

SECOND ORDER MODULAR FORMS

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1. Introduction

In some recent papers (cf. [G2], [O], [CG], [GG], [DO]) the properties of new types of Eisenstein series are investigated. Motivated by the *abc* conjecture, these series were originally introduced by Goldfeld ([G1], [G3]) in order to study the distribution of modular symbols. Let $f(z)$ be a fixed cusp form of weight 2 for $\Gamma = \Gamma_0(N)$, say, the Hecke congruence group of level N . Then the defining formula for the series is

$$E^*(z, s) = \sum_{\tau \in \Gamma_\infty \backslash \Gamma} \langle \tau, f \rangle \text{Im}(\tau z)^s,$$

for z in the upper half-plane \mathfrak{H} and complex s with $\text{Re}(s) > 2$. Here

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}$$

is the stabilizer of the cusp ∞ and

$$\langle \tau, f \rangle = \int_{w_0}^{\tau w_0} f(w) dw$$

is called a modular symbol. Its definition is independent of w_0 in $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{i\infty\}$. The function E^* satisfies the equation

$$E^*(\gamma z, s) = E^*(z, s) + \langle \gamma, f \rangle E(z, s) \quad \text{for all } \gamma \in \Gamma \tag{1.1}$$

where

$$E(z, s) = \sum_{\tau \in \Gamma_\infty \backslash \Gamma} \text{Im}(\tau z)^s,$$

the usual Eisenstein series, satisfies $E(\gamma z, s) = E(z, s)$.

From equation (1.1) and the above it is clear that

$$E^*(\gamma\delta z, s) - E^*(\gamma z, s) - E^*(\delta z, s) + E^*(z, s) = 0 \quad \text{for all } \gamma, \delta \in \Gamma.$$

The form of this equation motivated us to study functions with similar transformation properties.

We will work more generally with $\Gamma \subset PSL_2(\mathbb{R})$ a Fuchsian group of the first kind. See the explanation of these in Section 2.3 of [I]. As described there we may choose a fundamental polygon \mathcal{F} to represent $\Gamma \backslash \mathfrak{H}$. We are primarily interested in groups Γ that contain parabolic elements. The surface $\Gamma \backslash \mathfrak{H}$ will therefore not be compact and $\overline{\mathcal{F}} \cap \hat{\mathbb{R}}$ will be a finite set of inequivalent cusps $\mathfrak{a}, \mathfrak{b}, \dots$ for $\hat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$. For each cusp \mathfrak{a} we may choose a scaling matrix $\sigma_{\mathfrak{a}} \in SL_2(\mathbb{R})$ that maps the upper part of the strip $\mathcal{F}_{\infty} = \{z \in \mathfrak{H} : -1/2 \leq \text{Re}(z) \leq 1/2\}$ to the neighborhood of \mathfrak{a} in \mathcal{F} (and hence $\sigma_{\mathfrak{a}}\infty = \mathfrak{a}$).

Next we define the spaces of functions that we are concerned with. Let k be a fixed integer. If v is a character of Γ and F a function on \mathfrak{H} then for all $\gamma \in \Gamma$ we set $(F|_{k,v}\gamma)(z) = v(\gamma)F(\gamma z)j(\gamma, z)^{-k}$ and extend the action of Γ to $\mathbb{Z}[\Gamma]$ by linearity. Here $j\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}, z\right) = cz + d$.

Definition I. Let $M_k(\Gamma, v)$ be the space of maps $f : \mathfrak{H} \rightarrow \mathbb{C}$ with the following properties:

- (i) f is holomorphic,
- (ii) $f|_{k,v}(\gamma - 1) = 0$ for all γ in Γ ,
- (iii) f has at most polynomial growth at the cusps.

The precise meaning of (iii) is that $f|_{\sigma_{\mathfrak{a}}}(z) \ll \text{Im}(z)^n$ for each cusp \mathfrak{a} and some constant n with z in the upper part of the strip \mathcal{F}_{∞} . These are the modular forms of weight k and character v for Γ . We denote by $\tilde{M}_k(\Gamma, v)$ the space obtained by relaxing (i) to include smooth functions. The non-holomorphic Eisenstein series are examples. (In this paper \sim will always signify a smooth space and its absence a holomorphic space.)

Definition II. In a similar manner we may define the space $\tilde{M}_k^2(\Gamma, v)$ of maps $f : \mathfrak{H} \rightarrow \mathbb{C}$ satisfying:

- (i) f is smooth,
- (ii) $f|_{k,v}(\gamma_a - 1)(\gamma_b - 1) = 0$ for all γ_a, γ_b in Γ ,
- (iii) for each γ in Γ , $(f|_{k,v}\gamma)(z)$ has at most polynomial growth at the cusps,
- (iv) $f|_{k,v}(\pi - 1) = 0$ for all parabolic π in Γ .

It can be shown that $E^*(z, s)$ is an example of such a function. If we call the holomorphic subspace $M_k^2(\Gamma, v)$ then we have the inclusions

$$\begin{aligned} M_k(\Gamma, v) &\subset M_k^2(\Gamma, v) \subset \tilde{M}_k^2(\Gamma, v), \\ \tilde{M}_k(\Gamma, v) &\subset \tilde{M}_k^2(\Gamma, v). \end{aligned}$$

Remarks.

- Condition (iv) in the definition of $\tilde{M}_k^2(\Gamma, v)$ was included to simplify the statements of the results and because the examples we have in mind so far satisfy it. It also ensures the existence of Fourier expansions of the functions at each cusp provided v is trivial on the parabolic elements.

- Condition (iii) may be strengthened by replacing polynomial growth with exponential decay,

$$(f|_{k,v}\sigma_{\mathfrak{a}})(z) \ll e^{-c \operatorname{Im}(z)}$$

for each cusp \mathfrak{a} and some constant $c > 0$ with, as before, z in the upper part of the strip \mathcal{F}_{∞} . We obtain (in an obvious notation) the spaces of smooth functions $\tilde{S}_k(\Gamma, v)$, $\tilde{S}_k^2(\Gamma, v)$ and their holomorphic versions $S_k(\Gamma, v)$ and $S_k^2(\Gamma, v)$.

- It is also interesting to consider other spaces, for example smooth functions f such that $f|_{k,v}(\gamma - 1) \in M_k(\Gamma, v)$.

We call elements of $M_k^2(\Gamma, v)$ or $\tilde{M}_k^2(\Gamma, v)$ *second-order modular forms* and elements of $S_k^2(\Gamma, v)$ or $\tilde{S}_k^2(\Gamma, v)$ *second-order cuspforms*. The names were suggested by D. Zagier in whose work with P. Kleban on percolation theory such functions also appear.

In this paper we show that these functions are much more basic in terms of the usual modular forms than one might think at first. In fact, their role is analogous to that of Eichler integrals with respect to period polynomials.

Another reason for the interest of second order modular forms is that the action $|_{k,v}$ induces a natural representation of the abelianization of Γ in $\tilde{M}_k^2(\Gamma, v)$. Indeed, let $\rho : \Gamma \rightarrow \operatorname{End}(\tilde{M}_k^2(\Gamma, v))$ be such that $\rho(\gamma)(f) = f|_{k,v}\gamma$ for all $\gamma \in \Gamma$, $f \in \tilde{M}_k^2(\Gamma, v)$. By definition, $f|_{k,v}\gamma\delta = f|_{k,v}\gamma + f|_{k,v}\delta - f$, so

$$f|_{k,v}\gamma\delta(\delta\gamma)^{-1} = f|_{k,v}(\gamma + \delta - 1)(\delta\gamma)^{-1} = f|_{k,v}\delta\gamma(\delta\gamma)^{-1} = f.$$

Similarly for the other spaces of second-order modular forms we will examine. Thanks to our work in Section 2 we can then associate such a representation to a usual modular form.

The paper is organized as follows. In Section 2 we determine the structure of the spaces $\tilde{M}_k^2(\Gamma, v)$ and $\tilde{S}_k^2(\Gamma, v)$. For example, if g is the genus of $\Gamma \backslash \mathfrak{H}$ then an easy to state corollary of the more precise Theorem 2.3 is

Corollary 2.4. *As \mathbb{R} -vector spaces we have*

$$\tilde{M}_k^2(\Gamma, v) \cong \bigoplus_{i=0}^{2g} \tilde{M}_k(\Gamma, v).$$

Then we turn to functions that satisfy the more general equation

$$f|_{k,v}(\gamma - 1)(\delta - 1)(\epsilon - 1) = 0 \quad \text{for all } \gamma, \delta, \epsilon \in \Gamma$$

rather than $f|_{k,v}(\gamma - 1)(\delta - 1) = 0$, $(\gamma, \delta \in \Gamma)$. If the space of such functions for which $f|_{k,v}(\gamma\delta - \delta\gamma) = 0$ is called $\tilde{M}_k^3(\Gamma, v)_{ab}$ then a consequence of Theorem 2.5 is that

$$\tilde{M}_k^3(\Gamma, v)_{ab} \cong_{\mathbb{R}} \bigoplus_{i=1}^{(2g+1)(g+1)} \tilde{M}_k(\Gamma, v).$$

Furthermore, we give a partial description of the class of functions f such that $f|_{k,v}p(\gamma, \dots) = 0$ where p is an arbitrary polynomial in $\mathbb{Z}[x_1, \dots, x_n]$ with x_i non-commuting variables.

In Section 3 we give an analogous treatment of the subspace of second-order modular forms that are also eigenfunctions of the Laplacian for a particular eigenvalue. These *second-order Maass forms* arise as residues of the function

$$E^*(z, s) = \sum_{\tau \in \Gamma_\infty \backslash \Gamma} |\langle \tau, f \rangle|^2 \text{Im}(\tau z)^s$$

studied in [G2] for example. It is hoped that a deeper understanding of these residues will help establish new results about the distribution of modular symbols.

Finally in Section 4 we show that there is a natural extension of the definition of Hecke operators that applies to second-order modular forms. These Hecke operators have the same multiplicativity and commutativity properties as the usual Hecke operators and hence the Fourier coefficients of their eigenfunctions have multiplicativity properties analogous to those of the usual Hecke eigenforms.

2. The structure of $\tilde{M}_k^2(\Gamma, v)$

To obtain a description of the structure of $\tilde{M}_k^2(\Gamma, v)$ we use the set of generators of Γ given by Fricke and Klein in, say, [I]. Specifically, if $\Gamma \backslash \mathfrak{H}$ has genus g , r elliptic fixed

points and m cusps, then there are $2g$ hyperbolic elements γ_i , r elliptic elements ϵ_i and m parabolic elements π_i generating Γ . Furthermore, these generators satisfy the $r + 1$ relations:

$$[\gamma_1, \gamma_{g+1}] \cdots [\gamma_g, \gamma_{2g}] \epsilon_1 \cdots \epsilon_r \pi_1 \cdots \pi_m = 1, \quad \epsilon_j^{e_j} = 1 \quad (2.1)$$

for $1 \leq j \leq r$ and integers $e_j \geq 2$. Here $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$ of a and b .

Recall the definition of the modular symbol $\langle \cdot, \cdot \rangle : \Gamma \times M_2(\Gamma) \rightarrow \mathbb{C}$. If we take f_1 in $M_2(\Gamma)$ and f_2 in $S_2(\Gamma)$ then the map $L_{f_1, f_2} : \Gamma \rightarrow \mathbb{C}$ with

$$L_{f_1, f_2}(\gamma) = \langle \gamma, f_1 + f_2 \rangle + \overline{\langle \gamma, f_1 - f_2 \rangle}$$

is an element of $\text{Hom}(\Gamma, \mathbb{C})$. The Eichler-Shimura isomorphism theorem (for weight 2) states that the map

$$(f_1, f_2) \mapsto L_{f_1, f_2}$$

is actually an \mathbb{R} -vector space isomorphism:

$$M_2(\Gamma) \oplus S_2(\Gamma) \cong \text{Hom}(\Gamma, \mathbb{C}).$$

Also if we are only interested in homomorphisms $\Gamma \rightarrow \mathbb{C}$ that are zero on the parabolic elements (call this space $\text{Hom}_0(\Gamma, \mathbb{C})$) then the same map gives

$$S_2(\Gamma) \oplus S_2(\Gamma) \cong \text{Hom}_0(\Gamma, \mathbb{C}).$$

In particular for any L in $\text{Hom}_0(\Gamma, \mathbb{C})$ there exist f, g in $S_2(\Gamma)$ so that if we define

$$\Lambda(z) := \int_{i\infty}^z f(w)dw + \overline{\int_{i\infty}^z g(w)dw} \quad (2.2)$$

then $L(\gamma) = \Lambda(\gamma z) - \Lambda(z)$ for all γ in Γ and all z in \mathfrak{H}^* . With the $2g$ hyperbolic generators γ_i we next define corresponding homomorphisms L_i such that $L_i(\gamma_i) = 1$ and $L_i(\gamma) = 0$ for all other generators γ of Γ . Each L_i can be expressed as $L_i(\gamma) = \Lambda_i(\gamma z) - \Lambda_i(z)$ with Λ_i defined as before with two cusp forms.

Lemma 2.1. *For each $\gamma \in \Gamma$ we have the map $f \mapsto f|_{k,v}(\gamma - 1)$. This map sends $\tilde{M}_k^2(\Gamma, v)$ to $\tilde{M}_k(\Gamma, v)$.*

The proof follows directly from the definitions of these spaces. In a similar manner these maps send

$$M_k^2(\Gamma, v) \rightarrow M_k(\Gamma, v), \quad \tilde{S}_k^2(\Gamma, v) \rightarrow \tilde{S}_k(\Gamma, v), \quad S_k^2(\Gamma, v) \rightarrow S_k(\Gamma, v).$$

Lemma 2.2. For f a second-order modular form we have $f_{k,v}(\epsilon - 1) = 0$ for all elliptic elements of Γ .

Proof: If $\epsilon^n = 1$ then

$$\begin{aligned} f|_{k,v}(\epsilon - 1) &= f|_{k,v}(\epsilon^{n+1} - 1) \\ &= f|_{k,v}(\epsilon - 1)(1 + \epsilon + \epsilon^2 + \cdots + \epsilon^n) \\ &= (n + 1)f|_{k,v}(\epsilon - 1). \end{aligned}$$

Therefore $nf|_{k,v}(\epsilon - 1) = 0$ and the lemma is proved.

Theorem 2.3. (“Chinese Remainder Theorem”) For f in $\tilde{M}_k^2(\Gamma, v)$ we let ψ denote the map sending f to the vector $(f|_{k,v}(\gamma_1 - 1), \dots, f|_{k,v}(\gamma_{2g} - 1))$. Then the following sequence of maps is exact:

$$0 \rightarrow \tilde{M}_k(\Gamma, v) \hookrightarrow \tilde{M}_k^2(\Gamma, v) \xrightarrow{\psi} \bigoplus_{i=1}^{2g} \tilde{M}_k(\Gamma, v) \rightarrow 0.$$

In other words, for each set $\{f_i; i = 1, \dots, 2g\} \subset \tilde{M}_k(\Gamma, v)^{2g}$ there is a $h \in M_k^2(\Gamma, v)$ (unique up to addition by a form in $\tilde{M}_k(\Gamma, v)$) such that $f_i = h|_{k,v}(\gamma_i - 1)$, ($i = 1, \dots, 2g$), and conversely.

Proof: To prove the exactness of the middle term we observe that if f is in the kernel of ψ then we must have $f|_{k,v}(\gamma - 1) = 0$ for all γ in Γ since it is true for each of the parabolic, elliptic and hyperbolic generators of the group. Thus $\text{Ker}(\psi) = \tilde{M}_k(\Gamma, v)$.

Finally, to prove that ψ is surjective we note that for any vector $V = (f_1, \dots, f_{2g})$ in $\bigoplus_{i=1}^{2g} \tilde{M}_k(\Gamma, v)$ we have $\psi(\sum_{i=1}^{2g} f_i \Lambda_i) = V$. It is routine to check that $\sum_{i=1}^{2g} f_i \Lambda_i$ is in $\tilde{M}_k^2(\Gamma, v)$. This completes the proof of Theorem 2.3.

The same proof gives the exact sequence

$$0 \rightarrow \tilde{S}_k(\Gamma, v) \hookrightarrow \tilde{S}_k^2(\Gamma, v) \xrightarrow{\psi} \bigoplus_{i=1}^{2g} \tilde{S}_k(\Gamma, v) \rightarrow 0.$$

For the holomorphic spaces $M_k^2(\Gamma, v)$ and $S_k^2(\Gamma, v)$ the above proof fails since $\Lambda_i(z)$ is not always holomorphic. In light of this difficulty it is natural to define the hybrid subspace $\tilde{M}_k^2(\Gamma, v)^* \subset \tilde{M}_k^2(\Gamma, v)$ of smooth functions that satisfy

$$f|_{k,v}(\gamma - 1) \in M_k(\Gamma, v) \quad \text{for all } \gamma \in \Gamma$$

and similarly for $\tilde{S}_k^2(\Gamma, v)^*$. The proof of Theorem 2.3 then gives

$$\begin{aligned} 0 \rightarrow \tilde{M}_k(\Gamma, v) \hookrightarrow \tilde{M}_k^2(\Gamma, v)^* \xrightarrow{\psi} \bigoplus_{i=1}^{2g} M_k(\Gamma, v) \rightarrow 0, \\ 0 \rightarrow \tilde{S}_k(\Gamma, v) \hookrightarrow \tilde{S}_k^2(\Gamma, v)^* \xrightarrow{\psi} \bigoplus_{i=1}^{2g} S_k(\Gamma, v) \rightarrow 0. \end{aligned}$$

An easy consequence of Theorem 2.3 is

Corollary 2.4. *We have the \mathbb{R} -vector space isomorphism*

$$\tilde{M}_k^2(\Gamma, v) \cong \bigoplus_{i=0}^{2g} \tilde{M}_k(\Gamma, v)$$

and for any $f \in \tilde{M}_k^2(\Gamma, v)$ there exist unique $h_i \in \tilde{M}_k(\Gamma, v)$ for $0 \leq i \leq 2g$ such that

$$f = \sum_{i=0}^{2g} h_i \Lambda_i$$

where the functions Λ_i are as defined earlier and for convenience we set $\Lambda_0(z) = 1$. Similar results hold for the spaces $\tilde{S}_k(\Gamma, v)$, $\tilde{M}_k(\Gamma, v)^*$ and $\tilde{S}_k(\Gamma, v)^*$.

A natural generalization of second-order forms satisfying the transformation property:

$$f|_{k,v}(\gamma_a - 1)(\gamma_b - 1) = 0 \quad \text{for all } \gamma_a, \gamma_b \in \Gamma$$

in Definition II would be functions satisfying the new condition (ii),

$$f|_{k,v}(\gamma_a - 1)(\gamma_b - 1)(\gamma_c - 1) = 0 \quad \text{for all } \gamma_a, \gamma_b, \gamma_c \in \Gamma.$$

We might call such functions third-order modular forms and in a consistent notation write $\tilde{M}_k^3(\Gamma, v)$, $\tilde{S}_k^3(\Gamma, v)$ etc.

We may characterize third-order modular forms in an analogous way to Theorem 2.3 but there is an important difference. While it was true for f in $\tilde{M}_k(\Gamma, v)$ or $\tilde{M}_k^2(\Gamma, v)$ that $f|_{k,v}(\gamma_a \gamma_b - \gamma_b \gamma_a) = 0$ this is no longer necessarily the case for third-order modular forms.

If $f \in \tilde{M}_k^3(\Gamma, v)$ then it is easy to check that the analogs of Lemmas 2.1 and 2.2 are true. In other words $\tilde{M}_k^3(\Gamma, v) \rightarrow \tilde{M}_k^2(\Gamma, v)$ under the map $f \mapsto f|_{k,v}(\gamma - 1)$ for each $\gamma \in \Gamma$ and $f|_{k,v}(\epsilon - 1) = 0$ for all elliptic $\epsilon \in \Gamma$. Define the map

$$\psi^* : \tilde{M}_k^3(\Gamma, v) \rightarrow \bigoplus_{i=1}^{(2g)^2} \tilde{M}_k(\Gamma, v)$$

with

$$\psi^*(f) = (f|_{k,v}(\gamma_i - 1)(\gamma_j - 1))_{1 \leq i, j \leq 2g}.$$

Set $\delta_{ij} = 1$ if $i = j$ and zero otherwise. If there exist smooth functions $\Lambda_{ij}(z)$ (with at most polynomial growth at the cusps) satisfying, for $1 \leq i, j, m, n \leq 2g$,

$$\Lambda_{ij}(\gamma_m \gamma_n z) - \Lambda_{ij}(\gamma_m z) - \Lambda_{ij}(\gamma_n z) + \Lambda_{ij}(z) = \delta_{im} \delta_{jn} \quad (2.3)$$

then

$$0 \rightarrow \tilde{M}_k^2(\Gamma, v) \hookrightarrow \tilde{M}_k^3(\Gamma, v) \xrightarrow{\psi^*} \bigoplus_{i=1}^{(2g)^2} \tilde{M}_k(\Gamma, v) \rightarrow 0$$

by essentially the same proof as Theorem 2.3 and

$$f = \sum_{i=0}^{2g} h_i \Lambda_i + \sum_{1 \leq i, j \leq 2g} h_{ij} \Lambda_{ij}$$

for unique $h_i, h_{ij} \in \tilde{M}_k(\Gamma, v)$. Unfortunately the functions Λ_{ij} satisfying 2.3 remain to be found. Without them the above results are not valid.

The products $\Lambda_i(z)\Lambda_j(z)$ are very close to satisfying (2.3). Their only defect is that they fail to distinguish between $\gamma_a\gamma_b$ and $\gamma_b\gamma_a$. They do allow us to prove the following

Theorem 2.5. *Let $\tilde{M}_k^3(\Gamma, v)_{ab} \subset \tilde{M}_k^3(\Gamma, v)$ denote the subspace of third-order modular forms f that satisfy the additional abelian condition*

$$f|_{k,v}(\gamma_a\gamma_b) = f|_{k,v}(\gamma_b\gamma_a) \quad \text{for all } \gamma_a, \gamma_b \in \Gamma.$$

Then $f = \sum_{i=0}^{2g} \sum_{j=i}^{2g} h_{ij} \Lambda_i \Lambda_j$ for unique $h_{ij} \in \tilde{M}_k(\Gamma, v)$.

Proof: This theorem follows from our above discussion and the formula

$$\begin{aligned} \Lambda_i(\gamma_m\gamma_n z)\Lambda_j(\gamma_m\gamma_n z) - \Lambda_i(\gamma_m z)\Lambda_j(\gamma_m z) - \Lambda_i(\gamma_n z)\Lambda_j(\gamma_n z) + \Lambda_i(z)\Lambda_j(z) \\ = \Lambda_i\Lambda_j|_{0,1}(\gamma_m - 1)(\gamma_n - 1) = \delta_{im}\delta_{jn} + \delta_{jm}\delta_{in}. \end{aligned}$$

To prove this formula we write

$$\begin{aligned} \Lambda_i(\gamma_m z)\Lambda_j(\gamma_m z) - \Lambda_i(z)\Lambda_j(z) \\ = \Lambda_i(\gamma_m z)(\Lambda_j(\gamma_m z) - \Lambda_j(z)) + (\Lambda_i(\gamma_m z) - \Lambda_i(z))\Lambda_j(z) \\ \delta_{jm}\Lambda_i(\gamma_m z) + \delta_{im}\Lambda_j(z) \end{aligned}$$

and so

$$\begin{aligned} \delta_{jm}\Lambda_i(\gamma_m\gamma_n z) + \delta_{im}\Lambda_j(\gamma_n z) - \delta_{jm}\Lambda_i(\gamma_m z) - \delta_{im}\Lambda_j(z) \\ = \delta_{jm}(\Lambda_i(\gamma_m\gamma_n z) - \Lambda_i(\gamma_m z)) + \delta_{im}(\Lambda_j(\gamma_n z) - \Lambda_j(z)) \\ = \delta_{jm}\delta_{in} + \delta_{im}\delta_{jn} \end{aligned}$$

as required, completing the proof.

These ideas extend to higher order modular forms. More generally if p is a fixed polynomial in $\mathbb{Z}[x_1, x_2, \dots, x_n]$, for noncommuting variables x_1, \dots, x_n , consider replacing condition (ii) in Definition II with

$$f|_{k,v}p(\gamma_a, \gamma_b, \dots) = 0 \quad \text{for all } \gamma_a, \gamma_b \cdots \in \Gamma.$$

Simple examples have $p(\gamma) = \gamma^n - 1$ or $p(\gamma_a, \gamma_b) = \gamma_a\gamma_b - \gamma_a$. Label these spaces $\tilde{M}_k(N, v, p)$, $\tilde{S}_k(N, v, p)$ etc. For general polynomials we cannot give a simple characterization of them. However, some simple propositions may be proved.

Proposition 2.6. *For a fixed $p \in \mathbb{Z}[x_1, x_2, \dots, x_n]$ we have $\tilde{M}_k(\Gamma, v) \subset \tilde{M}_k(\Gamma, v, p)$ provided $\tilde{M}_k(\Gamma, v, p) \neq 0$.*

Proof: For every $f \in \tilde{M}_k(\Gamma, v)$ we have $f|_{k,v}\gamma = f$ for all $\gamma \in \Gamma$. Therefore $f|_{k,v}p(\gamma_a, \dots) = f|_{k,v}A$, where A is the sum of coefficients of p . However, A must be 0 because if $g \in \tilde{M}_k(\Gamma, v, p)$ is non-zero, then $g|_{k,v}p(\gamma_a, \gamma_b, \dots) = 0$ for all $\gamma_a, \gamma_b \cdots \in \Gamma$ and, in particular, for $\gamma_a = \gamma_b = \dots = 1$, the identity in Γ .

In the opposite direction we have

Proposition 2.7. *If p has exactly two terms with coefficients summing to zero then there exists a subgroup Γ_p of Γ such that $f|_{k,v}p(\gamma_a, \gamma_b, \dots) = 0$ for all $\gamma_a, \gamma_b, \dots$ in Γ if and only if $f|_{k,v}(\gamma - 1) = 0$ for all γ in Γ_p .*

Proof: The polynomial p has the form $n\delta_1 - n\delta_2$ with n in \mathbb{Z} and δ_1, δ_2 made up of combinations of elements of Γ . Clearly we may replace p by $\delta_1 - \delta_2$. Also replacing z by $\delta_2^{-1}z$ we see that the functions f must satisfy $f|_{k,v}(\delta_1\delta_2^{-1} - 1) = 0$. This is equivalent to $f|_{k,v}(\gamma - 1) = 0$ for all γ in the group generated by elements of the form $\delta_1\delta_2^{-1}$ since if $f|_{k,v}(\gamma_a - 1) = 0$ and $f|_{k,v}(\gamma_b - 1) = 0$ then

$$f|_{k,v}(\gamma_a\gamma_b - 1) = f|_{k,v}((\gamma_a - 1)\gamma_b + (\gamma_b - 1)) = 0.$$

This completes the proof.

It would be interesting to characterize $\tilde{M}_k(\Gamma, v, p)$ when p is a more complicated polynomial, for example

$$p(\gamma_a, \gamma_b) = (\gamma_a - 1)(\gamma_b - 1) + (\gamma_b - 1)(\gamma_a - 1).$$

3. Second-order Maass cusp forms

For simplicity in the following we restrict ourselves to the case $k = 0$ and $v \equiv 1$, so we write $|$ rather than $|_{0,v}$. In this situation we shall call $\tilde{S}_0^2(\Gamma, 1)$ simply $\mathcal{A}^2(\Gamma \backslash \mathfrak{H})$. Let

$$\Delta = -4y^2 \partial_z \partial_{\bar{z}}$$

be the hyperbolic Laplace operator. We call a function f in $\mathcal{A}^2(\Gamma \backslash \mathfrak{H})$ a *second-order Maass forms with eigenvalue* λ if $(\Delta + \lambda)f = 0$. The set of all such functions we denote by $\mathcal{A}_\lambda^2(\Gamma \backslash \mathfrak{H})$. Condition (iv) in the definition of $\tilde{S}_0^2(\Gamma, 1)$ implies that any member f has a Fourier expansion at every cusp. We call f a *second-order Maass cuspform* if the constant coefficient of f at each cusp is identically zero. Denote the space of second-order Maass cuspforms of eigenvalue λ by $\mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H})$.

To determine the structure of $\mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H})$ we first fix some notation. We let $\mathcal{L}(\Gamma \backslash \mathfrak{H})$ denote the space of automorphic functions on $\Gamma \backslash \mathfrak{H}$ which are square-integrable with respect to the measure

$$d\mu(z) = \frac{dx dy}{y^2}, \quad z = x + iy.$$

The subspace of automorphic eigenfunctions of the Laplacian with eigenvalue λ is denoted by $\mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H})$. We also fix orthonormal eigenbases $\{g_i\}$ and $\{u_i\}$ for $S_2(\Gamma)$ and $\mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H})$, respectively. These bases are orthonormal with respect to the usual Petersson scalar product

$$\langle g, h \rangle := \int_{\Gamma \backslash \mathfrak{H}} y^k g(z) \overline{h(z)} d\mu z$$

where k is the weight.

Since the hyperbolic Laplacian is $SL_2(\mathbb{R})$ -invariant, it follows that, for $f \in \mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H})$, the function $z \mapsto f(\gamma z) - f(z)$ is in $\mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H})$. Thus, as in the proof of surjectivity in Theorem 2.3 we can prove,

Proposition 3.1. *Let $f \in \mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H})$. Then there exist complex constants $\{\alpha_{ij}\}$ and $\{\beta_{ij}\}$ such that, for all $\gamma \in \Gamma$,*

$$f(\gamma z) = f(z) + \sum_{i,j} u_j(z) \left(\alpha_{ij} \langle \gamma, g_i \rangle + \beta_{ij} \overline{\langle \gamma, g_i \rangle} \right).$$

We now give a characterization of the quotient $\mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H})/\mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H})$ analogous to that given in Theorem 2.3.

Let first f be an element of $\mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H})$. If $\{\alpha_{ij}\}$ and $\{\beta_{ij}\}$ are the constants associated to f by Proposition 3.1, define the function

$$f_0(z) := \sum_{i,j} u_j(z) \left(\alpha_{ij} \int_{i\infty}^z g_i(w) dw + \beta_{ij} \overline{\left(\int_{i\infty}^z g_i(w) dw \right)} \right). \quad (3.1)$$

We note that, since we have fixed bases for $S_2(\Gamma)$ and $\mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H})$, the function

$$f_0(z) = f_0(z; \lambda, \{\alpha_{ij}\}, \{\beta_{ij}\})$$

is completely determined by the complex numbers $\{\alpha_{ij}\}$ and $\{\beta_{ij}\}$ and the eigenvalue λ .

Lemma 3.2. *The function $G := f - f_0$ is in $\mathcal{L}(\Gamma \backslash \mathfrak{H})$.*

Proof: The automorphy of G is obvious from Eq. (3.1) and Proposition 3.1. The square integrability follows from the rapid decay of both f and f_0 at the cusps. This completes the proof.

Thus we have characterized the function f in $\mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H})$ modulo the square integrable automorphic function G . To go further, we quickly review the spectral theory of $\mathcal{L}(\Gamma \backslash \mathfrak{H})$.

Let

$$E_{\mathfrak{a}}(z, s) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \text{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s$$

be the real analytic Eisenstein series for Γ associated to the stabilizer $\Gamma_{\mathfrak{a}}$ of the cusp \mathfrak{a} . Then the spectral theorem for $\mathcal{L}(\Gamma \backslash \mathfrak{H})$ says that there exists an orthonormal set of eigenforms η_1, η_2, \dots with corresponding eigenvalues $\lambda_1, \lambda_2, \dots$ such that any $u \in \mathcal{L}(\Gamma \backslash \mathfrak{H})$ has the decomposition

$$u(z) = \frac{\langle u, 1 \rangle}{\langle 1, 1 \rangle} + \sum_{j=1}^{\infty} \langle u, \eta_j \rangle \eta_j(z) + \frac{1}{4\pi} \sum_{\mathfrak{a}} \int_{-\infty}^{+\infty} \langle u, E_{\mathfrak{a}}(\cdot, 1/2 + ir) \rangle E_{\mathfrak{a}}(z, 1/2 + ir) dr, \quad (3.2)$$

where the second sum is over a set of inequivalent cusps.

Now, with $f = f_0 - G$ as above,

$$\begin{aligned} 0 &= (\Delta + \lambda)f(z) \\ &= (\Delta + \lambda)(f_0 - G)(z) \\ &= \sum 4y^2 (\alpha_{ij} g_i(z) \partial_{\bar{z}} u_j(z) + \beta_{ij} \bar{g}_i(z) \partial_z u_j(z)) - (\Delta + \lambda)G(z). \end{aligned}$$

Hence

$$(\Delta + \lambda)G = H,$$

say, where

$$H(z) = H(z; \lambda, \{\alpha_{ij}\}, \{\beta_{ij}\}) = - \sum 4y^2 (\alpha_{ij}g_i(z)\partial_{\bar{z}}u_j(z) + \beta_{ij}\bar{g}_i(z)\partial_zu_j(z)). \quad (3.3)$$

Expressing both G and H in the form (3.2) and equating the coefficients, we get

$$(\lambda - \lambda_j)\langle G, \eta_j \rangle = \langle H, \eta_j \rangle$$

and

$$(\lambda + (1/4 + r^2))\langle G, E_{\mathfrak{a}}(\cdot, 1/2 + ir) \rangle = \langle H, E_{\mathfrak{a}}(\cdot, 1/2 + ir) \rangle,$$

for all j, \mathfrak{a} and $r > 0$. In particular,

$$\langle H(\cdot; \lambda, \{\alpha_{ij}\}, \{\beta_{ij}\}), \eta \rangle = 0, \quad \text{for all } \eta \in \mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H}) \quad (3.4)$$

and

$$\langle H(\cdot; \lambda, \{\alpha_{ij}\}, \{\beta_{ij}\}), E_{\mathfrak{a}}(\cdot, 1/2 + i\kappa) \rangle = 0 \text{ for all cusps } \mathfrak{a}, \lambda = -(1/4 + \kappa^2). \quad (3.5)$$

The requirements given in Eqs. (3.4) and (3.5) impose certain linear relations on the $\{\alpha_{ij}\}$ and $\{\beta_{ij}\}$ which must be satisfied.

It turns out that the relation (3.5) always holds:

Proposition 3.3. *Let $\phi \in S_2(\Gamma)$ and $u \in \mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H}), \lambda = -(1/4 + \kappa^2)$. Then*

$$\langle 4y^2\phi\partial_{\bar{z}}u, E_{\mathfrak{a}}(\cdot, 1/2 + i\kappa) \rangle = \langle 4y^2\bar{\phi}\partial_zu, E_{\mathfrak{a}}(\cdot, 1/2 + i\kappa) \rangle = 0.$$

Proof: Without loss of generality assume $\mathfrak{a} = i\infty$. Also choose s with $\text{Re}(s) > 3$. Let

$$\Phi(z) := \int_{i\infty}^z \phi(w)dw$$

be an antiderivative of ϕ . Note that

$$(\Delta + \lambda)(u\Phi) = -4y^2\phi\partial_{\bar{z}}u.$$

Unfolding the integral,

$$\begin{aligned} -4\langle y^2\phi\partial_{\bar{z}}u, E_{\mathfrak{a}}(\cdot, \bar{s}) \rangle &= \langle (\Delta + \lambda)(u\Phi), E_{\mathfrak{a}}(\cdot, \bar{s}) \rangle \\ &= \int_0^\infty \int_0^1 (\Delta + \lambda)(u(z)\Phi(z))y^s \frac{dx dy}{y^2} \end{aligned}$$

Integrate by parts twice. The fact that the real part of s is sufficiently large ensures that the boundary terms vanish, leaving us with

$$-4\langle y^2 \phi \partial_{\bar{z}} u, E_{\mathfrak{a}}(\cdot, \bar{s}) \rangle = \int_0^\infty \int_0^1 u(z) \Phi(z) (\Delta + \lambda) y^s \frac{dx dy}{y^2}.$$

The proof is completed by analytically continuing to $s = 1/2 + i\kappa$, and using the fact that

$$(\Delta + \lambda) y^{1/2+i\kappa} = 0.$$

Conversely, given complex constants $\{\alpha_{ij}\}$ and $\{\beta_{ij}\}$ satisfying (3.4), we can construct a function $f \in \mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H}) / \mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H})$. We first set $H := (\Delta + \lambda)f_0$, where $f_0(z)$ is the function associated to $\{\alpha_{ij}\}, \{\beta_{ij}\}$ by Eq. (3.1). Thanks to the relation (3.4), there exists a function $G \in \mathcal{L}(\Gamma \backslash \mathfrak{H})$ such that $H := (\Delta + \lambda)G$. This function is well-defined modulo $\mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H})$. It follows that

$$f := f_0 - G$$

is an element of $\mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H})$. Furthermore, any two functions $f \in \mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H})$ associated to the complex constants $\{\alpha_{ij}\}$ and $\{\beta_{ij}\}$ as above must differ by a Maass form in $\mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H})$.

It is easy to see that the mappings defined in this way are linear and inverse to one another.

Let $M = 2 \dim(S_2(\Gamma)) \dim(\mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H}))$. We have shown

Theorem 3.4. *As an \mathbb{R} -vector space,*

$$\mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H}) / \mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H})$$

is isomorphic to

$$\{(\alpha_{ij}, \beta_{ij}) \in \mathbb{C}^M : \langle H(\cdot; \lambda, \{\alpha_{ij}\}, \{\beta_{ij}\}), \eta \rangle = 0, \text{ for all } \eta \in \mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H})\}.$$

In particular, $\dim \mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H}) \leq (2 \dim S_2(\Gamma) + 1) \dim(\mathcal{C}_\lambda(\Gamma \backslash \mathfrak{H}))$. It would be desirable to also find a strong lower bound for $\dim \mathcal{C}_\lambda^2(\Gamma \backslash \mathfrak{H})$.

4. Hecke operators

Using the above description of the structure of $\tilde{M}_k^2(\Gamma, v)$, we can define operators on it that are compatible with the usual Hecke operators on $\tilde{M}_k(\Gamma, v)$. We restrict

ourselves to the case $\Gamma = \Gamma_0(N)$, the Hecke congruence group of level N (where we have identified ± 1). The character v is induced by a character on $(\mathbb{Z}/N\mathbb{Z})^*$. In the following we will just indicate the level N instead of writing the full group $\Gamma_0(N)$.

According to Section 2, any f in $\tilde{M}_k^2(N, v)$ can be uniquely written in the form

$$f(z) = \sum_{i=0}^{2g} h_i(z) \Lambda_i(z)$$

for unique h_i in $\tilde{M}_k(N, v)$. We then naturally define

$$(T_n f)(z) := \sum_{i=0}^{2g} (T_n h_i)(z) \Lambda_i(z) \quad (4.1)$$

where the T_n is the usual Hecke operator on $\tilde{M}_k(N, v)$ given by the formula

$$T_n g := n^{k-1} \sum_{ad=nb} \sum_{b \pmod{d}} v(d) d^{-k} g\left(\frac{az+b}{d}\right).$$

Obviously the maps given by (4.1) map $\tilde{M}_k^2(N, v)$ to $\tilde{M}_k^2(N, v)$ and coincide with the usual Hecke operators on $\tilde{M}_k(N, v) \subset M_k^2(N, v)$.

It is possible to define other Hecke operators on these second-order spaces. For examples of such alternative operators in the special case of Eisenstein series formed with modular symbols and related functions, see [DO].

There is nothing special about the Λ_i functions used in the definition (4.1). If L'_i with $1 \leq i \leq 2g$ is any basis for $\text{Hom}_0(\Gamma, \mathbb{C})$ then there exist Λ'_i functions as in (2.2) such that $L'_i(\gamma) = \Lambda'_i(\gamma z) - \Lambda'_i(z)$ for all γ in Γ . Also set $\Lambda'_0 = \Lambda_0 = 1$.

Proposition 4.1. *For $f = \sum_{i=0}^{2g} h_i \Lambda_i = \sum_{j=0}^{2g} h'_j \Lambda'_j$ we have*

$$\sum_{i=0}^{2g} (T_n h_i) \Lambda_i = \sum_{j=0}^{2g} (T_n h'_j) \Lambda'_j.$$

Proof: We express the linear dependence of Λ'_j and Λ_i by writing

$$\Lambda'_j = \sum_i \alpha_{ij} \Lambda_i.$$

Thus

$$f = \sum_{i=0}^{2g} h_i \Lambda_i = \sum_{j=0}^{2g} h'_j \Lambda'_j = \sum_{i,j=0}^{2g} \alpha_{ij} h'_j \Lambda_i$$

so that $h_i - \sum_j \alpha_{ij} h'_j = 0$. Now

$$\begin{aligned} \sum_i (T_n h_i) \Lambda_i - \sum_j (T_n h'_j) \Lambda'_j &= \sum_i (T_n h_i) \Lambda_i - \sum_{i,j} (T_n h'_j) \alpha_{ij} \Lambda_i \\ &= \sum_i ((T_n h_i) - \sum_j \alpha_{ij} (T_n h'_j)) \Lambda_i \\ &= \sum_i (T_n (h_i - \sum_j \alpha_{ij} h'_j)) \Lambda_i = 0 \end{aligned}$$

as required, completing the proof.

It is obvious that these Hecke operators T_n ($(n, N) = 1$) inherit the multiplicativity properties of the usual Hecke operators. Furthermore, it is possible to give a simple characterization of the effect of T_p 's (p prime not dividing N) on the Fourier coefficients of a holomorphic $f \in \tilde{S}_k^2(N, v)$. Specifically, suppose that

$$f(z) = \sum_j f_j(z) \int_{i\infty}^z g_j(z)$$

for some $f_j \in S_k(N, v)$, $g_j \in S_2(N)$. If

$$\begin{aligned} f_j(z) &= \sum_{m=1}^{\infty} a_j(m) e^{2\pi i m z} \\ g_j(z) &= \sum_{m=1}^{\infty} c_j(m) e^{2\pi i m z} \end{aligned}$$

then

$$f(z) = \frac{1}{2\pi i} \sum_m \left(\sum_n \frac{\sum_j a_j(m-n) c_j(n)}{n} \right) e^{2\pi i m z}.$$

Therefore, for every prime p such that $(p, N) = 1$,

$$T_p f(z) = \frac{1}{2\pi i} \sum_m \left(\sum_n \frac{\sum_j \tilde{a}_j(m-n) c_j(n)}{n} \right) e^{2\pi i m z}$$

where $\tilde{a}_j(l) = a_j(pl) + p^{k-1} a_j(l/p)$ (with $a_j(\alpha) := 0$ if $\alpha \notin \mathbb{Z}$) is the l -th Fourier coefficient of $T_p f_j(z)$.

This implies, in particular, that if f is an eigenfunction of T_p with eigenvalue λ_p , then

$$\sum_j a_j(p) c_j(1) = \lambda_p \sum_j a_j(1) c_j(1).$$

It should finally be noted that in a similar manner we could construct operators induced by the Atkin-Lehner operators U_q ($q|N$) for which an analogous discussion applies.

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