Math 46 Abstract Algebra

Homework 4: Group homomorphisms. Due date: 04/17

1. Show that the groups $\mathbb{Z}_8, \mathbb{Z}_4 \times \mathbb{Z}_2$ and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are pairwise non-isomorphic. Hint: what are possible orders of elements of these groups?

2. If a group G has exactly one subgroup H of order k, prove that H is normal.

3. Let G be a group. Find all group homomorphisms form the cyclic group $\mathbb{Z}_n \longrightarrow G$. Use your result to:

(a) Show that for m, n relatively prime, there are not non-trivial group homomorphisms from $\mathbb{Z}_n \longrightarrow \mathbb{Z}_m$.
(b) Find all group homomorphisms from $\mathbb{Z}_3 \longrightarrow S_3$.
(c) Find all group homomorphisms from $\mathbb{Z}_4 \longrightarrow S_3$.

4. Show that the subgroup H of rotations is normal in the dihedral group $\mathbb{D}_n$. Find the quotient group $\mathbb{D}_n/H$.

5. Find all possible group homomorphisms from $\mathbb{D}_8$ to the cyclic group $\mathbb{Z}_{12}$. Determine the kernel of each homomorphism.

6. Find all normal subgroups in $\mathbb{D}_8$. Observe that some of them are given as kernel of homomorphisms from the previous question.

7. Consider the group $\mathbb{Z}_{24}$ and the subgroups $H = \langle 4 \rangle$ and $N = \langle 6 \rangle$:
(a) List the elements of $H.N$ (written as $H + N$) and $H \cap N$.
(b) List the elements of the quotients HN/N and $H/H \cap N$.
(c) Describe the correspondence given by the second isomorphism theorem.

8. Let $H = \{id, (12)(34), (13)(24), (14)(23)\}$.
(a) Check that H is a subgroup of S_4.
(b) Prove that H is normal in S_4 using that two permutations are conjugate in S_n iff they have the same cycle type.
(c) Show that the subgroup H generated by the 4-cycle (2314) is not normal in S_4.