcal{F}$ are flexible, they could just stay still for a turn whenever the robber did not move, rather than going into retreat. However forces with active cops do not have this option.

If $\mathcal{F} \not\longrightarrow G$, then $\mathcal{F}$ is not sufficient to catch a flexible robber. Showing $\mathcal{F} \rightsquigarrow G$, a sharp result for an active robber, almost obtains a sharp result in the flexible case–it means that $\mathcal{F}$ with any cop removed does not suffice to catch the flexible robber, while adding an active or flexible cop does suffice. These general comments leave open the question of whether $\mathcal{F}$ suffices to catch a flexible robber; the answer to this question will vary depending on the graph under consideration. Since the case of the active robber gets us so close to the case of the flexible robber, we will only consider the case of an active robber for the remainder of this paper.

3 Parity and the hypercube

In this section we consider what happens when the game begins in some arbitrary position, rather than beginning the game in the standard way, with the cops, then the robber choosing vertices to occupy. This section is of independent interest, but is also used in Section 4 when we return to the standard way of beginning the game. We obtain some results of [7] as a special case, discussed at the end of the section.

It will be useful in this section and throughout the paper to keep in mind some properties and terminology related to the hypercube. The hypercube is *vertex-transitive*, meaning that given any two vertices $v, w \in V(Q_n)$ there is a graph automorphism on Q_n that maps v to w; in practice we will use this fact to assume, without loss of generality, that whatever vertex is under consideration is in fact the vertex $(00\ldots0)$. When a cop or robber moves from one vertex to another in Q_n, the corresponding vectors differ in exactly one coordinate, and we refer to the move as **flipping** that coordinate from 0 to 1 or from 1 to 0.

By a **position** we mean some placement of the cops and the robber on the vertices of the hypercube so that the robber does *not* occupy the same vertex as a cop. A position is a **winning position** if from that position the cops can force a win (*we assume it is the robber's turn*). In this section we provide a full classification of the winning positions involving active and flexible cops.

The parity of the distance from a cop to the robber is a key property which will be exploited in this section and throughout the paper. We say that a cop is an **even cop** if her distance to the robber is of even parity, and a cop is an **odd cop** if this parity is odd; we measure the parity right before the robber moves. Note that if a cop is fixed active, her parity will be the same throughout the game.

Theorem 3.1 *A position on Q_n with d odd fixed active cops and e even fixed active cops is a winning position if and only if $2e + d \geq n$.*

Proof. We say a cop **covers** a vertex v if the cop occupies v or can move to v on her next turn. The robber is only forced to lose on the next cop turn if all of his neighbors are covered by cops. We show that it is possible for the cops to do this if and only if $2e + d \geq n$.

Suppose $2e + d < n$. A cop at distance 1 from the robber covers one of the robber's neighbors, a cop at distance 2 from the robber covers two of the robber's neighbors, and a cop at distance 3 or greater from the robber covers none of the robber's neighbors. Therefore on the robber's turn, even cops cover at most two neighbors and odd cops cover at most one, so a total of at most $2e + d < n$ of the robber's neighbors are covered. Since the robber has n neighbors, he can move to an uncovered neighbor and will not be caught.

Conversely, suppose $2e + d \geq n$. We describe a winning strategy for the cops. Each vertex of Q_n is a vector of length n. We will refer to the n coordinates of the vector as coordinates $1, 2, \ldots, n$. Each cop will have some coordinate between 1 and n as her "home." Let the d odd cops have their homes at coordinates $1, \ldots, d$, respectively, and the e even cops have their homes at coordinates $d+2, d+4, \ldots, d+2e$, respectively. We adopt the convention throughout the proof that if we refer to any coordinate $N > n$ that we interpret this as the coordinate k so that $k \equiv N \pmod{n}$. Since $2e + d \geq n$, the homes at least reach coordinate n. From each cop's home, say coordinate k, she regards the coordinates $k+1, k+2, \ldots, n, 1, 2, \ldots k-1$, in that order, as being to the right of k. Notice the convention that we wrap around, so the coordinate $k-1$ is the rightmost coordinate, or in other words, directly to the left of home.

Each cop follows the same strategy relative to her home. Define the **reach** of the cop with home at coordinate k to be the number of consecutive coordinates on which the cop's vector matches the robber's, starting at coordinate k and going right. When it is the cops' turn, each cop simply flips the coordinate which makes her reach as large as possible. For example, suppose $n = 6$ and the robber is at vertex (000000). Suppose a cop is at vertex (010010) and her home is coordinate 3. Then this cop's reach is currently 2, and if it were her turn, she would flip coordinate 5 to make her reach 5.

This strategy eventually catches the robber. We say that a cop has a **critical reach** if her reach is $n-2$ for an even cop, or her reach is $n-1$ for an odd cop; reach is always measured right before the robber moves. An even cop with critical reach has two coordinates that differ from the robber's; if the robber flips one of these two coordinates on his turn, then the cop flips the other on her turn to catch him. An odd cop with critical reach has one coordinate that differs from the robber's; if the robber flips that one, he has landed on the cop and is caught. Thus, when a cop has critical reach, we say that the coordinate to the left of her home (or the two coordinates to the left of her home, in the case of an even cop) are **closed** to the robber and assume that the robber never flips such a coordinate.

We show the strategy works, by showing that 1) Eventually all the cops have critical reach, or 2) The robber gets caught before (1) happens. To show this, it suffices to show that whatever the robber does, every cop's reach can at least be maintained, and some cop without a critical reach can have her reach extended. Suppose the robber flips coordinate k. Because of the positioning of the cop homes and the fact that the cop homes extend out to at least coordinate n, at least one of the coordinates, $k+1$ or $k+2$, is the home of some cop. If coordinate $k+1$ is the home of a cop, then since coordinate k was not yet closed to the robber, that means the cop with home $k+1$ did not yet have critical reach, but her reach can be extended. If coordinate $k+1$ is home to no cop, but $k+2$ is the home of a cop (in this case, it must be an even cop), then since coordinate k was not yet closed to the robber, that means the cop with home $k+2$ did not yet have critical reach, so her reach can be extended. In both cases, all other cops can at least maintain their reach just by flipping the same coordinate as the robber. ■

Remark. We note that the cop strategy used in the backwards direction of the last proof is similar to cop strategies used in [7, 10, 17, 19].

Corollary 3.2 *A position on Q_n with d odd fixed active cops, e even fixed active cops, and f flexible cops is a winning position if and only if $2(e + f) + d \geq n$.*

Proof. The forward direction follows from the forward direction of Theorem 3.1, since the best case for the flexible cops is to be even. For the backward direction, the flexible cops with odd parity can simply not move for a turn to obtain an even parity. Once the flexible cops are all even, the cops have a winning strategy due to the backwards direction of Theorem 3.1. ■

Corollary 3.2 provides a full classification of the winning positions for combinations of active and flexible cops in terms of an easily checkable arithmetic condition. Using the notion of an "optimal position" we can derive an interesting result from the corollary. By an **optimal position** we mean a winning position that is no longer a winning position if any cop is removed. Using this terminology, Theorems 1 and 2 in [7] state that the maximum optimal positions are exactly those with n odd cops. Their result is immediate from Corollary 3.2, by setting $e = f = 0$. We can go further, fully classifying the optimal positions in the following corollary; the **size** of a position is the number of cops it contains.

Corollary 3.3 *The possible sizes of optimal positions range from $\lceil n/2 \rceil$ to n (inclusive). Furthermore, for any k such that $\lceil n/2 \rceil < k \leq n$ the collection of optimal positions of size k are exactly the positions with $2k - n$ fixed active odd cops and $n - k$ other cops, each of which is even or flexible. If $k = \lceil n/2 \rceil$ and n is even, the optimal positions of size k are exactly those with $n/2$ cops, each of which is even or flexible. If $k = \lceil n/2 \rceil$ and n is odd, the optimal positions of size k are of two kinds: 1) Those with $\lceil n/2 \rceil$ cops, each of which is even or flexible, and 2) Those with one odd active cop along with $(n - 1)/2$ cops, each of which is even or flexible.*

4 Active and flexible cops

In this section we return to the standard method of starting the game, in which the cops place themselves, then the robber places himself. In Theorem 4.1 we give an arithmetic criterion that fully characterizes which forces with active and flexible cops can catch the robber. In Corollary 4.2 we determine the possible sizes of minimal winning forces.

In the arithmetic condition of Theorem 4.1, the coefficients 2 and 3/2 in can be interpreted in terms of the power of individual flexible and fixed active cops. Recall that in order for the cops to have a winning strategy on Q_n, they must be able to cover all of a robber's n neighbors. Since flexible cops can always become even, they can always move to a position to cover 2 of the robber's neighbors. Half of the fixed active cops can be forced to be odd (and thus cover at most 1 of the robber's neighbors) and the rest will be even, so the fixed active cops can each cover an average of 3/2 neighbors of the robber.

Theorem 4.1 *For all n, $(a_1\mathbf{A} + f_1\mathbf{F} + a_2\mathbf{a} + f_2\mathbf{f}) \longrightarrow Q_n$ if and only if either:*

1. *$f_2 = 0$ and $2f_1 + \lfloor (3/2) \cdot (a_1 + a_2) \rfloor \geq n$, or*
2. *$f_2 > 0$ and $2(f_1 + f_2 + a_2) + \lfloor (3/2) \cdot a_1 \rfloor \geq n$.*

Proof. Consider the first case, which is equivalent to having f_1 fixed flexible cops and $a = a_1 + a_2$ fixed active cops. Call a vertex of Q_n **odd** if its vector contains an odd number of 1's, and **even** otherwise. The robber's initial placement determines which of the a fixed active cops are even and which are odd: If a cop's vertex has the same parity as the robber's vertex, the cop will be odd, and otherwise she will be even (recall that the cop parity is determined when it is the robber's turn). Suppose the cops choose to initially place s of the fixed active cops on odd vertices and t of the fixed active cops on even vertices, where $s + t = a$, and without loss of generality assume $s \geq t$. Since the robber wants to minimize the number of even cops he will choose to start at an odd vertex. By Corollary 3.2, the robber is caught if and only if $2f_1 + 2t + s \geq n$. Since the cops want t as large as possible and the robber wants t as small as possible, the best initial placement for the cops is to split s and t as evenly as possible (perhaps by placing s cops on some vertex v, and t cops on a neighbor of v), with the robber making his initial placement so that the larger set is the odd cops; i.e. we can assume $t = \lfloor a/2 \rfloor$ and $s = \lceil a/2 \rceil$. By Corollary 3.2, the cops can catch the robber if and only if $2f_1 + 2\lfloor a/2 \rfloor + \lceil a/2 \rceil \geq n$. The result in Case 1 follows from the equation $2\lfloor a/2 \rfloor + \lceil a/2 \rceil = \lfloor (3/2) \cdot a \rfloor$.

In Case 2, when $f_2 > 0$, we note that the parity of the a_1 fixed active cops is determined at the beginning of the game, while all other cops can be made even: If an odd cop remains among the other cops, that cop can be chosen to stay still on a turn, thus changing her parity to even, while the other cops all maintain their parity. Thus the robber's best strategy is to maximize the number of odd cops among the a_1 fixed active cops. As above, the cops' best strategy is to split these a_1 cops as evenly as possible. The game is then equivalent to one with $f_1 + f_2 + a_2 + \lfloor a_1/2 \rfloor$ even active cops, and $\lceil a_1/2 \rceil$ odd active cops. Similar to Case 1, we apply Corollary 3.2 to obtain the result. ■

Recalling the essential point of the last theorem, that a flexible cop is worth 2, while an active cop is worth 3/2, we can see that 3 flexible cops should be of the same power as 4 active cops. That tradeoff is the key to proving the next corollary.

Corollary 4.2 *Suppose $(a_1\mathbf{A} + f_1\mathbf{F} + a_2\mathbf{a} + f_2\mathbf{f}) \rightsquigarrow Q_n$, and let $k = a_1 + f_1 + a_2 + f_2$, the total number of cops in the minimal force. Then $\lceil n/2 \rceil \leq k \leq \lceil 2n/3 \rceil$ and all such values for k are possible.*

Proof. The inequality $\lceil n/2 \rceil \le k \le \lceil 2n/3 \rceil$ follows immediately from the criterion in Theorem 4.1, so it remains to show that all values for k are possible.

We restrict our attention to forces with f fixed flexible cops and a fixed active cops. Suppose $(f\mathbf{F} + a\mathbf{A}) \rightsquigarrow Q_n$, i.e. there is a minimal force of size $f + a$, so by Theorem 4.1, $2f + \lfloor (3/2) \cdot a \rfloor$ equals n or $n + 1$. Since

$$2(f-3) + \lfloor (3/2) \cdot (a+4) \rfloor = 2f + \lfloor (3/2) \cdot a \rfloor,$$

Theorem 4.1 implies $((f-3)\mathbf{F} + (a+4)\mathbf{A}) \rightsquigarrow Q_n$ and thus there is a minimal force of size $(f-3) + (a+4) = f + a + 1$. In other words, we can increase the size of a minimal force by exactly one if we trade 3 flexible cops for 4 active cops.

Applying Theorem 4.1, we obtain a minimal force of $\lceil n/2 \rceil$ flexible cops. Then we repeatedly make the tradeoff to increase the size of the minimal force. Since we can make this tradeoff $\lfloor \lceil n/2 \rceil / 3 \rfloor$ times, all values of k from $\lceil n/2 \rceil$ to at least $\lceil 2n/3 \rceil - 1$ will be realized; we then observe that $(\lceil 2n/3 \rceil \mathbf{A}) \rightsquigarrow Q_n$ to finish the proof. ■

5 Active and passive cops

We now consider the surprises and challenges raised by introducing passive cops. In Subsection 5.1 we consider the game where one cop must be passive on each turn and the rest active, and prove in Theorem 5.2 that on the hypercube such a force is just as powerful as a fully flexible force. In Subsection 5.2, we move to the other end of the spectrum and examine games where only one cop is allowed to be active on a turn, and the rest are required to be passive. In this context the cop number becomes much larger and the problem becomes much more challenging. We find upper and lower bounds on the cop number in Theorems 5.4 and 5.5. In Subsection 5.3 we summarize our current knowledge about the tradeoff between active and passive cops. In contrast to the active/flexible setting of Section 4, we are quite far from being able to quantify this tradeoff so we highlight a few interesting open questions.

5.1 One passive cop

In Corollary 4.2, where we considered the active/flexible tradeoff, we found that the smallest cop number of $\lceil n/2 \rceil$ was achieved by a fully flexible force and the largest cop number of $\lceil 2n/3 \rceil$ was achieved by a fully active force. Surprisingly, a force with active and passive cops can have a significantly smaller cop number than a force with all active cops. For example, here is a simple cop strategy that proves $(\lceil n/2 \rceil \mathbf{a} + 1\mathbf{p}) \longrightarrow Q_n$:

> On each turn, while there is more than one odd cop, one odd cop stays still (thereby becoming even on the next turn) and the rest maintain their parity until there are $\lceil n/2 \rceil$ even cops. These even cops then remain active

for the rest of the game while the remaining odd cop stays passive. The $\lceil n/2 \rceil$ even active cops have a winning strategy by Theorem 3.1.

Thus the "extra" passive cop allows the remaining cops to become even, and so $\lceil n/2 \rceil + 1$ cops suffice to catch the robber. Going one step further, in Theorem 5.2, we give a cop strategy that proves for all $n \geq 3$, $((\lceil n/2 \rceil - 1)\mathbf{a} + 1\mathbf{p}) \longrightarrow Q_n$, i.e. a force with one passive cop and the rest active has the same cop number as the strongest force, one consisting of all flexible cops! Subsequent to this, we show in Proposition 5.3 that there is at least some increase in the cop number as we require more cops to be passive.

Given a force $\mathcal{F}$ and a graph G, we write $\mathcal{F} \stackrel{*}{\longrightarrow} G$ to indicate that not only is the force $\mathcal{F}$ sufficient to catch the robber on G, but that it can do so regardless of the initial position of the game. In the next proof, the "shadow" concept is used, a common technique, dating back to at least 1987 in [17].

Lemma 5.1 *Suppose $x, y \geq 1$ are integers and $(x\mathbf{a} + y\mathbf{p}) \stackrel{*}{\longrightarrow} Q_n$. Then $((x+1)\mathbf{a} + y\mathbf{p}) \stackrel{*}{\longrightarrow} Q_{n+2}$.*

Proof. Assume $(x\mathbf{a} + y\mathbf{p}) \stackrel{*}{\longrightarrow} Q_n$, and the force $((x+1)\mathbf{a} + y\mathbf{p})$ is placed arbitrarily on vertices in Q_{n+2}. We describe a winning strategy for the cops on Q_{n+2}.

Their first goal is to "catch" the robber on the first n coordinates–i.e. to have some cop whose vector is the same as the robber's on these coordinates. The cops designate one particular cop as a chaser. This cop will always be active, and will greedily try to match the robber's vector on the first n coordinates. Simultaneously, the remaining cops, i.e. $(x\mathbf{a} + y\mathbf{p})$, play their strategy for Q_n on the first n coordinates.

If the chaser succeeds in matching the robber's first n coordinates, we are done with the first goal, so let us assume otherwise. Any time the robber flips one of the last two coordinates, the chaser will get at least one step closer to her goal, so the robber can only play outside of the first n coordinates a fixed number of times. For the remaining cops of $(x\mathbf{a} + y\mathbf{p})$ a move outside of the first n coordinates looks like the robber is staying still; since the cops of $(x\mathbf{a} + y\mathbf{p})$ can catch the robber from any starting position, when the robber stays still, they can simply restart their strategy from the current position. Since there are only a fixed number of restarts, eventually the robber is active on the first n coordinates long enough for the cops of $(x\mathbf{a} + y\mathbf{p})$ to catch the robber on the first n coordinates. Thus the cops can accomplish their first goal of catching the robber on the first n coordinates.

With the first goal accomplished, we now have some cop whose distance to the robber is either 1 or 2. That cop can guarantee she is at distance exactly 2 (measured right before the robber is to move), by remaining stationary for a turn if necessary, which is possible since $y \geq 1$. Without loss of generality, we may assume a cop has caught the robber on the first n coordinates, and that the robber's last two coordinates are [11] and the cop's last two coordinates are [00]. The robber may not flip either of the last two coordinates, and this cop will be responsible for maintaining this state by acting as a "shadow" to the robber–every time the robber flips one of the first n coordinates, the shadow will flip the same one, maintaining distance two. Since $x \geq 1$, the remaining cops can all migrate to the copy of Q_n in Q_{n+2} where the last two coordinates are [11], and since the robber is confined to move in this copy of Q_n, the remaining cops can play their winning strategy there, which works from any start position. ■

We now prove that a force with one passive cop can catch the robber on Q_n with a total of only $\lceil n/2 \rceil$ cops.

Theorem 5.2 *For all $n \geq 3$, $((\lceil n/2 \rceil - 1)\mathbf{a} + 1\mathbf{p}) \rightsquigarrow Q_n$.*

Proof. We first show that $(1\mathbf{a} + 1\mathbf{p}) \xrightarrow{*} Q_3$ and $(1\mathbf{a} + 1\mathbf{p}) \xrightarrow{*} Q_4$. Since in these cases there is one active cop and one passive cop on each turn, the cops can move to any given position, and so the winning strategies do not depend on starting position.

On Q_3, the two cops move to occupy the vertices (000) and (111). Then wherever the robber chooses to go, the cop adjacent to the robber will be active on her next turn and catch the robber.

On Q_4, the cops move to occupy the vertices (0000) and (1111). The robber then must move to a vertex with two zeros and two ones. Without loss of generality, assume the robber is at vertex (1100). In this case, one cop should move to (1000), leaving the other cop at (1111). The only move where the robber is not caught on the next turn is to (0100). But then the cop at (1111) should move to (0111), and wherever the robber moves on his next turn, he will be caught.

To complete the proof, we proceed by induction. We have shown the theorem is true for $n = 3$ and $n = 4$, and since these cases satisfy the hypotheses of Lemma 5.1 (i.e. starting position is not important and we have at least one active and one passive cop), the lemma guarantees that if the theorem is true for n, then it is true for $n+2$. ■

The next proposition shows that with 2 passive cops the best possible cop number of $\lceil n/2 \rceil$ cannot be attained in general.

Proposition 5.3 *For all $n \geq 3$, $((\lceil n/2 \rceil - 2)\mathbf{a} + 2\mathbf{p}) \not\longrightarrow Q_n$ if $n \equiv 2 \pmod 4$.*

Proof. Since $n \equiv 2 \pmod 4$, the total number of cops, $\lceil n/2 \rceil = n/2$, is odd. Recalling the terminology in the proof of Theorem 4.1, we observe that the initial placement of the cops splits into two groups: The cops on even vertices and the cops on odd vertices. One of these groups is of odd size and the other of even size, since the cop total is odd. Thus the robber can choose an initial vertex that will result in an even number of cops at an odd distance, so that after the cops' first turn, there are an odd number of odd cops (recall that we measure cop parity on the robber's turn). For the rest of the game, since exactly two cops must remain passive on each turn, exactly two cops change parity on each turn. Thus there will always be at least one odd cop, and so by the backwards direction of Corollary 3.2, $n/2$ cops are not sufficient. ■

5.2 One active cop

In this section, we consider the game with one active cop and the rest passive. First we prove an upper bound on the cop number in this context. A **dominating set** in a graph G is a set of vertices $D \subseteq V(G)$ so that every element of $V(G)$ is either in D or a neighbor of at least one element of D. Since there exists a dominating set on Q_n of size $\Theta(2^n/n)$, this gives a trivial upper bound for the cop number with one active cop. Theorem 5.4 gives a slight improvement on this bound.

Theorem 5.4 $\left(1\mathbf{a} + O\left(\frac{2^n \ln n}{n^{3/2}}\right)\mathbf{p}\right) \longrightarrow Q_n$.

Proof. For an integer k, let **level** k refer to those vertices of Q_n with exactly k ones. We describe a cop strategy where we position the cops so that they dominate level $\lfloor n/2 \rfloor$ and then move up or down the levels in a phalanx in order to catch the robber. Let G_k denote the subgraph of Q_n induced by levels k and $k+1$. We claim that for all k, G_k has a dominating set of size $O\left(\frac{2^n \ln n}{n^{3/2}}\right)$. By symmetry it suffices to prove the claim for $k \leq \lfloor n/2 \rfloor$. It is well known (see for example Theorem 1.2.2 in [2]) that for a graph with N vertices and minimum degree δ, there exists a dominating set of size $N(1+\ln(\delta+1))/(\delta+1)$. Thus it suffices to show that $\binom{n}{k} \cdot \frac{\ln k}{k} \leq O\left(\frac{2^n \ln n}{n^{3/2}}\right)$. We proceed by considering two cases. If $n/4 \leq k \leq \lfloor n/2 \rfloor$, then $\binom{n}{k} \leq \binom{n}{n/2} \leq O(2^n/\sqrt{n})$, and (since $\frac{\ln x}{x}$ is decreasing) $\frac{\ln k}{k} \leq \frac{\ln(n/4)}{n/4}$, so we are done. In the case that $k < n/4$, we note that $\frac{\ln k}{k} \leq 1$, and proceed to bound $\binom{n}{k}$, by setting $k = n/4$. Using a Stirling bound, we see that $\binom{n}{n/4} \leq (\frac{3n}{n/4})^{n/4} = (12^{1/4})^n < (1.9)^n$, so we are done. That concludes the proof of the claim.

The cops should initially select a set of vertices, D, that dominates $G_{\lfloor n/2 \rfloor}$, and place two cops on each vertex in D, coloring one red and the other blue, thus supplying enough cops to dominate any pair of levels. If the robber chooses an initial position in level $\lfloor n/2 \rfloor$ then he will be caught immediately. Otherwise, we can assume without loss of generality that he chooses a vertex in some level $k > n/2$. Note that if the robber is at some level j, and so long as the cops dominate some level $i < j$, the robber is restricted to move only in levels greater than i. Thus at first, the blue cops should remain in place, while the red cops rearrange themselves one by one to dominate level $\lfloor n/2 \rfloor + 1$. After this is complete, the red cops should remain in place while the blue cops rearrange themselves to dominate level $\lfloor n/2 \rfloor + 2$. Proceeding in this manner, with cops of one level remaining in position to dominate some level while the cops of the other color proceed to dominate the next, the cops will force the robber to move only on higher and higher levels, until finally the cops dominate levels $n-1$ and n at which point the robber will be caught. ■

Now we prove the lower bound on the cop number.

Theorem 5.5 *For sufficiently large n,* $(1\mathbf{a} + 2^{\lfloor \sqrt{n}/20 \rfloor}\mathbf{p}) \not\longrightarrow Q_n$.

To prove the theorem, we will define a weight function, where the weight of a cop is related to her distance from the robber, and the weight of all the cops is the sum of their individual weights. We will show in Lemma 5.6 that the robber can choose a starting position of low weight. We then show in Lemmas 5.7 and 5.8 that regardless of the cop strategy, the robber can always move to maintain the total weight of the cops below a certain threshold, and that will guarantee that the robber cannot be caught. We first prove the theorem assuming Lemmas 5.6, 5.7, and 5.8; then we prove these three lemmas. For the remainder of this section we assume that n is very large, let $\omega = \lfloor \sqrt{n}/20 \rfloor$, let $C = 2^{\omega}+1$, and assume that the number of cops is C.

Proof of Theorem 5.5. A **distance vector** is any vector $\mathbf{A} = (A_1, \ldots, A_\omega)$ where for all d, A_d is a nonnegative real number. Given a position of the cops and the robber, the distance vector corresponding to the position is the vector $\mathbf{A} = (A_1, \ldots, A_\omega)$ where for $1 \le d < \omega$, $A_d =$ the number of cops at distance exactly d from the robber, and $A_\omega =$ the number of cops at distance ω or greater from the robber. Note that whenever a distance vector is used, it is assumed that there are no cops at distance 0.

To compute the weight function, we define the **maximum vector** $\mathbf{M} = (M_1, \ldots, M_\omega)$ to be a vector of nonnegative real numbers, where $M_1 = 42$, and for $d > 1$, $M_d = \dfrac{(\sqrt{n})^{(d-1)}}{(d-1)!10^{(d-1)}}$. We note some important properties of these numbers.

For $2 \le d < \omega$

$$\frac{M_d}{M_{d+1}} = \frac{(\sqrt{n})^{(d-1)}}{(d-1)!10^{(d-1)}} \cdot \frac{d!10^d}{(\sqrt{n})^d} = \frac{10d}{\sqrt{n}}. \tag{1}$$

This, along with the fact that M_1 is constant, implies that for all $d \ge 1$,

$$M_d < M_{d+1}. \tag{2}$$

Given a distance vector $\mathbf{A} = (A_1, \ldots, A_\omega)$, let

$$wt(\mathbf{A}) = \sum_{d=1}^{\omega} \frac{A_d}{M_d}.$$

We call $wt(\mathbf{A})$ the **weight** of $\mathbf{A}$. By a **zero-starting vector** we mean a vector whose first entry is 0.

For the remainder of the proof, we set the constants $t = 1/7$, and $r = 5/6$. By Stirling's approximation, for n sufficiently large

$$C = 2^{\lfloor \sqrt{n}/20 \rfloor} + 1 < t \cdot M_{\omega-1}. \tag{3}$$

To prove the theorem, it suffices to prove the following three lemmas.

Lemma 5.6 *Regardless of the initial position chosen by the cops, the robber can choose a vertex so that the position's distance vector* $\mathbf{D}$ *is zero-starting, and* $wt(\mathbf{D}) < r + t = 41/42$.

Lemma 5.7 *Suppose it is the cops' turn and the position has a zero-starting distance vector* $\mathbf{D}$ *such that* $wt(\mathbf{D}) < r+t = 41/42$*. Then no matter what they do, the resulting position will have a distance vector* $\mathbf{A}$ *with* $wt(\mathbf{A}) < 1$.

Lemma 5.8 *Suppose it is the robber's turn and the position has a distance vector* $\mathbf{A}$ *such that* $wt(\mathbf{A}) < 1$*. Then he has a move to a position with a zero-starting distance vector* $\mathbf{D}$ *such that* $wt(\mathbf{D}) < r + t = 41/42$.

By Lemma 5.6, the robber can choose a starting position satisfying the hypotheses of Lemma 5.7. By Lemma 5.7, whatever the cops do, the distance vector will have weight < 1. Thus by Lemma 5.8, the robber can put the game back in a position satisfying the hypotheses of Lemma 5.7. By alternately applying Lemmas 5.7 and 5.8, the robber can evade capture indefinitely, so the cop number must be larger than $C = 2^{\lfloor\sqrt{n}/20\rfloor} + 1$. ■

Proof of Lemma 5.6. Since Q_n is an n-regular graph, the number of vertices in Q_n at distance at most ω from any set of $C = 2^\omega + 1$ cops is at most $Cn^{\omega+1}$. By a simple calculation, $Cn^{\omega+1} = \exp(O(\sqrt{n}\log n)) < 2^n$ for large n, so there is some vertex at distance $\geq \omega$ from every cop. Thus the robber can start the game in a position where all cops are at distance ω or greater from the robber. The distance vector of this position is certainly zero-starting, and has weight $C/M_\omega < t \cdot (M_{\omega-1}/M_\omega) < t$, where the first inequality follows from Equation 3 and the second inequality follows from Equation 2. ■

Proof of Lemma 5.7. The cops can move at most one cop, and since the values of M_d are strictly increasing, the cops can add at most weight $1/M_1$=$1/42$. Thus if the previous weight was less than $r + t$, the new total weight is less than $r + t + 1/M_1 = 5/6 + 1/7 + 1/42 = 1$. Also note that since the position has a zero-starting vector, whatever the cops do, the resulting position has all cops at distance at least one from the robber. ■

Proof of Lemma 5.8. To prove the lemma, we require some further definitions with regard to vectors. Suppose $\mathbf{A} = (A_1, \ldots, A_\omega)$ and $\mathbf{D} = (D_1, \ldots, D_\omega)$ are both distance vectors. Then $\mathbf{A}+\mathbf{D} = (A_1+D_1, \ldots, A_\omega+D_\omega)$. For integers $k \geq 0$ and $m \geq 1$ (such that $k + m \leq \omega$) and real numbers $u_0, \ldots, u_k$, we denote by $[u_0, \ldots, u_k]_m$ the distance vector $(A_1, \ldots, A_\omega)$ such that $A_m = u_0, \ldots, A_{m+k} = u_k$ and the rest of the A_d are zero. For example: $[7, 8, 9]_3 = (0, 0, 7, 8, 9, 0, 0, \ldots)$. Finally, by $\mathbf{B}$ we denote the vector of positive real numbers $\mathbf{B} = (B_1, \ldots, B_{\omega-1})$ such that $B_1 = B_2 = 1$, and for $d \geq 3, B_d = M_{d-1}/3$.

The next definition (of the "Splitting transformation") represents a key concept in our proof. When the robber moves, some cops get one step closer to the robber and some become one step further from the robber. In particular, for the cops at distance d, some will be at distance $d-1$ and the rest will be at distance $d+1$; none of them will be at distance d. The robber's goal is to have as few cops as possible get closer. We will see that the splitting transformation represents an upper bound on the worst case for the robber.

Definition 5.9 *(Splitting Transformation) We define a function* **spl** *from distance vectors to distance vectors. We first define the function on vectors with one nonnegative real entry, u.*

- *Let* $\mathbf{spl}([u]_1) = [0, u]_1$.
- *Let* $\mathbf{spl}([u]_\omega) = [u]_{\omega-1}$.
- *For* $1 < d < \omega$,

$$\mathbf{spl}([u]_d) = \left[\frac{u}{M_d}B_d, 0, \left(u - \frac{u}{M_d}B_d\right)\right]_{d-1}.$$

For any vector $\mathbf{A} = (A_1, \ldots, A_\omega)$*, we define* $\mathbf{spl}(\mathbf{A}) = \mathbf{spl}([A_1]_1) + \ldots + \mathbf{spl}([A_\omega]_\omega)$

The splitting transformation has the following key property.

Claim 5.10 *If* $\mathbf{A}$ *is a distance vector with* $wt(\mathbf{A}) < 1$*, then* $wt(\mathbf{spl}(\mathbf{A})) \leq wt(\mathbf{A})r + t$.

Proof of Claim 5.10. We show this by considering $wt(\mathbf{spl}([A_d]_d))$ for each value of d. We will repeatedly use the observation that since $wt(\mathbf{A}) < 1$, then $A_d < M_d$ for any d.

- $d = 1$: In this case, we first note that $M_1/M_2 = 420/\sqrt{n} < 5/6 = r$. Thus,

$$wt(\mathbf{spl}([A_1]_1)) = \frac{A_1}{M_2} \leq \frac{A_1}{M_1}r$$

- $2 < d < \omega$:

$$\begin{aligned} wt(\mathbf{spl}([A_d]_d)) &= \frac{(A_d/M_d)B_d}{M_{d-1}} + \frac{A_d - (A_d/M_d)B_d}{M_{d+1}} \\ &= \frac{(A_d/M_d)B_d}{M_{d-1}} + \frac{(A_d/M_d)M_d - (A_d/M_d)B_d}{M_{d+1}} \\ &= (A_d/M_d)\left(\frac{B_d}{M_{d-1}} + \frac{M_d - B_d}{M_{d+1}}\right) \end{aligned}$$

Now,

$$\begin{aligned}\frac{B_d}{M_{d-1}} + \frac{M_d - B_d}{M_{d+1}} &= \frac{(1/3)M_{d-1}}{M_{d-1}} + \frac{M_d - (1/3)M_{d-1}}{M_{d+1}} \\ &= (1/3) + \frac{M_d}{M_{d+1}} - (1/3)\frac{M_{d-1}}{M_{d+1}} \\ &\leq (1/3) + \frac{10d}{\sqrt{n}} - 0\end{aligned}$$

using Equation 1. The last expression is largest when $d = \omega - 1$, so making this substitution we conclude

$$\begin{aligned}wt(\mathbf{spl}([A_d]_d)) &\leq (A_d/M_d)\left((1/3) + \frac{10(\omega-1)}{\sqrt{n}}\right) \\ &\leq (A_d/M_d)\left((1/3) + \frac{10(\sqrt{n}/20)}{\sqrt{n}}\right) \\ &= (A_d/M_d)(1/3 + 1/2) \\ &= (A_d/M_d)r\end{aligned}$$

- $d = 2$: This case is basically identical to the previous case except (since $B_d = 1$ when $d = 2$)
$$\frac{B_d}{M_{d-1}} + \frac{M_d - B_d}{M_{d+1}} \leq (1/42) + \frac{10d}{\sqrt{n}} \leq r.$$
- $d = \omega$: In this case, $\mathbf{spl}([A_\omega]_\omega) = [A_\omega]_{\omega-1}$. Since the total number of cops is $\leq t \cdot M_{\omega-1}$, by Equation 3, so is A_ω, so $wt(\mathbf{spl}([A_\omega]_\omega) = A_\omega/M_{\omega-1} \leq t$.

Now we finish the argument for $\mathbf{A} = (A_1, \ldots, A_\omega)$.

$$\begin{aligned}wt(\mathbf{spl}(\mathbf{A})) &= wt(\mathbf{spl}([A_1]_1)) + \cdots + wt(\mathbf{spl}([A_\omega]_\omega)) \\ &\leq (A_1/M_1)r + \cdots + (A_{\omega-1}/M_{\omega-1})r + t \\ &= ((A_1/M_1) + \ldots + (A_{\omega-1}/M_{\omega-1}))r + t \\ &\leq wt(\mathbf{A})r + t\end{aligned}$$

■

We will now show that the robber has a move to a position with a zero-starting distance vector $\mathbf{D}$ such that $wt(\mathbf{D}) \leq wt(\mathbf{spl}(\mathbf{A}))$. The following series of inequalities then finishes the proof of Lemma 5.8:

$$\begin{aligned}wt(\mathbf{D}) &\leq wt(\mathbf{spl}(\mathbf{A})) & \\ &\leq wt(\mathbf{A}) \cdot r + t, & \text{by Claim 5.10} \\ &< r + t, & \text{since } wt(\mathbf{A}) < 1.\end{aligned}$$

Now we find such a vector $\mathbf{D}$. Consider some fixed integer d such that $1 \le d < \omega$. Label the n neighbors of the robber by $1, 2, \ldots, n$ and let $z_i =$ the number of cops at distance d from the robber such that vertex i is on a shortest path from her to the robber. Consider the sum $z_1 + \ldots + z_n$. Note that each cop at distance d contributes exactly d to this sum, since a cop at distance d has exactly d neighbors of the robber on a shortest path between her and the robber (To see this, it is helpful to recall that we may assume that the robber is at vertex $(00\ldots0)$). Thus

$$z_1 + \ldots + z_n = dA_d.$$

Since $\left(\frac{d \cdot M_d}{B_d}\right)\left(\frac{A_d}{M_d} \cdot B_d\right) = d \cdot A_d$, at most $\frac{d \cdot M_d}{B_d}$ of the z_i can be $\frac{A_d}{M_d} \cdot B_d$ or more; we call the corresponding i values the **bad neighbors** for d.

As we range over all $d < \omega$, the total number of bad neighbors is at most

$$\frac{1 \cdot M_1}{B_1} + \ldots + \frac{(\omega - 1) \cdot M_{\omega-1}}{B_{\omega-1}}.$$

For all $d < \omega$, using Equation 1, we conclude that $\frac{d \cdot M_d}{B_d} < \sqrt{n}$, so the total number of bad neighbors is at most $\omega \cdot \sqrt{n} = \lfloor \sqrt{n}/20 \rfloor \cdot \sqrt{n} < n$. Thus the robber can move to a vertex that is not bad for any d; suppose this new position has distance vector $\mathbf{D} = (D_1, \ldots, D_\omega)$.

To understand the change in weight from $\mathbf{A}$ to $\mathbf{D}$, we examine what happens to each coordinate of $\mathbf{A}$. Of the A_d cops at distance d, some number of these, say L_d will get closer to the robber and thus be at distance $d-1$ and the rest, $A_d - L_d$, must get further away, arriving at distance $d+1$. Since $\mathbf{D}$ is not bad for any d, we obtain for all d, that $L_d < \frac{A_d}{M_d} \cdot B_d$. Notice that in the case that $d = 1$ or $d = 2$, $\frac{A_d}{M_d} \cdot B_d = \frac{A_d}{M_d} < 1$, where the inequality holds because $wt(A) < 1$. Thus $L_d = 0$ for $d = 1$ and $d = 2$, and $\mathbf{D}$ is a zero-starting distance vector. We can write $(D_1, \ldots, D_\omega) = D_1^* + \ldots + D_\omega^*$, where

- For $1 < d < \omega$: $D_d^* = [L_d, 0, A_d - L_d]_{d-1}$
- For $d = \omega$: $D_\omega^* = [L_\omega, A_\omega - L_\omega]_{\omega-1}$
- For $d = 1$: $D_1^* = [0, A_1]_1$

Since $L_d < \frac{A_d}{M_d} \cdot B_d$ for all d, and the values of M_d increase with d, for $1 < d < \omega$

$$wt(D_d^*) = \frac{L_d}{M_{d-1}} + \frac{A_d - L_d}{M_{d+1}} \le \frac{(A_d/M_d)B_d}{M_{d-1}} + \frac{A_d - (A_d/M_d)B_d}{M_{d+1}} = wt(\mathbf{spl}([A_d]_d))$$

It is also easy to see that $wt(D_d^*) \le wt(\mathbf{spl}([A_d]_d))$ for $d = 1$ and $d = \omega$, so we can conclude

$$\begin{aligned} wt(\mathbf{D}) &= wt(D_1^*) + \ldots + wt(D_\omega^*) \\ &\le wt(\mathbf{spl}([A_1]_1)) + \ldots + wt(\mathbf{spl}([A_\omega]_\omega)) \\ &= wt(\mathbf{spl}(\mathbf{A})). \end{aligned}$$

■

Remark. The proof of Theorem 5.5 can be adapted to give a lower bound of $2^{c\sqrt{n}}$ (with a smaller constant $c > 0$) on the cop number of a game with any constant number of active cops and the rest passive.

5.3 Active versus passive tradeoff

In the last two subsections we considered the game with active and passive cops, focusing on the game in which only one cop is active, and on the game in which all cops are active but one. We now consider the full range of possibilities, summarizing our knowledge and presenting some open questions.

For an integer k, let $C_{k\mathbf{a}}(n)$ be $k + r$, where r is the smallest integer such that $k\mathbf{a} + r\mathbf{p} \longrightarrow Q_n$. In this notation, Theorem 5.2 states that $C_{(\lceil n/2\rceil-1)\mathbf{a}}(n) = \lceil n/2\rceil$. However, for $C_{1\mathbf{a}}(n)$ our understanding is not complete; Theorems 5.4 and 5.5 show that for n sufficiently large,

$$2^{\lfloor \sqrt{n}/20 \rfloor} < C_{1\mathbf{a}}(n) \le O(2^n \ln n / n^{3/2}).$$

We are interested in the values of $C_{k\mathbf{a}}$ for k such that $1 \le k < \lceil n/2\rceil$. We conjecture that

$$C_{(\lceil n/2\rceil-1)\mathbf{a}}(n) \le C_{(\lceil n/2\rceil-2)\mathbf{a}}(n) \le \ldots \le C_{1\mathbf{a}}(n). \tag{4}$$

At the right end of this conjectured linear order, i.e. for $C_{k\mathbf{a}}(n)$ with k constant, we found, in Theorem 5.5 and the remark following it, that for large n the value is large, at least $2^{c\cdot\sqrt{n}}$ (for some constant $c > 0$). At the left end of the linear order, the next proposition shows that $C_{(\lceil n/2\rceil-m)\mathbf{a}}(n)$ with a constant m is asymptotically $n/2$, i.e. as small as the most powerful force, the fully flexible force.

Proposition 5.11 *For m constant,*

$$\lim_{n \longrightarrow \infty} \frac{C_{(\lceil n/2\rceil-m)\mathbf{a}}(n)}{(n/2)} = 1.$$

Proof. We conclude that $C_{2\mathbf{a}}(n+2) \le C_{1\mathbf{a}}(n)+1$ by observing that $(1\mathbf{a}+m\mathbf{p}) \longrightarrow Q_n$ implies $(1\mathbf{a}+m\mathbf{p}) \stackrel{*}{\longrightarrow} Q_n$, which in turn, by Lemma 5.1, implies $(2\mathbf{a}+m\mathbf{p}) \stackrel{*}{\longrightarrow} Q_{n+2}$. Similarly, for $1 \le k \le (\lceil n/2\rceil - 3)$, we can bound $C_{(k+1)\mathbf{a}}(n)$ by applying Lemma 5.1 k times, to imply that

$$C_{(k+1)\mathbf{a}}(n + 2k) \le C_{1\mathbf{a}}(n) + k,$$

or equivalently,

$$C_{(k+1)\mathbf{a}}(n) \leq C_{1\mathbf{a}}(n-2k) + k. \tag{5}$$

If $k = k(n)$, we can use inequality (5) to give the bound

$$C_{(k(n)+1)\mathbf{a}}(n) \leq C_{1\mathbf{a}}(n-2k(n)) + k(n). \tag{6}$$

For $k(n) = n/2 - m$, inequality (6) becomes

$$C_{(n/2-m+1)\mathbf{a}}(n) \leq C_{1\mathbf{a}}(2m) + k(n)$$

which suffices to prove the proposition because $C_{1\mathbf{a}}(2m)$ is a constant. ■

This leaves very open what happens to $C_{k\mathbf{a}}(n)$ for values of k like $n/4$ or $\log n$, which are neither constant, nor within a constant of $n/2$.

Open Question 1 *Determine the value of* $C_{k\mathbf{a}}(n)$ *for* $1 \leq k < \lceil n/2 \rceil - 1$.

Even if we cannot determine the values of $C_{k\mathbf{a}}(n)$ exactly, it would still be interesting to understand how this quantity behaves as the parameters are changed. We would expect that $C_{k\mathbf{a}}(n)$ would decrease in k and increase in n, and in the next proposition we show that there is an increase in n. However we cannot prove that there is a decrease in k, though we note the trivial claim that $C_{(k+1)\mathbf{a}}(n) \leq C_{k\mathbf{a}}(n)+1$, i.e. we almost have the conjectured linear order (4).

Open Question 2 *How big is the gap between* $C_{k\mathbf{a}}(n)$ *and* $C_{(k+1)\mathbf{a}}(n)$*?*

Proposition 5.12 *For* $k \geq 1$, $C_{k\mathbf{a}}(n) \geq C_{k\mathbf{a}}(n-1)$.

Proof. It suffices to show that if $k\mathbf{a} + m\mathbf{p} \longrightarrow Q_n$ then $k\mathbf{a} + m\mathbf{p} \longrightarrow Q_{n-1}$. We give a strategy for the cops to catch the robber on Q_{n-1} (in the "$(n-1)$-game") by using their winning strategy on Q_n (in the "n-game"). If a cop has initial position $(x_1, \ldots, x_{n-1}, 0)$ in the n-game, she should choose initial position $(x_1, \ldots, x_{n-1})$ in the $(n-1)$-game. Otherwise, if a cop has initial position $(x_1, x_2, \ldots, x_{n-1}, 1)$ in the n-game, she should choose initial position $(1-x_1, x_2, \ldots, x_{n-1})$ in the $(n-1)$-game.

The robber moves in Q_{n-1} correspond to robber moves in Q_n where the last coordinate is 0. On the cops' turn, if the robber is at vertex $(x_1, \ldots, x_{n-1})$ in Q_{n-1}, they should use the n-game strategy for the robber placed at $(x_1, \ldots, x_{n-1}, 0)$ in Q_n. Whenever the n-game strategy calls for a cop to flip one of the first $n-1$ coordinates, she should do that in the $(n-1)$-game, and if the strategy calls for a cop to flip the nth coordinate, she should instead flip the first one.

This strategy guarantees the following condition: If a cop's position in the n-game is $(x_1, \ldots, x_{n-1}, 0)$, then her position in the $(n-1)$-game is $(x_1, \ldots, x_{n-1})$. This condition is enough to guarantee that when the robber is caught in the n-game, he is also caught in the $(n-1)$-game. ■

6 The relationship to Graph Searching

We now relate Cops and Robber to the Graph Searching game. One version of the Graph Searching game has all flexible cops, but allows the robber to move "infinitely fast," that is on the robber's turn, he may move any number of steps along any path, provided that no cop occupies a vertex on the path. Let $C^{\infty}(n)$ be the number of cops required to catch the robber on Q_n under these rules. In Lemma 6.1 we prove $C_{\mathbf{1a}}(n) \leq C^{\infty}(n)$, and use this to derive the lower bound for $C^{\infty}(n)$ given in Corollary 6.2.

Lemma 6.1 $C_{\mathbf{1a}}(n) \leq C^{\infty}(n)$.

Proof. We refer to the game associated with $C_{\mathbf{1a}}(n)$ as the "1-active game" and the game associated with $C^{\infty}(n)$ as the "$\infty-$game". It suffices to show that if the robber has a strategy to evade m cops in the 1-active game, then he has a strategy to evade m cops in the $\infty-$game.

We now describe a robber strategy for the $\infty-$game, using his strategy in the 1-active game. After the cops' initial placement in the $\infty-$game, we place the cops on the same vertices in the $1-$active game to see the robber's initial placement. We choose the robber's initial placement in the $\infty-$game to be the same as his placement in the $1-$active game.

The two games start out in the same position. To show that the robber can win, it suffices to show that the games persist in the same position. Assume the two games are in the same position. Some r of the m cops move in the $\infty-$game. The robber needs to respond. He considers the corresponding 1-active game, selecting the same r cops that moved, but now he moves them one at a time, seeing what the sequence of robber moves are in this case. Since the robber has an evading strategy in the 1-active game, his sequence of moves can never take him to a place where he could be caught. In particular, this sequence of robber moves never overlaps with one of the m original vertices of the cops, nor any of the r vertices that the r cops moved to. Thus in one turn of the $\infty-$game the robber can safely copy this sequence of moves. The two games are thus in the same position again, and the robber can continue in a similar fashion. ■

The following is an immediate corollary of Lemma 6.1 and Theorem 5.5.

Corollary 6.2 *For n sufficiently large,* $2^{\lfloor \sqrt{n}/20 \rfloor} \leq C^{\infty}(n)$.

We remark that the inequality in Lemma 6.1 would hold for the respective games played on any graph, since there are no properties of the hypercube used in the proof. Most other versions of the Graph Searching game, for example the version where the robber is infinitely fast and also invisible, make it even more difficult for the cops to catch the robber. Thus $C_{\mathbf{1a}}(n)$ is also a lower bound for those cop numbers. Though Lemma 6.1 is a first step toward connecting Cops and Robber with Graph Searching, we do not know how close the connection is.

Open Question 3 *How big is the gap between $C_{\mathbf{1a}}(n)$ and $C^{\infty}(n)$?*

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