Ramanujan J (2006) 12:327–347
DOI 10.1007/s11139-006-0147-2

L-functions of second-order cusp forms

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Received: 8 September 2003 / Accepted: 9 June 2004

Abstract We discuss equivalent definitions of holomorphic second-order cusp forms and prove bounds on their Fourier coefficients. We also introduce their associated L-functions, prove functional equations for twisted versions of these L-functions and establish a criterion for a Dirichlet series to originate from a second order form. In the last section we investigate the effect of adding an assumption of periodicity to this criterion.

Keywords L-functions · Converse theorems · Cuspforms · Second-order automorphic forms

2000 Mathematics Subject Classification Primary—11F12, 11F66

G. Mason: Research supported in part by NSF Grant DMS 0245225.

C. O'Sullivan: Research supported in part by PSC CUNY Research Award No. 65453-00 34.

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1 Introduction

The study of second-order modular forms has been initiated in connection with percolation theory [7] and Eisenstein series formed with modular symbols (cf. [1]). More recently, second-order modular forms have appeared in research on converse theorems.

Specifically, the pursuit of converse theorems for L-functions requiring the minimum number of twists possible has been a long-standing project of great interest. One of the approaches, due to B. Conrey and D. Farmer, has been successful in small levels (cf. [2]). It transpires that, for the extension of this approach to higher levels, it is necessary to study a kind of second-order modular form that involves two groups. In particular, proving that, in some cases, there are no such functions (besides the usual modular forms) is enough to prove a converse theorem without twists for some levels (cf. [3]).

Motivated by this relation between such forms and converse theorems of L-functions and by the success of L-functions in the study of usual modular forms, in this paper we initiate a study of L-functions of second-order modular forms.

In Section 2, we first define and classify the holomorphic second-order modular forms. Although the structure of general second-order modular forms has already been determined in [1], a separate discussion is necessary here mainly because we require precise information about growth in the sequel. Moreover, for our investigations on converse theorems mentioned above, we are also interested in holomorphic second-order modular forms that are not invariant under all parabolics.

In Section 3 we see that the L-function of a second-order modular form satisfies the usual functional equation. We did not find a functional equation for the L-function of a second-order modular form with Fourier coefficients twisted by a Dirichlet character. Instead, we used the classification theorem to define two twisting operators which do yield a functional equation (Theorem 11).

Given that we do not have a functional equation of the classical type, we should not expect a converse theorem for second-order modular forms. Nevertheless, we managed to apply Razar's method to obtain a criterion for functions satisfying certain 4-term functional equations to be L-functions of second-order modular forms. Section 4 is devoted to the statement and proof of this criterion (Theorem 14).

The paper ends in Section 5 with a discussion of the effect of the periodicity on functions satisfying this criterion and on second-order modular forms in general. We do believe that the twisted L-function of a second-order modular form should have a functional equation of the usual type. It seems likely that it will require two Dirichlet characters and be a 4-term functional equation similar to the one in Theorem 14. In future work we hope to find it along with its converse theorem.

The eventual goal is to extend these results to more general cases and, especially, to cases related to the converse theorem. For this reason, we have tried to minimize the dependence of our proofs on the specific features of the functions in [1].

2 The space of holomorphic second-order cusp forms

The definitions and some basic properties of the holomorphic automorphic forms under study are now given. We try to make the conditions and definitions for these spaces as flexible as possible. Some of this material is standard (see for example [5, 8]) but is included for comparison with the corresponding facts for second-order modular forms.

We first describe the group, group action and other concepts on which all subsequent definitions are based. In subsection 2.1 we discuss two equivalent definitions of the classical cusp forms. Next, in 2.2 we give several equivalent definitions of the second-order cusp forms and prove some basic properties. Finally, in 2.3 we introduce a variation of these second-order forms that dispenses with one of the equivariance conditions and we give a characterization of their space.

Let Γ be a Fuchsian group of the first kind with parabolic elements and of genus g. We use the set of generators of Γ given by Fricke and Klein. Specifically, there are $2g$ hyperbolic elements $\gamma_1, \dots, \gamma_{2g}$, r elliptic elements $\epsilon_1, \dots, \epsilon_r$ and m parabolic elements $\pi_1, \dots, \pi_m$ generating Γ which satisfy the $r+1$ relations:

$$[\gamma_1, \gamma_{g+1}] \cdots [\gamma_g, \gamma_{2g}] \epsilon_1 \cdots \epsilon_r \pi_1 \cdots \pi_m = 1,\ \epsilon_j^{e_j} = 1$$

for $1 \leqslant j \leqslant r$ and integers $e_j \geqslant 2$. Here [a, b] denotes the commutator $a^{-1}b^{-1}ab$ of a and b.

Now, fix a fundamental domain $\mathfrak{F}$ for $\Gamma\backslash\mathfrak{h}$, where $\mathfrak{h}$ is the upper-half plane. Since Γ is a Fuchsian group of the first kind we assume its boundary is a polygon and label the finite number of inequivalent cusps with Gothic letters such as $\mathfrak{a}$, $\mathfrak{b}$. The corresponding scaling matrices $\sigma_{\mathfrak{a}}, \sigma_{\mathfrak{b}}$ in $\mathrm{SL}_2(\mathbb{R})$ map the neighborhood of each cusp to the upper part of the vertical strip of width one. This means that $\sigma_{\mathfrak{a}}^{-1}\Gamma_{\mathfrak{a}}\sigma_{\mathfrak{a}} = \Gamma_\infty$ for

$$\Gamma_{\mathfrak{a}} = \{\gamma \in \Gamma \mid \gamma\mathfrak{a} = \mathfrak{a}\},$$
$$\Gamma_\infty = \left\{ \pm \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \,\middle|\, n \in \mathbb{Z} \right\},$$

where Γ_∞ is not necessarily in Γ and ∞ may not be a cusp of $\mathfrak{F}$. See [6], Chapter 2 for this notation and more details.

For every even $k \in \mathbb{Z}$, we define an action of $\mathrm{GL}_2(\mathbb{R})^+$ on the space of functions on $\mathfrak{h}$, setting

$$(f|_k\gamma)(z) := f(\gamma z)(cz+d)^{-k}(\det(\gamma))^{k/2}$$

for all $f : \mathfrak{h} \to \mathbb{C}$, $z = x + iy \in \mathfrak{h}$ and $\gamma = [\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}] \in \mathrm{GL}_2(\mathbb{R})^+$. We extend the action to $\mathbb{C}[\mathrm{GL}_2(\mathbb{R})^+]$ by linearity. Throughout the paper we use T and S for the generators $[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}]$ and $[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]$ of $\mathrm{SL}_2(\mathbb{Z})$, respectively.

2.1 The classical cusp forms

Definition of $S_k(\Gamma)$. Let k be a positive even integer. Define $S_k(\Gamma)$ to be the $\mathbb{C}$-vector space of functions f such that

A1. $f : \mathfrak{h} \to \mathbb{C}$ is holomorphic,

A2. $f|_k(\gamma - 1) = 0$ for all γ in Γ,
A3. ("vanishing at the cusps") for each cusp $\mathfrak{a}$, $(f|_k\sigma_{\mathfrak{a}})(z) \ll e^{-cy}$ as $y \to \infty$ uniformly in x for a constant $c > 0$.

We call the elements of this space the holomorphic weight k cusp forms. The growth condition **A3** is simple and natural but it is often useful to know the growth of f in the entire upper half plane without referring to the cusps. To achieve this goal we need the following proposition.

Proposition 1. *Suppose f is holomorphic on $\mathfrak{h}$ and that $y^r|f(z)| \ll 1$ for some $r \geqslant k/2$. If $(f|_k\sigma_{\mathfrak{a}})(z+1) = (f|_k\sigma_{\mathfrak{a}})(z)$ for some cusp $\mathfrak{a}$ then*

$$(f|_k\sigma_{\mathfrak{a}})(z) = \sum_{n=0}^{\infty} b_{\mathfrak{a}}(n)e(nz) \tag{1}$$

where $b_{\mathfrak{a}}(n) \ll n^r$ for $n \geqslant 1$. If $r < k$ then $b_{\mathfrak{a}}(0) = 0$.

Since the proof would interrupt the exposition we put it at the end of Section 2.

We may now replace **A3** by
A3.1. $y^{k/2}|f(z)| \ll 1$ for all z in $\mathfrak{h}$.

Lemma 2. *We have $f \in S_k(\Gamma)$ if and only if f satisfies* **A1, A2** *and* **A3.1.**

Proof: Any f in $S_k(\Gamma)$ has exponential decay at each cusp by **A3** and hence $y^{k/2}|f(z)|$ is bounded on $\mathfrak{F}$, Implicitly here, and throughout this paper, we are thinking of the fundamental domain $\mathfrak{F}$ as a central compact set attached to noncompact cuspidal zones. Each of these zones is homeomorphic, by way of a scaling matrix, to the upper part of a vertical strip. See [6] Section 2.2 for a more complete discussion of this. Now if $y^{k/2}|f(z)| \leqslant C$ on $\mathfrak{F}$ then it is necessarily bounded by C on $\gamma\mathfrak{F}$ also, for any γ in Γ, since $y^{k/2}|f(z)|$ has weight 0. The images of $\mathfrak{F}$ tessellate $\mathfrak{h}$ and it follows that f satisfies **A3.1.**.

In the other direction, suppose f satisfies **A1, A2** and **A3.1.** By Proposition 1, f has the Fourier expansion (1) at any cusp $\mathfrak{a}$ with Fourier coefficients $b_{\mathfrak{a}}(n) \ll n^{k/2}$ and $b_{\mathfrak{a}}(0) = 0$. Therefore f has exponential decay at any cusp. $\square$

Incidentally, in the proof we showed Hardy's 'trivial' bound of $n^{k/2}$ for the nth Fourier coefficient of f in $S_k(\Gamma)$.

2.2 The second-order cusp forms

Definition of $S^2_k(\Gamma)$. We define the space $S^2_k(\Gamma)$ to consist of functions f such that

B1. $f : \mathfrak{h} \to \mathbb{C}$ is holomorphic,
B2. $f|_k(\gamma - 1) \in S_k(\Gamma)$ for all γ in Γ,
B3. ("vanishing at the cusps") for each cusp $\mathfrak{a}$, $(f|_k\sigma_{\mathfrak{a}})(z) \ll e^{-cy}$ as $y \to \infty$ uniformly in x for a constant $c > 0$,
B4. $f|_k(\pi - 1) = 0$ for all parabolic π in Γ.

This is the space of holomorphic, weight k, (parabolic) second-order cusp forms. It is similar to $S_k(\Gamma)$, the only difference being the transformation rule **B2.**

We recall the definitions of the functions Λ_i of [1]. For $1 \leqslant i \leqslant 2g$ we may define $L_i \in \mathrm{Hom}(\Gamma, \mathbb{C})$ such that

$$L_i(\gamma_j) = \delta_{ij}$$

for the hyperbolic generators and

$$L_i(\gamma) = 0$$

for the parabolic and elliptic generators γ of Γ. Each L_i vanishes at the parabolic elements of Γ. Therefore, by the Eichler-Shimura isomorphism (cf. e.g. [1]), for each $i \in \{1, \dots, 2g\}$ there exist $g_i, h_i \in S_2(\Gamma)$ such that the function Λ_i defined by

$$\Lambda_i(z) := \int_{z_0}^{z} g_i(w)dw + \overline{\int_{z_0}^{z} h_i(w)dw}$$

for a fixed $z_0 \in \mathfrak{h}$, satisfies

$$\Lambda_i(\gamma z) - \Lambda_i(z) = L_i(\gamma)$$

for all $z \in \mathfrak{h}$ and all $\gamma \in \Gamma$. The fixed point z_0 is usually taken to be the imaginary number i. For convenience we also set $\Lambda_0 \equiv 1$.

We need the following result.

Lemma 3. *For $1 \leqslant i \leqslant 2g$, all z in $\mathfrak{h}$, all $\gamma \in \Gamma$ and any cusp $\mathfrak{a}$ we have*

$$\Lambda_i(z) \ll \left|\log \mathrm{Im}\big(\sigma_{\mathfrak{a}}^{-1}\gamma z\big)\right| + 1,$$

with the implied constant independent of z.

Proof: We have

$$\begin{aligned}\Lambda_i(\gamma^{-1}\sigma_{\mathfrak{a}}z) &= \int_{z_0}^{\gamma^{-1}\sigma_{\mathfrak{a}}z} g_i(w)dw + \overline{\int_{z_0}^{\gamma^{-1}\sigma_{\mathfrak{a}}z} h_i(w)dw} \\ &= \int_{\sigma_{\mathfrak{a}}^{-1}\gamma z_0}^{z} (g_i|_2\sigma_{\mathfrak{a}})(w)dw + \overline{\int_{\sigma_{\mathfrak{a}}^{-1}\gamma z_0}^{z} (h_i|_2\sigma_{\mathfrak{a}})(w)dw}.\end{aligned}$$

Note that $g_i|_2\sigma_{\mathfrak{a}}$ and $h_i|_2\sigma_{\mathfrak{a}}$ are elements of $S_2(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}})$ and by **A3.1** satisfy $(g_i|_2\sigma_{\mathfrak{a}})(w), (h_i|_2\sigma_{\mathfrak{a}})(w) \ll \mathrm{Im}(w)^{-1}$ for all w in $\mathfrak{h}$. Also, by the Fourier expansion (1), we have

$$\int_{z}^{z+1} (g_i|_2\sigma_{\mathfrak{a}})(w)dw = 0$$

and the same for $h_i|_2\sigma_{\mathfrak{a}}$ so that

$$\Lambda_i(\gamma^{-1}\sigma_{\mathfrak{a}}z) \ll \int_{\sigma_{\mathfrak{a}}^{-1}\gamma z_0}^{iy} \mathrm{Im}(w)^{-1}dw \ll |\log y| + 1.$$

Replacing z by $\sigma_{\mathfrak{a}}^{-1}\gamma z$ completes the proof. □

Theorem 4. *We have f in $S_k^2(\Gamma)$ if and only if $f : \mathfrak{h} \to \mathbb{C}$ is holomorphic and may be written as*

$$f = \sum_{i=0}^{2g} f_i\Lambda_i$$

*where f_i is in $S_k(\Gamma)$ for $i > 0$ and f_0 is a smooth function on $\mathfrak{h}$ of weight k that satisfies **A2** and **A3**, i.e. $(f_0|_k\sigma_{\mathfrak{a}})(z) \ll e^{-cy}$ as $y \to \infty$ uniformly in x for some $c > 0$. Also, for fixed Λ_i, the functions f_i are uniquely defined by f.*

Proof: In one direction, if $f = \sum_{i=0}^{2g} f_i\Lambda_i$ then $f|_k(\gamma_i - 1) = f_i$ for all hyperbolic generators γ_i. Also $f|_k(\gamma - 1) = 0$ for γ a parabolic or elliptic generator. Conditions **B2** and **B4** now hold since they are true for the generators of the group. To verify **B3** we see that, by Lemma 3, f will have exponential decay at the cusps if each f_i does.

In the other direction, given any $f \in S_k^2(\Gamma)$ set $f_i = f|_k(\gamma_i - 1)$ for $1 \leqslant i \leqslant 2g$ and $f_0 = f - \sum_{i=1}^{2g} f_i\Lambda_i$ It is clear that f_0 is smooth, has weight k and has exponential decay at each cusp.

Finally, that the functions f_i are uniquely determined by f is obvious. □

A weaker condition than **B2** is:

B2.1. $f|_k(\gamma - 1)(\delta - 1) = 0$ for all γ, δ in Γ.

The combination **B1, B2.1, B3** and **B4** does not give $S_k^2(\Gamma)$. We need to strengthen **B3** in this case to have exponential decay on all the images of $\mathfrak{I}$ under the group action. Note that if $\mathfrak{a}$ is a cusp of $\mathfrak{F}$ then $\gamma\mathfrak{a}$ will be a cusp of $\gamma\mathfrak{F}$ and $\sigma_{\gamma\mathfrak{a}} = \gamma\sigma_{\mathfrak{a}}$ since $\sigma_{\gamma\mathfrak{a}}\infty = \gamma\sigma_{\mathfrak{a}}\infty = \gamma\mathfrak{a}$ and $\sigma_{\gamma\mathfrak{a}}^{-1}\Gamma_{\gamma\mathfrak{a}}\sigma_{\gamma\mathfrak{a}} = \Gamma_\infty$. Therefore exponential decay for all images of $\mathfrak{F}$ means the following:

B3′. for all $\gamma \in \Gamma$ we have $(f|_k(\gamma\sigma_{\mathfrak{a}})(z) \ll e^{-cy}$ as $y \to \infty$, uniformly in x with c and the implied constant depending on γ.

It is easy to see that we then have

Lemma 5. *$f \in S_k^2(\Gamma)$ if and only if f satisfies* **B1, B2.1, B3′** *and* **B4**.

This was the definition of $S_k^2(\Gamma)$ given in [1]. The analog of **A3.1** is

B3.1. $y^{k/2}(|\log \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)| + 1)^{-1}|f(z)| \ll 1$ for all z in $\mathfrak{h}$, all γ in Γ and any cusp $\mathfrak{a}$ with an implied constant independent of z.

Lemma 6. *We have $f \in S_k^2(\Gamma)$ if and only if f satisfies* **B1, B2.1, B3.1** *and* **B4**.

Proof: For $f \in S_k^2(\Gamma)$, **B2.1** is clearly true and we need only check that **B3.1** holds. By Theorem 4 and Lemma 3

$$\begin{aligned}
&\frac{y^{k/2}}{|\log \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)|+1}|f(z)| \\
&\leqslant \frac{y^{k/2}|f_0(z)|}{|\log \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)|+1} + \sum_{i=1}^{2g} y^{k/2}|f_i(z)|\frac{|\Lambda_i(z)|}{|\log \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)|+1} \\
&\ll 1.
\end{aligned}$$

Conversely, suppose f satisfies **B1, B3.1** and **B4**. Proposition 1 can be adapted to show that f satisfies (1) for $b_a(n) \ll n^{k/2}\log n$ and $b_{\mathfrak{a}}(0) = 0$. Thus f has exponential decay on the cusps of $\mathfrak{F}$: $(f|_k\sigma_{\mathfrak{a}})(z) \ll e^{-2\pi y}$ as $y \to \infty$ uniformly in x. Replace $\sigma_{\mathfrak{a}}$ by $\sigma_{\gamma\mathfrak{a}} = \gamma\sigma_{\mathfrak{a}}$ in the above proof to show that it also has exponential decay on the cusps of $\gamma\mathfrak{F}$. Thus the difference $(f|_k\gamma)(z) - f(z)$ has exponential decay at the cusps and with **B2.1** it has weight k. Conditions **B2** and **B3** (or alternatively **B3′**) now follow. □

The proof of Lemma 6 also gives

Lemma 7 ('Trivial' bound). *The nth Fourier coefficient of a holomorphic second-order cusp form of weight k is* $\ll n^{k/2}\log n$.

2.3 A variation of $S_k^2(\Gamma)$

In view of the 'twist-less' converse theorem we seek, we are also interested in the larger space of second-order cusp forms that do not necessarily satisfy **B4**, i.e. for parabolic elements π we may not have $f|_k(\pi - 1) = 0$.

Definition of $PS_k^2(\Gamma)$. Define this space to be all functions satisfying

B1. $f : \mathfrak{h} \to \mathbb{C}$ is holomorphic,
B2.1. $f|_k(\gamma - 1)(\delta - 1) = 0$ for all γ, δ in Γ,
B3*. ("vanishing at the cusps") for each cusp $\mathfrak{a}$ and all $\gamma \in \Gamma$ we have $(f|_k(\gamma\sigma_{\mathfrak{a}}))(z) \ll e^{-cy}(1 + |x|)$ as $y \to \infty$ with c and the implied constant depending on γ.

To formulate Theorem 9 (the analog of the classification Theorem 4) we need to define the space of all modular forms, not necessarily with exponential decay at the cusps.

Definition of $M_k(\Gamma)$. Let $M_k(\Gamma)$ denote functions satisfying the conditions **A1, A2** and **C3** where

C3. $(f|_k\sigma_{\mathfrak{a}})(z) \ll 1$ as $y \to \infty$ uniformly on $\mathfrak{h}$.
With Proposition 1 we may check that **C3** can be replaced by the equivalent condition **C3.1**. $y^k|f(z)| \ll 1$ for all z in $\mathfrak{h}$.

We also need to associate functions to the parabolic generators, which satisfy equations similar to those of Λ_i of Subsection 2.1. For $2g + 1 \leqslant i \leqslant 2g + m - 1$ we may define $L_i \in \mathrm{Hom}(\Gamma, \mathbb{C})$ such that

$$L_i(\pi_j) = \delta_{(i-2g)j}, \quad L_i(\pi_m) = -1$$

and

$$L_i(\gamma) = 0,$$

for all nonparabolic generators γ of Γ. These give well-defined maps because of the relations the chosen generators of Γ satisfy. By the Eichler-Shimura isomorphism, for each $i \in \{2g+1, \dots, 2g+m-1\}$ there exist $g_i \in M_2(\Gamma)$, $h_i \in S_2(\Gamma)$ such that the function Λ_i defined by

$$\Lambda_i(z) := \int_{z_0}^{z} g_i(w)dw + \overline{\int_{z_0}^{z} h_i(w)dw}$$

where z_0 is a fixed element of $\mathfrak{h}$, satisfies

$$\Lambda_i(\gamma z) - \Lambda_i(z) = L_i(\gamma),$$

for all $z \in \mathfrak{h}$ and all $\gamma \in \Gamma$.

Now, as in the proof of Lemma 3, for $g_i \in M_2(\Gamma)$ we see (with C3.1) that

$$\int_{z_0}^{\gamma^{-1}\sigma_{\mathfrak{a}} z} g_i(w)dw \ll \frac{1}{y} + \frac{|x|}{y^2} + 1.$$

Therefore,

$$\Lambda_i(\gamma^{-1}\sigma_{\mathfrak{a}} z) \ll \frac{1}{y} + \frac{|x|}{y^2} + |\log y| + 1$$

and we have the following analog of Lemma 3.

Lemma 8. *For $2g+1 \leqslant i \leqslant 2g+m-1$, all $z \in \mathfrak{h}$, $\gamma \in \Gamma$ and any cusp $\mathfrak{a}$ we have*

$$\Lambda_i(z) \ll \frac{1}{\mathrm{Im}\left(\sigma_{\mathfrak{a}}^{-1}\gamma z\right)} + \frac{\left|\mathrm{Re}\left(\sigma_{\mathfrak{a}}^{-1}\gamma z\right)\right|}{\mathrm{Im}\left(\sigma_{\mathfrak{a}}^{-1}\gamma z\right)^2} + \left|\log \mathrm{Im}\left(\sigma_{\mathfrak{a}}^{-1}\gamma z\right)\right| + 1$$

with the implied constant independent of z.

In a way similar to the proof of Theorem 4, we can then use Lemma 8 to show the next result.

Theorem 9. *We have f in $PS_k^2(\Gamma)$ if and only if $f : \mathfrak{h} \to \mathbb{C}$ is holomorphic and may be written as*

$$f = \sum_{i=0}^{2g+m-1} f_i \Lambda_i$$

where f_i is in $S_k(\Gamma)$ for $i > 0$ and f_0 is a smooth function on $\mathfrak{h}$ weight k that satisfies **A2** *and* **A3**, *i.e.* $(f_0|_k\sigma_{\mathfrak{a}})(z) \ll e^{-cy}$ *as $y \to \infty$ uniformly in x for some $c > 0$. Also, for fixed Λ_i, the functions f_i are uniquely defined by f.*

We close this section with the promised proof of Proposition 1.

Proof of Proposition 1: If f is holomorphic then so is $f|_k\sigma_{\mathfrak{a}}$ and, if it is periodic, it must have the Fourier expansion (1) since any terms $e(nz)$ with $n < 0$ would violate $y^r|f(z)| \ll 1$. For $n \geqslant 1$ we have

$$\begin{aligned} b_a(n) &= e^{-2\pi} \int_0^1 (f|_k\sigma_{\mathfrak{a}})(x + i/n)e^{-2\pi i n x} dx \\ &\ll n^{k/2} \int_0^1 \mathrm{Im}\,(\sigma_{\mathfrak{a}}(x + i/n))^{k/2}|f(\sigma_{\mathfrak{a}}(x + i/n))|dx \\ &\ll n^{k/2} \mathrm{Im}\,(\sigma_{\mathfrak{a}}(x + i/n))^{k/2-r}. \end{aligned}$$

Now

$$\mathrm{Im}\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix}(x + i/n)\right)^{-1} = n((cx + d)^2 + c^2/n^2) \ll n$$

for $x \in [0, 1]$ and the implied constant depending on c, d. Hence $b_{\mathfrak{a}}(n) \ll n^r$.

Also

$$\begin{aligned} b_{\mathfrak{a}}(0) &= \int_0^1 (f|_k\sigma_{\mathfrak{a}})(x + iy)dx \\ &\leqslant \int_0^1 y^{-k/2} \mathrm{Im}\,(\sigma_{\mathfrak{a}}(x + iy))^{k/2}|f(\sigma_{\mathfrak{a}}(x + iy))|dx. \end{aligned}$$

As $y \to \infty$ we have $1/y \asymp \mathrm{Im}(\sigma_{\mathfrak{a}}(x + iy))$ if $\sigma_{\mathfrak{a}}$ is not upper triangular. Therefore, as $y \to \infty$,

$$\begin{aligned} b_{\mathfrak{a}}(0) &\ll \int_0^1 \mathrm{Im}(\sigma_{\mathfrak{a}}(x + iy))^k|f(\sigma_{\mathfrak{a}}(x + iy))|dx \\ &\ll \int_0^1 y^{r-k} \mathrm{Im}(\sigma_{\mathfrak{a}}(x + iy))^r|f(\sigma_{\mathfrak{a}}(x + iy))|dx \\ &\ll y^{r-k}. \end{aligned}$$

If $\sigma_{\mathfrak{a}}$ is upper triangular then $y \asymp \text{Im}\,(\sigma_{\mathfrak{a}}(x+iy))$ as $y \to \infty$ and

$$\begin{aligned} b_{\mathfrak{a}}(0) &\ll \int_0^1 |f(\sigma_{\mathfrak{a}}(x+iy))|dx \\ &\ll y^{-r}. \end{aligned}$$

Either way, $b_{\mathfrak{a}}(0) = 0$, and the proof is complete. $\square$

3 Functional equations

In this section, we first prove a functional equation for the L-function we associate to second-order cusp forms. Next, we define two twisting operators, one playing the role of the contragredient of the other. Finally, we prove a functional equation (Theorem 11) for the second-order cusp forms of PS_k^2 twisted according to these operators.

We specialize to the case $\Gamma = \Gamma_0(N)$, for a fixed positive integer N and write $S_k(N)$ for $S_k(\Gamma_0(N))$ etc. Set $W_N := [\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}]$ and define $\hat{f} := f|_k W_N$ for f a weight k, first or second order modular form.

Proposition 10. *If $f \in S_k(N)$, then $\hat{f} \in S_k(N)$. Also, if $F \in S_k^2(N)$, then $\hat{F} \in S_k^2(N)$.*

Proof: Since W_N normalizes $\Gamma_0(N)$, $\hat{f}$ satisfies **A2**. It also satisfies **A3.1** because

$$y^{\frac{k}{2}}|\hat{f}(z)| = y^{\frac{k}{2}} N^{-\frac{k}{2}} |z|^{-k} \left| f\left(\frac{-1}{Nz}\right)\right| \ll y^{\frac{k}{2}} N^{-\frac{k}{2}} |z|^{-k} \text{Im}\left(\frac{-1}{Nz}\right)^{-\frac{k}{2}} \ll 1.$$

Therefore, by Lemma 2, $\hat{f} \in S_k(N)$.

In a similar way, if $F \in S_k^2(N)$, then $\hat{F}$ satisfies **B1, B2.1** and **B4**. On the other hand,

$$\begin{aligned} &y^{\frac{k}{2}}\left(\left|\log\left(\text{Im}\,\left(\sigma_{\mathfrak{a}}^{-1}\gamma z\right)\right)\right| + 1\right)^{-1} |\hat{F}(z)| \\ &= y^{\frac{k}{2}}\left(\left|\log\left(\text{Im}\,\left(\sigma_{\mathfrak{a}}^{-1}\gamma z\right)\right)\right| + 1\right)^{-1} N^{-\frac{k}{2}} |z|^{-k} \left| F\left(\frac{-1}{Nz}\right)\right| \\ &\ll \left(\left|\log\left(\text{Im}\,\left(\sigma_{\mathfrak{a}}^{-1}\gamma z\right)\right)\right| + 1\right)^{-1} \left(\left|\log(\text{Im}\,\left(\sigma_{\mathfrak{b}}^{-1}\delta W_N z\right)\right)\right| + 1\right) \end{aligned}$$

for every cusp $\mathfrak{b}$ and each $\delta \in \Gamma_0(N)$. The final step is to choose $\mathfrak{b}$ and δ so that $\sigma_{\mathfrak{a}}^{-1}\gamma = \sigma_{\mathfrak{b}}^{-1}\delta W_N$: Since $\gamma^{-1}\mathfrak{a}$ and $W_N\gamma^{-1}\mathfrak{a}$ are also cusps of $\Gamma_0(N)$ we must have $\mathfrak{b} = \delta W_N \gamma^{-1}\mathfrak{a}$ for a $\mathfrak{b}$ in the set of inequivalent cusps and a $\delta \in \Gamma_0(N)$. Because of the relation $\sigma_{\tau\mathfrak{a}} = \tau\sigma_{\mathfrak{a}}$, this implies that $\sigma_{\mathfrak{a}}^{-1}\gamma = \sigma_{\mathfrak{b}}^{-1}\delta W_N$. Thus $\hat{F}$ satisfies **B3.1** and by Lemma 6 we are done. $\square$

If $F(z) = \sum_{n=1}^{\infty} \mathfrak{a}_n e^{2\pi n z} \in S_k^2(N)$, then its L-function is defined by

$$L(s, F) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = (2\pi)^s \Gamma(s)^{-1} \Lambda(s, F)$$

for $\Lambda(s, F) = \int_0^{\infty} F(iy) y^{s-1} dy$. Since ∞ and 0 are cusps of $\Gamma_0(N)$, where F has exponential decay, we see that $L(s, F)$ has a meromorphic continuation to $s \in \mathbb{C}$. The functional equation

$$i^{-k} N^{\frac{k}{2}-s} \Lambda(k-s, F) = \Lambda(s, \hat{F})$$

follows by a simple change of variables in the above integral just as in the case of L-functions of first order cusp forms.

In order to get a functional equation for the L-function of F twisted by a Dirichlet character we may define two twisting operators as follows.

First of all, for any function g on $\mathfrak{h}$ and a Dirichlet character $\chi \mod N$, we set

$$g_\chi(z) = \sum_{0<M<N} \chi(m) g\left(\frac{z+m}{N}\right)$$

Here, as in all the sums appearing in the sequel, the sum ranges only over integers that are relatively prime to the modulus. Set $S = [\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]$. It is known (cf. [9]) that a holomorphic function f on $\mathfrak{h}$ is $\Gamma_0(N)$-invariant if and only if it is periodic with period 1 and

$$f_\chi|_k S = \chi(-1) f_{\bar{\chi}} \tag{2}$$

for all Dirichlet characters $\chi \mod Nc$, $(c \in \{1, \ldots, N\})$.

For $F \in PS_k^2(N)$ decomposed as in Theorem 9, we define

$$F^\chi = \sum_{i=1}^{2g+m-1} (f_i)_\chi \Lambda_i + (f_0)_\chi$$

and its "contragredient"

$$\check{F}^\chi(z) = \sum_{i=1}^{2g+m-1} (f_i)_\chi(z) \left(\int_i^z (g_i|_2 S)(w) dw + \overline{\int_i^z (h_i|_2 S)(w) dw} \right) + (f_0)_\chi(z).$$

We also set

$$L(s, F, \chi) := (2\pi)^s \Gamma(s)^{-1} \Lambda(s, F, \chi) = (2\pi)^s \Gamma(s)^{-1} \int_0^{\infty} F^\chi(iy) y^{s-1} dy \text{ and}$$

$$L^*(s, F, \chi) := (2\pi)^s \Gamma(s)^{-1} \Lambda^*(s, F, \chi) = (2\pi)^s \Gamma(s)^{-1} \int_0^{\infty} \check{F}^\chi(iy) y^{s-1} dy.$$

Theorem 11. *For $F \in PS_k^2(N)$ the following functional equation holds:*

$$\Lambda(k-s, F, \chi) = i^k \chi(-1)\Lambda^*(s, F, \bar{\chi}).$$

Proof: Apply S to F^χ to get

$$\begin{aligned} F^\chi|_k S &= \sum_{i=0}^{2g+m-1} ((f_i)_\chi|_k S)(z)\Lambda_i(Sz) \\ &= \sum_{i=1}^{2g+m-1} ((f_i)_\chi|_k S)(z)\left(\int_i^{Sz} g_i(w)dw + \overline{\int_i^{Sz} h_i(w)dw}\right) + (f_0)_\chi|_k S. \end{aligned}$$

Since S fixes i we can make the change of variables $w \to Sw$ in the integrals. This, in combination with (2), gives

$$\begin{aligned} &\chi(-1)\left(\sum_{i=1}^{2g+m-1} (f_i)_{\bar{\chi}}(z)\int_i^z (g_i|_2 S)(w)dw + \overline{\int_i^z (h_i|_2 S)(w)dw} + (f_0)_{\bar{\chi}}(z)\right) \\ &\quad = \chi(-1)\check{F}^{\bar{\chi}}. \end{aligned}$$

It should be noted that, although f_0 is not necessarily holomorphic, (2) applies to it too because the derivation of (2) depends only on algebraic manipulations on $\mathbb{Z}[GL_2(\mathbb{R})]$. Applying the Mellin transform to this equality we obtain

$$i^{-k}\int_0^\infty F^\chi\left(\frac{i}{y}\right)y^{s-k}\frac{dy}{y} = \chi(-1)\int_0^\infty \check{F}^{\bar{\chi}}(iy)y^s\frac{dy}{y}.$$

The change of variables $y \to \frac{1}{y}$ in the first integral implies the functional equation. $\square$

We cannot use this theorem directly in order to obtain a meaningful converse theorem for second-order modular forms. The reason is that, in this approach, we must be able to isolate the functions f_i from the given function F for the above functional equation to even be set up. The functions f_i are $F|_k(\gamma_i - 1)$ for certain elements γ_i of Γ and in the next section we give a criterion for a function to be in PS_k^2 which is based on each $F|_k(\gamma_i - 1)$ separately.

4 A converse theorem

In this section, we will use Razar's method (cf. [9]) to prove a criterion for a function to be a second-order cusp form. After selecting a set of generators for $\Gamma_0(N)$ in Lemma 12, we review the proof of an unpublished result by Flood (Lemma 13) that considers the effect of twisting on functions from a more general standpoint than that of [9]. After these preliminaries, we state and prove our criterion, Theorem 14.

For $c, c_1 \in \{1, \ldots, N\}$, let ψ be a Dirichlet character mod(Nc_1) and let χ, w be Dirichlet characters mod (Nc). Set

$$F_{\chi,\psi,\omega} := \sum_{\substack{0<m,b<Nc \\ 0<a<Nc_1}} \psi(a)\chi(m)\omega(b)F\bigg|_k \begin{bmatrix} c & ac-bc_1+mc_1 \\ 0 & Ncc_1 \end{bmatrix} \quad \text{and}$$

$$F^{\chi,\psi,\omega} := \sum_{\substack{0<m,b<Nc \\ 0<a<Nc_1}} \psi(a)\chi(m)\omega(b)F\bigg|_k \begin{bmatrix} mc & m(ac-bc_1)-c_1 \\ Nc^2 & Nc(ac-bc_1) \end{bmatrix}.$$

We now fix a set of generators for $\Gamma_0(N)$. To this end we define a family of elements of $\Gamma_0(N)$ which we will use to prove a lemma from [9].

Let c be a positive integer. For each $0 < a < Nc((a, Nc) = 1)$ choose one matrix

$$V_a = \begin{bmatrix} a & b_a \\ Nc & d_a \end{bmatrix} \in \Gamma_0(N)$$

such that $-Nc < d_a < 0$. For each positive integer c, we denote the set of all such matrices by S_c.

Lemma 12 ([9]). *(i) The set* $\bigcup_{c=1}^N S_c \cup \{[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}]\} \cup \{[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}]\}$ *generates* $\Gamma_0(N)$.

(ii) If $N = p^r$ *(p prime), then* $\Gamma_0(N)$ *is generated by* $[\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}]$, $[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}]$ *and* $[\begin{smallmatrix} a & b \\ N & d \end{smallmatrix}] \in \Gamma_0(N)$*, as a ranges over a system of residues mod N prime to N.*

We will also need a lemma from [4]. Since it has not been published and since our statement is somewhat more general than that in [4], we give a proof here.

Lemma 13 ([4]). *Set* $S = [\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}]$*. Let* χ *be a character mod(Nc) and let F be a function on* $\mathfrak{h}$*. Then,*

$$F_\chi|_k S = \chi(-1)F_{\bar{\chi}} + \chi(-1)q_\chi$$

where $q_\chi = \sum_{0<a<Nc} \bar{\chi}(d_a)q_{V_a}(\frac{z-d_a}{Nc})$ *and* $q_{Va} = F|_k V_a - F$.

Proof: We have

$$V_a\left(\frac{z-d_a}{Nc}\right) = \frac{-z^{-1}+a}{Nc} \quad \text{and} \quad Nc\left(\frac{z-d_a}{Nc}\right) + d_a = z.$$

Therefore,

$$(F_\chi|_k S)(z) = z^{-k} F_\chi(-z^{-1}) = \sum_{0<a<Nc} \chi(a) z^{-k} F\left(\frac{-z^{-1}+a}{Nc}\right)$$
$$= \sum_{0<a<Nc} \chi(a)(F|_k V_a)\left(\frac{z-d_a}{Nc}\right)$$
$$= \chi(-1) \sum_{a<a<Nc} \bar{\chi}(-d_a)(F|_k V_a)\left(\frac{z-d_a}{Nc}\right). \tag{3}$$

Here we use the fact that $ad_a - Nb_a c = 1$ and therefore $\chi(a)\chi(d_a) = 1$.

On the other hand, $-d_a$ ranges over the elements of $\{1, \dots, Nc\}$ prime to Nc, as a ranges over the same set. Hence,

$$F_{\bar{\chi}}(z) = \sum_{0<a<Nc} \bar{\chi}(-d_a) F\left(\frac{z-d_a}{Nc}\right).$$

On subtraction from (3) we obtain

$$(F_\chi|_k S)(z) - \chi(-1)F_{\bar{\chi}}(z) = \chi(-1) \sum_{0<a<Nc} \bar{\chi}(-d_a) q_{Va}\left(\frac{z-d_a}{Nc}\right).$$

□

Theorem 14. *Let F be a holomorphic function on $\mathfrak{h}$ such that, for all $\gamma \in \Gamma_0(N)$, $(F|_k(\gamma\sigma_{\mathfrak{a}}))(z) \ll e^{-cy}(1+|x|)$ as $y \to \infty$, with c and the implied constant depending on γ. Suppose that for all Dirichlet characters χ, ω mod(Nc), ψ mod$(Nc_1)(c, c_1 \in \{1, \dots, N\})F_{\chi,\psi,\omega}(iy)$ and $F^{\chi,\psi,\omega}(iy)$ decay exponentially as $y \to \infty$ and $y \to 0$. Set*

$$\Phi^1(s, \chi, \psi, \omega) := \int_0^\infty F_{\chi,\psi,\omega}(iy) y^{s-1} dy \quad \textit{and}$$
$$\Phi^2(s, \chi, \psi, \omega) := \int_0^\infty F^{\chi,\psi,\omega}(iy) y^{s-1} dy.$$

If, for all $\gamma \in \Gamma_0(N)$,

$$F|_k\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 1\right)(\gamma - 1) = F|_k(\gamma - 1)\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 1\right) = 0$$

and the functional equation

$$\chi(-1)i^{-k}\Phi^2(k-s, \chi, \psi, \omega) - i^k \Phi^1(k-s, \bar{\chi}, \psi, \omega)$$
$$= \psi(-1)\chi(-1)\Phi^2(s, \chi, \bar{\psi}, \omega) - \Phi^1(s, \bar{\chi}, \bar{\psi}, \omega)$$

is true for all characters χ, ψ, ω then $F \in PS_k^2(N)$.

Proof: We see that Φ^1 and Φ^2 converge to analytic functions of s for all $s \in \mathbb{C}$. It follows by the Mellin inversion formula that

$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Phi^1(s,\chi,\psi,\omega)y^{-s}ds = F_{\chi,\psi,\omega}(iy),$$
$$\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \Phi^2(s,\chi,\psi,\omega)y^{-s}ds = F^{\chi,\psi,\omega}(iy)$$

are valid for every $\sigma = \mathrm{Re}(s) \in \mathbb{R}$ Replace s by $k-s$ and y by $\frac{1}{y}$ in the above to see that the functional equation assumed in the theorem implies that

$$(iy)^{-k}\chi(-1)F^{\chi,\psi,\omega}\left(\frac{-1}{iy}\right) = (iy)^{-k}F_{\bar{\chi},\psi,\omega}\left(\frac{-1}{iy}\right) + \psi(-1)\chi(-1)F^{\chi,\bar{\psi},\omega}(iy)$$
$$-F_{\bar{\chi},\bar{\psi},\omega}(iy).$$

Since all the functions involved are analytic, this equality is true on the entire upper-half plane and we can rewrite it in the form

$$\left[\chi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\chi(m)F\Big|_k\begin{bmatrix} mc & m(ac-bc_1)-c_1\\ Nc^2 & Nc(ac-bc_1)\end{bmatrix}\right]\Bigg|_k S$$
$$-\left[\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\bar{\chi}(m)F\Big|_k\begin{bmatrix} c & ac-bc_1+mc_1\\ 0 & Ncc_1\end{bmatrix}\right]\Bigg|_k S$$
$$-\chi(-1)\psi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)\sum_{0<m<Nc}\chi(m)F\Big|_k\begin{bmatrix} mc & m(ac-bc_1)-c_1\\ Nc^2 & Nc(ac-bc_1)\end{bmatrix}$$
$$+\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)\sum_{0<m<Nc}\bar{\chi}(m)F\Big|_k\begin{bmatrix} c & ac-bc_1-mc_1\\ 0 & Ncc_1\end{bmatrix} = 0.$$

This is equivalent to

$$\chi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\chi(m)F\Big|_k\begin{bmatrix} 1 & m\\ 0 & Nc\end{bmatrix}S\begin{bmatrix} c & ac-bc_1\\ 0 & c_1\end{bmatrix}S$$
$$-\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\bar{\chi}(m)F\Big|_k\begin{bmatrix} 1 & m\\ 0 & Nc\end{bmatrix}\begin{bmatrix} c & ac-bc_1\\ 0 & c_1\end{bmatrix}S$$
$$-\chi(-1)\psi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)\sum_{0<m<Nc}\chi(m)F\Big|_k\begin{bmatrix} 1 & m\\ 0 & Nc\end{bmatrix}S\begin{bmatrix} c & ac-bc_1\\ 0 & c_1\end{bmatrix}$$
$$+\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)\sum_{0<m<Nc}\bar{\chi}(m)F\Big|_k\begin{bmatrix} 1 & m\\ 0 & Nc\end{bmatrix}\begin{bmatrix} c & ac-bc_1\\ 0 & c_1\end{bmatrix} = 0. \qquad (4)$$

We can further use the definition of F_χ (of Section 3) to write the last equality in the form:

$$\chi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)F_\chi\Big|_k S\begin{bmatrix} c & ac-bc_1\\ 0 & c_1\end{bmatrix}S$$
$$=\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)F_{\bar\chi}\Big|_k\begin{bmatrix} c & ac-bc_1\\ 0 & c_1\end{bmatrix}S$$
$$-\chi(-1)\psi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar\psi(a)F_\chi\Big|_k S\begin{bmatrix} c & ac-bc_1\\ 0 & c_1\end{bmatrix}$$
$$+\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar\psi(a)F_{\bar\chi}\Big|_k\begin{bmatrix} c & ac-bc_1\\ 0 & c_1\end{bmatrix}=0$$

or,

$$\chi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\big(F_\chi|_k S-\chi(-1)F_{\bar\chi}\big)|_k\begin{bmatrix} Nc & Nac-Nbc_1\\ 0 & Nc_1\end{bmatrix}S$$
$$=\chi(-1)\psi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar\psi(a)\big(F_\chi|_k S-\chi(-1)F_{\bar\chi}\big)|_k\begin{bmatrix} Nc & Nac-Nbc_1\\ 0 & Nc_1\end{bmatrix}.$$

Since $\begin{bmatrix} Nc & Nac-Nbc_1\\ 0 & Nc_1\end{bmatrix}=\begin{bmatrix} Nc & -b\\ 0 & 1\end{bmatrix}\begin{bmatrix} 1 & d\\ 0 & Nc_1\end{bmatrix}$ we have

$$\sum_{0<b<Nc}\omega(b)\sum_{0<a<Nc_1}\psi(a)\left(\big(F_\chi|_k S-\chi(-1)F_{\bar\chi}\big)|_k\begin{bmatrix} Nc & -b\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix} 1 & a\\ 0 & Nc_1\end{bmatrix}S$$
$$=\psi(-1)\sum_{0<b<Nc}\omega(b)\sum_{0<a<Nc_1}\bar\psi(a)\left(\big(F_\chi|_k S-\chi(-1)F_{\bar\chi}\big)|_k\begin{bmatrix} Nc & -b\\ 0 & 1\end{bmatrix}\right)\begin{bmatrix} 1 & a\\ 0 & Nc_1\end{bmatrix}.$$

Therefore, by the defining formula for the twist of a function,

$$\sum_{0<b<Nc}\omega(b)\left(\big(F_\chi|_k S-\chi(-1)F_{\bar\chi}\big)\Big|_k\begin{bmatrix} Nc & -b\\ 0 & 1\end{bmatrix}\right)_\psi\Big|_k S$$
$$=\psi(-1)\sum_{0<b<Nc}\omega(b)\left(\left(F_\chi|_k S-\chi(-1)F_{\bar\chi}\right)\Big|_k\begin{bmatrix} Nc & -b\\ 0 & 1\end{bmatrix}\right)_{\bar\psi}.$$

By character summation (over characters ω mod(Nc)) we obtain

$$\left(\left(F_\chi\big|_k S-\chi(-1)F_{\bar{\chi}}\right)\Big|_k \begin{bmatrix} Nc & -b \\ 0 & 1 \end{bmatrix}\right)_\psi \Big|_k S$$
$$=\psi(-1)\left(\left(F_\chi\big|_k S-\chi(-1)F_{\bar{\chi}}\right)\Big|_k \begin{bmatrix} Nc & -b \\ 0 & 1 \end{bmatrix}\right)_{\bar{\psi}}$$

for all $b\in\{1,\dots,Nc\}$ with $((b,Nc)=1)$.

Now, for $a=1,\dots,Nc$, prime to Nc and $b=-d_a>0$, this implies

$$\left(\sum_{\chi(\mathrm{mod}\ Nc)}\chi(d_a)\left(F_\chi\big|_k S-\chi(-1)F_{\bar{\chi}}\right)\Big|_k \begin{bmatrix} Nc & d_a \\ 0 & 1 \end{bmatrix}\right)_\psi \Big|_k S$$
$$=\psi(-1)\left(\sum_{\chi(\mathrm{mod}\ Nc)}\chi(d_a)\left(F_\chi\big|_k S-\chi(-1)F_{\bar{\chi}}\right)\Big|_k \begin{bmatrix} Nc & d_a \\ 0 & 1 \end{bmatrix}\right)_{\bar{\psi}}.$$

By the usual character summation argument, Lemma 13 then implies that the sum inside the parentheses equals $\phi(Nc)q_{Va}$, where ϕ denotes Euler's function. So the last equality can be rewritten as

$$(q_{V_a})_\psi|_k S=\psi(-1)(q_{V_a})_{\bar{\psi}},$$

for all $V_a\in S_c$.

From (2) (together with our assumption on the periodicity of $F|_k(\gamma-1)$'s) we can then deduce that $q_{Va}=F|_k V_a-F$ is invariant under $\Gamma_0(N)$. Therefore, $F|_k(V_a-1)\gamma=F|_k(V_a-1)$ for all $\gamma\in\Gamma_0(N)$. On the other hand, by assumption, $F|_k([\begin{smallmatrix}1&1\\0&1\end{smallmatrix}]-1)(\gamma-1)=0$ for all $\gamma\in\Gamma_0(N)$. Now, if $F|_k(\gamma_1-1)(\gamma-1)=0$ and $F|_k(\gamma_2-1)(\gamma-1)=0$ for all $\gamma\in\Gamma_0(N)$ then $F|_k(\gamma_1\gamma_2-1)(\gamma-1)=0$ because

$$(\gamma_1\gamma_2-1)(\gamma-1)=(\gamma_1-1)(\gamma_2\gamma-1)-(\gamma_1-1)(\gamma_2-1)+(\gamma_2-1)(\gamma-1).$$

According to Lemma 12(i), $\Gamma_0(N)$ is generated by the V_a's, the translations and $[\begin{smallmatrix}-1&0\\0&-1\end{smallmatrix}]$. Therefore, F satisfies **B2.1** and thus by definition, $F\in PS_k^2(N)$. □

The following corollary of the proof allows us to distinguish the case that F is a "trivial" second-order cusp form, that is, a usual cusp form.

Proposition 15. *With the assumptions of Theorem 14, the left-hand side of the functional equation vanishes if and only if F is a usual cusp form.*

Proof: We repeat the first steps of the proof of Theorem 14 up to Eq. (4). The left-hand side of the functional equation vanishes if and only if

$$\chi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\chi(m)F\Big|_k\begin{bmatrix}1 & m\\0 & Nc\end{bmatrix}S\begin{bmatrix}c & ac-bc_1\\0 & c_1\end{bmatrix}S$$
$$=\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\bar{\chi}(m)F\Big|_k\begin{bmatrix}1 & m\\0 & Nc\end{bmatrix}\begin{bmatrix}c & ac-bc_1\\0 & c_1\end{bmatrix}S.$$

A character summation over ω mod(Nc) and ψ mod(Nc_1) together with the definition of F_χ implies that this is equivalent to $F_\chi|_kS=\chi(-1)F_{\bar{\chi}}$ for all χ mod(Nc). With our assumptions, this equality, according to [9], holds if and only if $F\in S_k(N)$. □

5 Periodicity

Let $T=[\begin{smallmatrix}1 & 1\\0 & 1\end{smallmatrix}]$. In Theorem 14 we had to include the assumption

$$F|_k(\gamma-1)(T-1)=F|_k(T-1)(\gamma-1)=0$$

for all $\gamma\in\Gamma_0(N)$. The second equality is clearly satisfied if F has period 1. In this section we examine how the imposition of this stronger assumption of periodicity can affect F.

Indeed, suppose that $F|_k(T-1)=0$ and that $F|_k(\gamma-1)(T-1)=0$ for all $\gamma\in\Gamma_0(N)$. Then $F|_k\gamma T=F|_k\gamma$, so that F is invariant under the group $\tilde{\Gamma}_0(N)$ generated by the set of $\gamma T\gamma^{-1}$, $\gamma\in\Gamma_0(N)$. It is reasonable to ask whether this invariance implies the modularity of F, thus making the remaining assumptions of the theorem redundant. We will show that, for $N\geq 4$, this is far from being the case.

Specifically, for

$$\Gamma_1(N)=\left\{\gamma\in SL_2(\mathbb{Z})\,\middle|\,\gamma\equiv\begin{bmatrix}1 & *\\0 & 1\end{bmatrix}\mod N\right\}$$

set

$$\tilde{\Gamma}_1(N)=\langle\gamma^{-1}T\gamma|\gamma\in\Gamma_1(N)\rangle.$$

As usual, we identify the groups with their images in PSL(2, $\mathbb{Z}$).

Theorem 16. $\tilde{\Gamma}_0(N)$ *has infinite index in* $SL(2,\mathbb{Z})$ *for* $N\geq 4$.

Proof: It is well-known that, for $N \geq 4$, $\Gamma_1(N)$ is free and its rank equals

$$r := 1 + \frac{N^2}{12} \prod_{p|N} \left(1 - \frac{1}{p^2}\right). \tag{5}$$

Next note that $\Gamma_1(N) \trianglelefteq \Gamma_0(N)$ and $|\Gamma_0(N) : \Gamma_1(N)| = \frac{1}{2}\phi(N) =: \nu$, say. Let $g_1 = 1, g_2, \ldots, g_\nu$ be a set of coset representatives of $\Gamma_1(N)$ in $\Gamma_0(N)$.

Observe that

$$\begin{aligned}\tilde{\Gamma}_0(N) &= \left\langle g_i^{-1}\gamma^{-1}T\gamma g_i \mid \gamma \in \Gamma_1(N), 1 \leqslant i \leqslant \nu \right\rangle. \\ &= \left\langle g_i^{-1}\tilde{\Gamma}_1(N) g_i \mid 1 \leqslant i \leqslant \nu \right\rangle.\end{aligned}$$

Set $\Delta_i(N) = g_i^{-1}\tilde{\Gamma}_1(N) g_i$. Because $\Delta_1(N) = \tilde{\Gamma}_1(N) \trianglelefteq \Gamma_1(N) \trianglelefteq \Gamma_0(N)$ then $\Delta_i(N) \trianglelefteq \Gamma_1(N)$ for each i, and therefore

$$\tilde{\Gamma}_0(N) = \Delta_1(N) \ldots \Delta_\nu(N) \leqslant \Gamma_1(N). \tag{6}$$

Set $T_i = g_i^{-1} T g_i$. We then also have

$$\Delta_i(N) = \langle \gamma^{-1} T_i \gamma | \gamma \in \Gamma_1(N) \rangle = \langle T_i [T_i, \gamma] | \gamma \in \Gamma_1(N) \rangle$$

recalling the standard notation $[x, y] = x^{-1}y^{-1}xy$.

Now let A be the abelianization of the group $\Gamma_1(N)$ i.e. the quotient of $\Gamma_1(N)$ by its commutator subgroup. Because $\Gamma_1(N)$ is free then

$$A \cong \mathbb{Z}^r$$

where r is given by (5). It follows from the second equality in (7) that the image of each $\Delta_i(N)$ in A is cyclic, being generated by the image of T_i. Hence by (6), the image of $\tilde{\Gamma}_0(N)$ in A has rank no greater than ν. If we can show that

$$r > \nu$$

then it follows immediately that the theorem holds. But this is a triviality: it says that

$$1 + \frac{N^2}{12} \prod_{p|N} \left(1 - \frac{1}{p^2}\right) > \frac{1}{2}\phi(N) = \frac{N}{2} \prod_{p|N} \left(1 - \frac{1}{p}\right),$$

that is

$$\frac{2}{\phi(N)} + \frac{N}{6} \prod_{p|N} \left(1 + \frac{1}{p}\right) > 1,$$

which is obvious. $\square$

On the other hand, we have

Proposition 17. *For* $1 \leqslant N \leqslant 3$, $\tilde{\Gamma}_0(N) = \Gamma_0(N^2)$.

Proof: $N = 1$. We want to prove that $\Gamma_0(1) = \Gamma(1)$ can be generated by $\gamma T \gamma^{-1} (\gamma \in \Gamma(1))$. A simple check shows that $S = -T^2 P$, where $P = [\begin{smallmatrix} 2 & 1 \\ -1 & 0 \end{smallmatrix}] = [\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}] T [\begin{smallmatrix} 1 & 1 \\ -1 & 0 \end{smallmatrix}]^{-1}$. Since $T^2 \in \tilde{\Gamma}_0(1)$, this settles the case $N = 1$.

$N = 2$. $\Gamma_0(4)$ (or, more precisely, its projection onto $\mathrm{PSL}_2(\mathbb{Z})$) is generated by T, $P_1 = [\begin{smallmatrix} -1 & 0 \\ 4 & -1 \end{smallmatrix}]$ and $P_2 = [\begin{smallmatrix} 1 & -1 \\ 4 & -3 \end{smallmatrix}]$ This can be seen by Lemma 12(ii). However, $P_1 P_2 = -T^{-1}$. Thus, since $T^{-1} \in \tilde{\Gamma}_0(2)$ (obviously), it suffices to prove that $P_2 \in \tilde{\Gamma}_0(2)$. Indeed,

$$P_2 = -\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} T \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1}$$

and our result follows for $N = 2$.

$N = 3$. By Lemma 12(ii), $\Gamma_0(9)$ is generated by T, $P_1 = [\begin{smallmatrix} -1 & 0 \\ 9 & -1 \end{smallmatrix}]$, $P_2 = [\begin{smallmatrix} 2 & -1 \\ 9 & -4 \end{smallmatrix}]$, $P_3 = [\begin{smallmatrix} 5 & -4 \\ 9 & -7 \end{smallmatrix}]$, $P_4 = [\begin{smallmatrix} 7 & -4 \\ 9 & -5 \end{smallmatrix}]$, $P_5 = [\begin{smallmatrix} 4 & -1 \\ 9 & -2 \end{smallmatrix}]$ and $P_6 = [\begin{smallmatrix} 1 & 0 \\ 9 & 1 \end{smallmatrix}]$. Since $P_4 = -P_3^{-1}$, $P_5 = -P_2^{-1}$, $P_6 = -P_1^{-1}$ and $P_1 P_2 P_3 = T^{-1}$ it is sufficient to show that P_2 and P_3 are in $\tilde{\Gamma}_0(3)$. Indeed,

$$P_2 = -\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} T \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}^{-1} \quad \text{and} \quad P_3 = -\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} T \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1}.$$

□

We also observe that the invariance under $\Gamma_0(N^2)$ implied by Proposition 17 (when $N = 1, 2, 3$) for functions satisfying the assumptions of Theorem 14, in fact implies modularity for $\Gamma_0(N)$. This is a consequence (with $\Gamma_1 = \Gamma_0(N)$, $\Gamma_2 = \Gamma_0(N^2)$) of the next claim.

Proposition 18. *If F satisfies **B2.1** for* Γ_1 *and is invariant under a group* Γ_2 *with* $[\Gamma_1 : \Gamma_2] < \infty$, *then F is invariant under* Γ_1.

Proof: Since Γ_2 contains a subgroup of finite index which is normal in Γ_1, we can assume, without loss of generality, that Γ_2 is normal in Γ_1. Let $\mu = [\Gamma_1 : \Gamma_2]$. Then, for $\gamma \in \Gamma_1$, $\gamma^\mu \in \Gamma_2$ Thus, $F|_k \gamma^\mu - F = 0$. On the other hand, $F|_k \gamma - F$ is invariant under Γ_1, therefore we have:

$$0 = F|_k(\gamma^\mu - 1) = F|_k(\gamma - 1)(\gamma^{\mu-1} + \cdots + 1) = \mu F|_k(\gamma - 1)$$

for all $\gamma \in \Gamma_1$. □

We should finally remark that the discussion of this paragraph applies more generally to all periodic second-order modular forms and therefore can be carried out independently of Theorem 14. This is a consequence of

Proposition 19. *Every periodic second-order modular form G on $\Gamma_0(N)$ is $\tilde{\Gamma}_0(N)$-invariant.*

Proof: Let F be a periodic second-order modular form. For all $\gamma, \delta, \epsilon \in \Gamma_0(N)$ we have:

$$F|_k(\gamma\delta\epsilon - \gamma - \delta - \epsilon + 2) = F|_k((\gamma\delta - 1)(\epsilon - 1) + (\gamma - 1)(\delta - 1)) = 0.$$

For $\delta = T$ and $\epsilon = \gamma^{-1}$ this gives $F|_k(\gamma T\gamma^{-1} - \gamma - T - \gamma^{-1} + 2) = 0$. This, in turn, in combination with $F|_k(-\gamma - \gamma^{-1} + 2) = F|_k(\gamma - 1)(\gamma^{-1} - 1) = 0$ implies $F|_k(\gamma T\gamma^{-1} - 1) = 0$ for all $\gamma \in \Gamma_0(N)$. □

Therefore, we can deduce from Propositions 17 and 18 that, for $N = 1, 2, 3$, if $F \in S_k^2(N)$ and $F(z + 1) = F(z)$ then $F \in S_k(N)$.

Acknowledgments The authors thank the referee for a careful reading of the paper and suggestions that substantially improved the exposition.

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