Math 46 Abstract Algebra.

Homework 2: Examples and properties of groups. Cyclic groups. Due 3/8/2023

1. Consider groups G and G'. Prove that the direct product $G \times G'$ is a group. Show that if G, G' are abelian, the product is also abelian.

2. Find all subgroups for the groups V_4 and $\mathbb{Z}_4$.

3. Let $S = \mathbb{R} \setminus \{-1\}$ and define a binary operation on S by $a * b = ab + a + b$. Prove that $(S, *, 0)$ is an abelian group.

4. Show that if $(a * b)^2 = a^2 * b^2$ for all $a, b \in G$, then the group $(G, *)$ is abelian.

5. Show that a group in which every element is its own inverse must be abelian.

6. Let G be the group of $n \times n$ matrices with addition. Show that the set

$$\{A \in G \mid \operatorname{tr}(A) = 0\}$$

is a subgroup of G. (The trace $\operatorname{tr}(A)$ of a matrix A is the sum of elements in the diagonal.)

7. Find all orders of elements in the group $\mathbb{Z}_{12}$.

8. Let G be a group and $S \subset G$ a subset of G. Show that $C_S(G)$ is a subgroup of G. Show that the center $Z(G)$ is an abelian subgroup of G. (regardless of whether G is abelian or not)

9.[10 pts] Consider the cyclic group C_{40} with generator a. Find the orders of elements a^2, a^{12}, a^{-5} and a^{11} Does C_{40} have a subgroup of order 8? What about order 12?

10. This questions is about some multiplicative groups of units:
(a) Consider the group $\mathbb{Z}_9^*$ of invertible residues modulo 9 under multiplication. List all elements of this group, how many are there? Pick one or more elements of this group and find their orders. Is the group cyclic (can you find a generator)?
(b) Same as (a) for the group $\mathbb{Z}_{12}^*$.