Math 46: Abstract Algebra

Homework 1: Sets, maps and relations. Due date: 3/8/2022

1. [10 pts] List all subsets of the 3-element set $A = \{1, 2, 3\}$. How many subsets does a set with n elements have? Prove your result.

2. [10 pts] Consider sets X and Y with n and m elements respectively:
(a) How man elements are there in the Cartesian product $X \times Y$?
(b) How many relations are there between X and Y?
(c) How many functions are there from X to Y?

3. [10 pts] Prove, using induction, that if a, b, c are integers values corresponding to the sides of a right triangle. Then for any $n > 2$, $a^n + b^n \neq c^n$.

4. [10 pts] Find the last 2 digits in the decimal representation of the following numbers:

$$(a)\ 7^4 \qquad (b)\ 7^{102} \qquad (c)\ 9^{10} \qquad (d)\ 9^{103}.$$

5. [10 pts] Find necessary conditions for the maps $f(x) = mx + b$ and $g(x) = ax^2 + c$ to commute by composition: $f \circ g = g \circ f$.

6. [10 pts] Let A be a finite set. Prove that a map $f \colon A \longrightarrow A$ that is injective is also bijective. Give an example of an infinite set B and an injective map $f \colon B \longrightarrow B$ which is not surjective.

7. (a) [5 pts] Show that, for any set A, there is exactly one map f from the empty set $\emptyset$ to A. When is f injective? Surjective?
(b) [5 pts] Describe all maps from a set A to the empty set $\emptyset$ (the answer will depend on A).

8.. [10 pts] Let $x, y \in \mathbb{N}$ be relatively prime. If the product xy is a perfect square, prove that x and y must both be perfect squares. Hint: use the Fundamental Theorem of Arithmetic.

9. [10 pts] Prove that the integers q, r obtained in the division algorithm are unique.

10. [10 pts] Show that if S is a finite set, and $f \colon S \longrightarrow S$ is bijective, there exist $n > 0$ such that

$$f^{(n)} = f \circ f \cdots \circ f = 1_S.$$