GEOMETRY OF FOUR-FOLDS WITH THREE NON-COMMUTING INVOLUTIONS

JORGE PINEIRO

Abstract. In this paper we adapt some techniques developed for K3 surfaces, to study the geometry of a family of projective varieties in $\mathbb{P}^2_K \times \mathbb{P}^2_K \times \mathbb{P}^2_K$ defined as the intersection of a form of degree $(2,2,2)$ and a form of degree $(1,1,1)$. Members of the family will be equipped with dominant rational self-maps and we will study the actions of those maps on divisors and compute the first dynamical degrees of the composition of any pair.

1. Introduction

As a generalization of the work of Silverman and others [10], [6] on families of K3 surfaces with infinite groups of automorphisms, we study dynamics on a family of varieties $X^{A,B}$ in $\mathbb{P}^2_K \times \mathbb{P}^2_K \times \mathbb{P}^2_K$ defined as the intersection of a form of degree $(2,2,2)$ and a form of degree $(1,1,1)$. Individual members of the family $X^{A,B}$ come equipped with $(2:1)$-projections $p_1, p_2, p_3 : X^{A,B} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ that generate involutions $\sigma_1, \sigma_2, \sigma_3$ on $X^{A,B}$. In this situation however the maps σ_i for $i = 1,2,3$ are not morphisms of the whole $X^{A,B}$, but only rational dominant maps. Still it is possible to induce maps $\sigma_i^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X)$ and $\tilde{\sigma}_i^* : NS(X)_{\mathbb{Q}} \longrightarrow NS(X)_{\mathbb{Q}}$, on divisors modulo linear and numerical equivalence. The computations with divisors in the case of three involutions is going to be similar to the K3 surfaces of type $(2,2,2)$ in $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$ studied by several authors like Wang [12] and Baragar [2], [3] and [4].

The following degree associated to the dynamics was initially studied by Arnold in [1], and particularly for dominant rational maps by Silverman in [11].

Definition 1.1. *Let X be an algebraic variety and $\varphi : X \dashrightarrow X$ a dominant rational map. The first dynamical degree of φ is*

$$\delta_\varphi = \limsup_{n\to\infty} \rho(\widetilde{\varphi^{n*}})^{1/n},$$

2010 *Mathematics Subject Classification.* Primary: 37P55 ; Secondary: 14J35, 14J28, 32J15 .

where $\rho(\widetilde{\varphi^{n*}})$ represents the spectral radius or maximal eigenvalue of the map $\widetilde{\varphi^{n*}} : NS(X)_{\mathbb{Q}} \longrightarrow NS(X)_{\mathbb{Q}}$.

It is also possible to extend the notion of polarization, with respect to one rational map or, more general, in the sense of Kawaguchi [9], associated to several rational maps:

Definition 1.2. *Let X be a projective variety and $\varphi_i : X \dashrightarrow X$ for $i = 1, \ldots, k$ dominant rational maps. We say that the system $(X, \{\varphi_1, \ldots, \varphi_k\}, \mathcal{L}, d)$ is a polarized dynamical system of k maps if there exist an ample line bundle $\mathcal{L} \in \mathrm{Pic}(X) \otimes \mathbb{R}$ such that $\bigotimes_{i=1}^{k} \varphi_i^* \mathcal{L} \cong \mathcal{L}^d$ for some $d > k$.*

The action of the maps σ_1^*, σ_2^* and σ_3^* on $\mathrm{Pic}(X)$ will provide a polarization for the system of three maps $\{\sigma_1, \sigma_2, \sigma_3\}$. Also, under the condition that the Picard number is the least possible value $p(X) = 3$, the first dynamical degree of any of the maps $\sigma_{ij} = \sigma_i \circ \sigma_j$ will be computed. The computations will produce the same dynamical degree as the dynamical degree of the maps on K3 surfaces (Section 12 of [11]).

2. Four dimensional Varieties with three involutions

Let $\mathbf{L}^A \subset \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ be a family of varieties defined over a field K by a single equation linear on each variable,

$$\mathbf{L}^A = \{P \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(x, y, z) = \sum_{i,j,k=0}^{2} a_{i,j,k} x_i y_j z_k = 0\},$$

where $A = (a_{ijk})_{0 \leq i,j,k \leq 2}$. A member of the family $\mathbf{L}$ comes equipped with projections

$$p_3 = p_{xy} : \mathbf{L} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2,$$

$$p_2 = p_{xz} : \mathbf{L} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2,$$

$$p_1 = p_{yz} : \mathbf{L} \longrightarrow \mathbb{P}^2 \times \mathbb{P}^2,$$

and the $\mathrm{Pic}(\mathbf{L}) \cong \mathbb{Z}^3$ from the embedding $\mathbf{L} \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Using the adjunction formula we can get its canonical line bundle

$$\omega_{\mathbf{L}} \cong \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2}(-3, -3, -3) \otimes \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2}(\mathbf{L}) = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2}(-2, -2, -2).$$

By choosing a section $Q = Q^A$ of $\mathcal{O}_{\mathbf{L}}(2, 2, 2)$ and consider the variety $X = \mathrm{Var}(Q)$ we get a variety with trivial canonical divisor $K_X \sim 0$. Besides, by the weak lefschetz theorem, we have an injective map $\mathbb{Z}^3 \cong \mathrm{Pic}(L) \hookrightarrow \mathrm{Pic}(X)$ and we will get three distinct classes even in $NS(X)$ and therefore a Picard number $p(X) \geq 3$.

By varying the coefficients A, B one obtains a family $X^{A,B}$ defined in $\mathbb{P}^2_K \times \mathbb{P}^2_K \times \mathbb{P}^2_K$ by equations

$$L(x,y,z) = \sum_{i,j,k=0}^{2} a_{i,j,k} x_i y_j z_k = 0,$$

$$Q(x,y,z) = \sum_{i,j,k,l,m,n=0}^{2} b_{i,j,k,l,m,n} x_i x_l y_j y_m z_k z_n = 0,$$

where $A = (a_{ijk})$, $B = (b_{i,j,k,l,m,n})$ and all indices are moving in the set $\{0,1,2\}$. The projections p_1, p_2, p_3 restricted to X represent generically $(2:1)$ coverings of $\mathbb{P}^2 \times \mathbb{P}^2$. Indeed when we fix two of the variables we get the intersection on $\mathbb{P}^2$ of a quadric and a line, which is general, will give two points $P_i, P_i' \in X$ for $i = 1,2,3$ and will determine involutions $\sigma_1, \sigma_2, \sigma_3 : X \dashrightarrow X$. The involutions σ_i for $i = 1,2,3$, will not be in general morphisms but just rational maps defined on certain open sets $U_i \subset X$. We are interesting in studying the dynamics of the maps σ_i, but first we should devote some time to get familiar with the geometry of $X = X^{A,B}$. We collect the coefficients of our variables using the following notation for i, j, k in the set $\{0,1,2\}$

$$L_k^{x,y}(x,y) = \sum_{i,j=0}^{2} a_{i,j,k} x_i y_j, \quad Q_{k,n}^{x,y}(x,y) = \sum_{i,j,l,m=0}^{2} b_{i,j,k,l,m,n} x_i x_l y_j y_m,$$

$$L_j^{x,z}(x,z) = \sum_{i,k=0}^{2} a_{i,j,k} x_i z_k, \quad Q_{i,l}^{y,z}(y,z) = \sum_{j,k,m,n=0}^{2} b_{i,j,k,l,m,n} y_j y_m z_k z_n,$$

$$L_i^{y,z}(y,z) = \sum_{j,k=0}^{2} a_{i,j,k} y_j z_k, \quad Q_{j,m}^{x,z}(x,z) = \sum_{i,k,l,n=0}^{2} b_{i,j,k,l,m,n} x_i x_l z_k z_n.$$

Suppose, with the above notation in mind, that we want to study the action of σ_3 computing the solutions $(z_0, z_1, 1)$ of the system

$$\begin{aligned} 0 =& L_0^{x,y} z_0 + L_1^{x,y} z_1 + L_2^{x,y}, \\ 0 =& Q_{0,0}^{x,y} z_0^2 + Q_{1,1}^{x,y} z_1^2 + Q_{2,2}^{x,y} + Q_{0,1}^{x,y} z_0 z_1 + Q_{0,2}^{x,y} z_0 + Q_{1,2}^{x,y} z_1, \end{aligned}$$

assuming that $L_1^{x,y} \neq 0$ and replacing $z_1 = \frac{-L_2^{x,y} - L_0^{x,y} z_0}{L_1^{x,y}}$ in the second equation gives $G_0^{x,y} + H_{0,2}^{x,y} z_0 + G_2^{x,y} z_0^2 = 0$ where,

$$G_0^{x,y} = (L_1^{x,y})^2 Q_{2,2}^{x,y} - L_1^{x,y} L_2^{x,y} Q_{1,2}^{x,y} + (L_2^{x,y})^2 Q_{1,1}^{x,y},$$

$$G_2^{x,y} = (L_1^{x,y})^2 Q_{0,0}^{x,y} - L_1^{x,y} L_0^{x,y} Q_{0,1}^{x,y} + (L_0^{x,y})^2 Q_{1,1}^{x,y},$$

$$H_{0,2}^{x,y} = 2 L_0^{x,y} L_2^{x,y} Q_{1,1}^{x,y} - L_0^{x,y} L_1^{x,y} Q_{1,2}^{x,y} - L_2^{x,y} L_1^{x,y} Q_{0,1}^{x,y} + (L_1^{x,y})^2 Q_{0,2}^{x,y},$$

and the map σ_3 that sends $(z_0, z_1, 1) \mapsto (z_0', z_1', 1)$ will be defined unless all the three coefficients $G_0^{x,y}, H_{0,2}^{x,y}, G_2^{x,y}$ vanish. So, we are forced, by a codimension checking, to work with rational maps $\sigma_i : X \dashrightarrow X$ and our first task will be, to locate where are these maps well defined morphisms.
Motivated by the above discussion we define for any permutation (i, j, k) of $(0, 1, 2)$ the $(4, 4)$-bi-homogeneous forms

$$G_k^{x,y} = (L_i^{x,y})^2 Q_{j,j}^{x,y} - L_i^{x,y} L_j^{x,y} Q_{i,j}^{x,y} + (L_j^{x,y})^2 Q_{i,i}^{x,y},$$
$$G_k^{y,z} = (L_i^{y,z})^2 Q_{j,j}^{y,z} - L_i^{y,z} L_j^{y,z} Q_{i,j}^{y,z} + (L_j^{y,z})^2 Q_{i,i}^{y,z},$$
$$G_k^{x,z} = (L_i^{x,z})^2 Q_{j,j}^{x,z} - L_i^{x,z} L_j^{x,z} Q_{i,j}^{x,z} + (L_j^{x,z})^2 Q_{i,i}^{x,z},$$
$$H_{i,j}^{x,y} = 2L_i^{x,y} L_j^{x,y} Q_{kk}^{x,y} - L_i^{x,y} L_k^{x,y} Q_{jk}^{x,y} - L_j^{x,y} L_k^{x,y} Q_{ik}^{x,y} + (L_k^{x,y})^2 Q_{ij}^{x,y},$$
$$H_{i,j}^{x,z} = 2L_i^{x,z} L_j^{x,z} Q_{kk}^{x,z} - L_i^{x,z} L_k^{x,z} Q_{jk}^{x,z} - L_j^{x,z} L_k^{x,z} Q_{ik}^{x,z} + (L_k^{x,z})^2 Q_{ij}^{x,z},$$
$$H_{i,j}^{y,z} = 2L_i^{y,z} L_j^{y,z} Q_{kk}^{y,z} - L_i^{y,z} L_k^{y,z} Q_{jk}^{y,z} - L_j^{y,z} L_k^{y,z} Q_{ik}^{y,z} + (L_k^{y,z})^2 Q_{ij}^{y,z},$$

For any $a, b, c \in \mathbb{P}_K^2$, the fibres of the projections p_1, p_2 and p_3 will be defined as $X_{a,b}^z = p_3^{-1}(a, b) = L_{a,b}^z \cap Q_{a,b}^z$, $X_{b,c}^x = p_1^{-1}(b, c) = L_{b,c}^x \cap Q_{b,c}^x$ and $X_{a,c}^y = p_2^{-1}(a, c) = L_{a,c}^y \cap Q_{a,c}^y$; where

$$L_{a,b}^z = \{(a, b, z) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(a, b, z) = 0\},$$
$$Q_{a,b}^z = \{(a, b, z) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : Q(a, b, z) = 0$$
$$L_{b,c}^x = \{(x, b, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(x, b, c) = 0\},$$
$$Q_{b,c}^x = \{(x, b, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : Q(x, b, c) = 0$$
$$L_{a,c}^y = \{(a, y, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : L(a, y, c) = 0\},$$
$$Q_{a,c}^y = \{(a, y, c) \in \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 : Q(a, y, c) = 0.$$

Definition 2.1. *We say that a fibre $X_{a,b}^z, X_{b,c}^x$ or $X_{a,c}^y$ is degenerate if it has positive dimension.*

If the fibres $X_{a,b}^z, X_{b,c}^x$ or $X_{a,c}^y$ are non-degenerate at (a, b, c), they will consist of two points and the maps σ_1, σ_2 and σ_3 will be well defined morphisms at $(a, b, c) \in X$. Following the outline of [6] we have the following result characterizing the degenerate fibres.

Proposition 2.2. *Let $[a, b, c] \in X$.*

(1) *$X_{a,b}^z$ is degenerate if and only if*

$$G_0^{x,y}(a, b) = G_1^{x,y}(a, b) = G_2^{x,y}(a, b) = H_{0,1}^{x,y}(a, b) = H_{0,2}^{x,y}(a, b) = H_{1,2}^{x,y}(a, b) = 0.$$

(2) *$X_{a,c}^y$ is degenerate if and only if*

$$G_0^{x,z}(a, c) = G_1^{x,z}(a, c) = G_2^{x,z}(a, c) = H_{0,1}^{x,z}(a, c) = H_{0,2}^{x,z}(a, c) = H_{1,2}^{x,z}(a, c) = 0.$$

(3) *$X^x_{b,c}$ is degenerate if and only if*

$$G_0^{y,z}(b,c)=G_1^{y,z}(b,c)=G_2^{y,z}(b,c)=H_{0,1}^{y,z}(b,c)=H_{0,2}^{y,z}(b,c)=H_{1,2}^{y,z}(b,c)=0.$$

Proof. The proof is identical to the proof of proposition 1.4 in [6]. We do the proof of (1). When we substitute $z_0=(L-L_1^{x,y}z_1-L_2^{x,y}z_2)/L_0^{x,y}, z_1=(L-L_0^{x,y}z_0-L_2^{x,y}z_2)/L_1^{x,y}$ and $z_2=(L-L_1^{x,y}z_1-L_0^{x,y}z_0)/L_2^{x,y}$ into Q respectively we get formulas:

$$(L_0^{x,y})^2Q(x,y,z)\equiv G_2^{x,y}z_1^2+H_{1,2}^{x,y}z_1z_2+G_1^{x,y}z_2^2 \quad (modL(x,y,z)),$$

$$(L_1^{x,y})^2Q(x,y,z)\equiv G_2^{x,y}z_0^2+H_{0,2}^{x,y}z_0z_2+G_0^{x,y}z_2^2 \quad (modL(x,y,z)),$$

$$(L_2^{x,y})^2Q(x,y,z)\equiv G_1^{x,y}z_0^2+H_{0,1}^{x,y}z_0z_1+G_0^{x,y}z_1^2 \quad (modL(x,y,z)).$$

Now, the proof is divided into two parts, depending on whether or not for the point $[a,b,c]\in\mathbb{P}^2\times\mathbb{P}^2\times\mathbb{P}^2$ we have $L(a,b,z)\equiv 0$.
If $L(a,b,z)\equiv 0$, then $X^z_{a,b}=Q^z_{a,b}$ and the fibre is degenerate. In this case $L_0^{a,b}=L_1^{a,b}=L_2^{a,b}=0$ will force $H_{i,j}^{x,y}(a,b)=G_k^{x,y}(a,b)=0$ and the proof is finished.
If $L(a,b,z)\neq 0$, one of the $L_i^{x,y}(a,b)\neq 0$ and the fact that $G_0^{x,y}(a,b)=G_1^{x,y}(a,b)=G_2^{x,y}(a,b)=H_{0,1}^{x,y}(a,b)=H_{0,2}^{x,y}(a,b)=H_{1,2}^{x,y}(a,b)=0$ forces $Q(a,b,z)\equiv 0 \quad (modL(a,b,z))$ and hence $X^z_{a,b}$ is degenerate containing the entire line $L^z_{a,b}$.
If $L(a,b,z)\neq 0$ and the fibre $X^z_{a,b}$ is degenerate we must have $L^z_{a,b}\subset Q^z_{a,b}$. We are going to proof that $G_0^{x,y}(a,b)=G_1^{x,y}(a,b)=G_2^{x,y}(a,b)=H_{0,1}^{x,y}(a,b)=H_{0,2}^{x,y}(a,b)=H_{1,2}^{x,y}(a,b)=0$. First let's do $G_0^{x,y}(a,b)=0$. If $L_1^{x,y}(a,b)=L_2^{x,y}(a,b)=0$, this follows from the definition, otherwise $(0,L_2^{x,y}(a,b),-L_1^{x,y}(a,b))\in L^z_{a,b}$ and therefore must belong to $Q^z_{a,b}$, when we evaluate we get

$$0=Q_{1,1}^{x,y}(a,b)(L_2^{x,y}(a,b))^2-Q_{1,2}^{x,y}L_2^{x,y}(a,b)L_1^{x,y}(a,b)+Q_{2,2}^{x,y}(L_1^{x,y}(a,b))^2$$

So $G_0^{x,y}(a,b)=0$. In a similar way we do $G_1^{x,y}(a,b)=G_2^{x,y}(a,b)=0$. The substitution of the results $G_i^{x,y}(a,b)=0$ in the equations and evaluations at $x=a, y=b$ will give

$$H_{1,2}^{x,y}(a,b)z_1z_2=H_{0,2}^{x,y}(a,b)z_0z_2=H_{1,0}^{x,y}(a,b)z_1z_0=0$$

for all points $(z_0,z_1,z_2)\in L^z(a,b)$. If $L^z(a,b)$ is the line $z_1=0$, then $L_0^{x,y}(a,b)=L_2^{x,y}(a,b)=0$ and $H_{1,2}^{x,y}(a,b)=0$ using the definition. If $L^z(a,b)$ is the line $z_2=0$, then $L_1^{x,y}(a,b)=L_2^{x,y}(a,b)=0$ and $H_{1,2}^{x,y}(a,b)=0$ will be again equal to zero. Otherwise if $L^z_{a,b}$ is none of the lines $z_1=0$ or $z_2=0$, then $H_{1,2}^{x,y}(a,b)=0$ from the previous line. The other cases for $H_{i,j}^{x,y}(a,b)=0$ are solved similarly. $\square$

We can now define open sets U_1, U_2, U_3 in such a way that the dominant rational maps $\sigma_i : X \dashrightarrow X$ are bijective morphisms

$$\sigma_i : U_i \longrightarrow U_i.$$

$$\begin{aligned} U_1 = X - \{(a,b,c) \in X : G_0^{y,z}(b,c) = G_1^{y,z}(b,c) = G_2^{y,z}(b,c) = 0 \\ H_{0,1}^{y,z}(b,c) = H_{0,2}^{y,z}(b,c) = H_{1,2}^{y,z}(b,c) = 0\}, \\ U_2 = X - \{(a,b,c) \in X : G_0^{x,z}(a,c) = G_1^{x,z}(a,c) = G_2^{x,z}(a,c) = 0 \\ H_{0,1}^{x,z}(a,c) = H_{0,2}^{x,z}(a,c) = H_{1,2}^{x,z}(a,c) = 0\}, \\ U_3 = X - \{(a,b,c) \in X : G_0^{x,y}(a,b) = G_1^{x,y}(a,b) = G_2^{x,y}(a,b) = 0 \\ H_{0,1}^{x,y}(a,b) = H_{0,2}^{x,y}(a,b) = H_{1,2}^{x,y}(a,b) = 0\}. \end{aligned}$$

The maps $\sigma_1, \sigma_2, \sigma_3$ induce maps on divisors: Let's consider Y a closed subvariety of codimension one and $\sigma_i^* Y = \overline{\sigma_i^{-1} Y}$, the Zariski closure of the pre-image. In this way we induce maps on Weil divisors, that respect linear and numerical equivalence and descend to maps

$$\sigma_i^* : \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X) \qquad \tilde{\sigma}_i^* : NS(X)_{\mathbb{Q}} \longrightarrow NS(X)_{\mathbb{Q}}.$$

To study the action of the σ_i^* on $\mathrm{Pic}(X)$ we denote by H, H' hyperplane sections representing the two fundamental classes in $\mathrm{Pic}(\mathbb{P}^2 \times \mathbb{P}^2)$,

$$\begin{aligned} H &= \{((a_0 : a_1 : a_2), (b_0 : b_1 : b_2)) \in \mathbb{P}^2 \times \mathbb{P}^2 : a_0 = 0\}, \\ H' &= \{((a_0 : a_1 : a_2), (b_0 : b_1 : b_2)) \in \mathbb{P}^2 \times \mathbb{P}^2 : b_0 = 0\}. \end{aligned}$$

and the divisors D_x, D_y, D_z on X defined by:

$$D_x = \{P \in X : x_0 = 0\}, \quad D_y = \{P \in X : y_0 = 0\}, \quad D_z = \{P \in X : z_0 = 0\}.$$

The pullbacks of H, H' by the different projections give back the D_x, D_y, D_z,

$$\begin{aligned} p_{xy}^* H &= p_3^* H = D_x, & p_{xy}^* H' &= p_3^* H' = D_y, \\ p_{xz}^* H &= p_2^* H = D_x, & p_{xz}^* H' &= p_2^* H' = D_z, \\ p_{yz}^* H &= p_1^* H = D_y, & p_{yz}^* H' &= p_1^* H' = D_z. \end{aligned}$$

Lemma 2.3. *We have the following equivalences of divisors in* $\mathrm{div}(X)$*:*

(a) $p_{1*} p_2^* H \sim 4H + 4H'$; (b) $p_{2*} p_1^* H \sim 4H + 4H'$;
(c) $p_{3*} p_1^* H' \sim 4H + 4H'$.

Proof. The prove of all parts will be analogous and straightforward from the definition of H, H' and the p_i's. Let's see for example the proof of (a). The pull-back $p_2^* H = \{P \in X : x_0 = 0\}$ is given by the two equations

$$L_1^{y,z} x_1 + L_2^{y,z} x_2 = 0, \qquad Q_{1,1}^{y,z} x_1^2 + Q_{1,2}^{y,z} x_1 x_2 + Q_{2,2}^{y,z} x_2^2 = 0.$$

When we project onto (y,z) we eliminate x_1, x_2 and get the equation

$$G_0^{y,z} = (L_1^{y,z})^2 Q_{2,2}^{y,z} - L_1^{y,z} L_2^{y,z} Q_{1,2}^{y,z} + (L_2^{y,z})^2 Q_{1,1}^{y,z} = 0.$$

where $G_0^{y,z}$ is a $(4,4)$-bihomogeneous form in y and z, and therefore $p_{1*}p_2^*H \sim 4H + 4H'$. ☐

Applying lemma 2.3 we obtain the pushforwards:

$$p_{1*}(D_x) \sim 4H+4H', \qquad p_{2*}(D_y) \sim 4H+4H', \qquad p_{3*}(D_z) \sim 4H+4H',$$

and the action of the σ_i^*'s on the divisors D_x, D_y, D_z:

$$\begin{aligned}
\sigma_1^*(D_x) &= p_1^* p_{1*} D_x - D_x \sim 4D_y + 4D_z - D_x,\\
\sigma_1^*(D_y) &= \sigma_1^* p_1^* H = (p_1 \circ \sigma_1)^* H = D_y,\\
\sigma_1^*(D_z) &= \sigma_1^* p_1^* H' = (p_1 \circ \sigma_1)^* H' = D_z,\\
\sigma_2^*(D_x) &= \sigma_2 p_2^* H = (p_2 \circ \sigma_2)^* H' = D_x,\\
\sigma_2^*(D_y) &= p_2^* p_{2*} D_y - D_y \sim 4D_x + 4D_z - D_y,\\
\sigma_2^*(D_z) &= \sigma_2^* p_2^* H' = (p_2 \circ \sigma_2)^* H' = D_z,\\
\sigma_3^*(D_x) &= \sigma_3^* p_3^* H = (p_3 \circ \sigma_3)^* H = D_x,\\
\sigma_3^*(D_y) &= \sigma_3^* p_3^* H' = (p_3 \circ \sigma_3)^* H' = D_y,\\
\sigma_3^*(D_z) &= p_3^* p_{3*} D_z - D_z \sim 4D_x + 4D_y - D_z.
\end{aligned}$$

Using the actions of the σ_i^* we can get a polarizations by a very ample line bundle for the system of involutions $\sigma_1, \sigma_2, \sigma_3$.

Proposition 2.4. *Suppose that r_x, r_y, r_z are positive real numbers and we have the polarization by three maps*

$$\sum_i \sigma_i^*(r_x D_x + r_y D_y + r_z D_z) \sim d(r_x D_x + r_y D_y + r_z D_z),$$

in $\mathrm{Pic}(X) \otimes \mathbb{R}$. *Then $d = 9$ and $r_x = r_y = r_z = 1$.*

Proof. When we add up the actions of σ_i^* on $r_x D_x + r_y D_y + r_z D_z$, and equal that to $d(r_x D_x + r_y D_y + r_z D_z)$ for some $d > 3$, we get the system of linear equations:

$$\begin{aligned}
r_x + 4r_y + 4r_z &= d r_x,\\
4r_x + r_y + 4r_z &= d r_y,\\
4r_x + 4r_y + r_z &= d r_z.
\end{aligned}$$

The determinant is $(9-d)(3+d)^3$ and the value of $d = 9$ gives $r_x = r_y = r_z = 1$. ☐

Proposition 2.5. *The maps σ_i and $\sigma_{ij} = \sigma_i \circ \sigma_j$, for $i, j \in \{0,1,2\}$, satisfy the properties:*

(1) $(\sigma_i \circ \sigma_j)^* = \sigma_j^* \circ \sigma_i^*$,

(2) $(\sigma_{ij}^n)^* = (\sigma_{ij}^*)^n$.

Proof. In general, given two rational maps $\tau : X \dashrightarrow X$ and $\tau' : X \dashrightarrow X$ defining involutions $\tau : U_\tau \longrightarrow U_\tau$ and $\tau' : U_{\tau'} \longrightarrow U_{\tau'}$ on open sets U_τ and $U_{\tau'}$ respectively, we will have $(\tau \circ \tau')^* = \tau'^* \circ \tau^*$. Let Y be an irreducible subvariety. If $P \in \overline{\tau(Y \cap U_\tau)} \cap U_{\tau'}$, there exist a sequence $P_n \to P$, with $P_n \in \tau(Y \cap U_\tau) \cap U_{\tau'}$. Therefore $\tau'(P_n) \to \tau'(P)$ and $\tau'(P) \in \overline{\tau'(\tau(Y \cap U_\tau)) \cap U_{\tau'})}$. In other words $\overline{\tau'(\overline{\tau(Y \cap U_\tau)} \cap U_{\tau'})} \subset \overline{\tau'(\tau(Y \cap U_\tau) \cap U_{\tau'})}$, so this two sets must be equal and $(\tau \circ \tau')^* = \tau'^* \circ \tau^*$. For the first part of the theorem we take $\sigma_i = \tau$ and $\sigma_j = \tau'$. For the second part we proceed by induction and use the result to proof the induction step. If we suppose that $(\sigma_{ij}^n)^* = (\sigma_{ij}^*)^n$ is true, then $(\sigma_{ij}^*)^{n+1} = \sigma_{ij}^*((\sigma_{ij}^*)^n) = \sigma_{ij}^*((\sigma_{ij}^n)^*)$ By our result above with $\tau = \sigma_{ij}$ and $\tau' = \sigma_{ij}^n$, the last equals to $(\sigma_{ij}^{n+1})^*$. □

2.1. **Computation of dynamical degree.** In this subsection we study the action induced by the maps $\sigma_{ij} = \sigma_i \circ \sigma_j$ on the subspace $V = \text{Span}(D_x, D_y, D_z)$ of $\text{Pic}(X) \otimes \mathbb{R}$. As an application we will be able to get the dynamical degree of those maps for members of the family with Picard number $p(X) = 3$.

Theorem 2.6. *Let σ_{ij} be the rational dominant map $\sigma_i \circ \sigma_j : X \dashrightarrow X$. Let V be the subspace of $\text{Pic}(X) \otimes \mathbb{R}$ spanned by D_x, D_y, D_z and consider the action of $\sigma_{ij}^{*n} : V \longrightarrow V$. The eigenvalues of $\sigma_{ij}^{*n}|V$ belong to the set $\{1, \beta^n, \beta'^n\}$, where $\beta = 7 + 4\sqrt{3}$ and $\beta' = \frac{1}{\beta}$.*

Proof. The action of the maps σ_{12}^*, σ_{31}^*, σ_{31}^*, σ_{32}^*, σ_{13}^* and σ_{23}^* with respect to that base $\{D_x, D_y, D_z\}$ is given respectively by the matrices

$$\sigma_{12}^* = \begin{pmatrix} -1 & -4 & 0 \\ 4 & 15 & 0 \\ 4 & 20 & 1 \end{pmatrix} \qquad \sigma_{13}^* = \begin{pmatrix} 15 & 0 & 4 \\ 20 & 1 & 4 \\ -4 & 0 & -1 \end{pmatrix}$$

$$\sigma_{12}^* = \begin{pmatrix} 15 & 4 & 0 \\ -4 & -1 & 0 \\ 20 & 4 & 1 \end{pmatrix} \qquad \sigma_{23}^* = \begin{pmatrix} 1 & 20 & 4 \\ 0 & 15 & 4 \\ 0 & -4 & -1 \end{pmatrix}$$

$$\sigma_{31}^* = \begin{pmatrix} -1 & 0 & -4 \\ 4 & 1 & 20 \\ 4 & 0 & 15 \end{pmatrix} \qquad \sigma_{32}^* = \begin{pmatrix} 1 & 4 & 20 \\ 0 & -1 & -4 \\ 0 & 4 & 15 \end{pmatrix}$$

With the help of SAGE we find that the six matrices are sharing the same characteristic polynomial $p = -(\lambda - 1)(\lambda^2 - 14\lambda + 1)$. The roots of $p(\lambda)$ are $\{1, \beta, \beta'\}$ with $\beta = 7 + 4\sqrt{3}$ and $\beta' = 1/\beta$, therefore all the six matrices are diagonalizable and the eigenvalues of the the powers are from the set $\{1, \beta^n, \beta'^n\}$. □

Corollary 2.7. *Suppose that the Picard number $p(X) = 3$, then the first dynamical degree $\delta_{\sigma_{ij}}$ of σ_{ij} is $\delta_{\sigma_{ij}} = \beta$.*

Proof. The divisors D_x, D_y, D_z represent three distinct classes in $NS(X)_{\mathbb{Q}}$. If the Picard number $p(X) = 3$, then we have $NS(X)_{\mathbb{Q}} \cong V_{\mathbb{Q}}$. The first dynamical degree of any of the maps σ_{ij} is:

$$\delta_{\sigma_{ij}} = \limsup_{n\to\infty} \rho((\sigma_{ij}^n)^*)^{1/n} = \limsup_{n\to\infty} \rho((\sigma_{ij}^*)^n)^{1/n} = \limsup_{n\to\infty} (\beta^n)^{1/n} = \beta.$$

□

References

[1] V. I. Arnold, *Dynamics of complexity of intersections.* Bol. Soc. Brasil. Mat. (N.S.), 21(1) (1990), 1-10.

[2] A. Baragar, *Canonical vector heights on K3 surfaces with Picard number three: an argument for nonexistence*, Math. Comp. **73** (2004), 2019-2025.

[3] ——— *Rational points on K3 surfaces in* $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$, Math. Ann. **305** (1996), 541-558.

[4] A. Baragar and van Luijk, *K3 surfaces with Picard number three and canonical heights*, Math. Comp. **76** (2007), 1493-1498.

[5] G. S. Call and J. Silverman, *Canonical heights on varieties with morphisms*, Compositio Math. **89** (1993), 163–205.

[6] ——— *Computing the canonical height on K3 surfaces.* Math. Comp., **65** (1996), 259-290.

[7] R. Hartshorne, *Algebraic geometry*, Springer-Verlag, Graduate Texts in Mathematics, vol. **52**, New York, (1977).

[8] M. Hindry and J. Silverman, *Diophantine Geometry: an introduction*, Graduate Texts in Mathematics **201** (2000).

[9] S. Kawaguchi, *Canonical heights, invariant currents, and dynamical eigensystems of morphisms for line bundles*, J. Reine Angew. Math. **597** (2006), 135-173

[10] J. Silverman, *Rational points on K3 surfaces: a new canonical height*, Invent. Math. **105** (1991), 347-373.

[11] ——— *Dynamical degrees, arithmetic degrees, and canonical heights for dominant rational self-maps of projective space*, to appear in Ergodic Th. and Dyn. Sys.

[12] L. Wang, *Rational points and canonical heights on K3-surfaces in* $\mathbb{P}_1 \times \mathbb{P}_1 \times \mathbb{P}_1$, Contemp. Math. **186**, AMS, (1994) 273-289.

Department of Mathematics and Computer Science. Bronx Community College of CUNY. 2155 University Ave. Bronx, NY 10453

E-mail address: `jorge.pineiro@bcc.cuny.edu`