PROPERTIES OF EISENSTEIN SERIES FORMED WITH MODULAR SYMBOLS

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Abstract. In this work a new kind of non-holomorphic Eisenstein series, first introduced by Goldfeld, is studied. For an arbitrary Fuchsian group of the first kind we fix a holomorphic cusp form and consider Eisenstein series constructed with the modular symbol associated with this cusp form. We develop the theory analogously with that of the usual Eisenstein series starting with its meromorphic continuation to the entire complex plane. A functional equation is then obtained relating the values at s to those at $1-s$.

Introduction

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}\, z > 0\}$ be the upper half plane and let $\Gamma \subset SL_2(\mathbb{R})$ be a fixed non co-compact Fuchsian group of the first kind, (for example $\Gamma(N)$, $\Gamma_0(N)$), acting on $\mathbb{H}$. For simplicity assume that Γ has a unique cusp at infinity with stability group

$$\Gamma_\infty = \left\{ \pm \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, m \in \mathbb{Z} \right\}.$$

For each γ in Γ we shall label its matrix elements $\begin{pmatrix} \gamma_a & \gamma_b \\ \gamma_c & \gamma_d \end{pmatrix}$. Let $f(z)$ be an element of $S_2(\Gamma)$, the space holomorphic cusp forms of weight 2 for Γ. Following [Go] we define a modified Eisenstein series

$$E^*(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \langle \gamma, f \rangle \mathrm{Im}(\gamma z)^s, \quad z \in \mathbb{H}, \tag{0.1}$$

where for $\gamma \in \Gamma$ the modular symbol is given by

$$\langle \gamma, f \rangle = -2\pi i \int_w^{\gamma w} f(\tau)\, d\tau,$$

the definition being independent of $w \in \mathbb{H}$. Note that since $\langle \gamma_1 \gamma_2, f \rangle = \langle \gamma_1, f \rangle + \langle \gamma_2, f \rangle$ the series is not automorphic. The transformation rule is

$$E^*(\gamma z, s) = E^*(z,s) - \langle \gamma, f \rangle E(z,s),$$

for all $\gamma \in \Gamma$ where $E(z,s)$ is the usual Eisenstein series for Γ.

This new type of non-holomorphic Eisenstein series was introduced by Goldfeld in order to study the distribution properties of modular symbols $\langle \gamma, f \rangle$ as γ ranges over the group Γ. The series (0.1) converges for $\mathrm{Re}(s) > 2$ and Goldfeld hypothesised that it should have an analytic continuation and a functional equation. In this paper Selberg's method, (described in [Iw], [He]), is extended to establish the following results.

Theorem 0.1. *The Eisenstein series $E^*(z,s)$ defined by equation (0.1) for* $\mathrm{Re}(s) > 2$ *has a meromorphic continuation to the entire complex s-plane.*

$E^*(z,s)$ can also be expressed using its Fourier expansion. This will be built up from a number of parts:

- a generalized Kloosterman sum

$$S^*(m,n,f;c) = \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_\infty \\ \gamma_c = c}} \langle \gamma, f \rangle \, e^{2\pi i (n \frac{\gamma_a}{c} + m \frac{\gamma_d}{c})}$$

  defined for $c \in C = \left\{ c > 0 : \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \Gamma \right\}$,

- the K-Bessel function

$$K_v(z) = \frac{1}{2} \int_0^\infty e^{\frac{-z}{2}(u + \frac{1}{u})} u^{v-1} \, du \quad \text{for} \quad \mathrm{Re}(z) > 0,$$

- lastly the Whittaker function (from now on $z = x + iy$, $z' = x' + iy'$ ...)

$$W_s(nz) = 2|n|^{\frac{1}{2}} y^{\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi |n| y) e^{2\pi i n x}.$$

Then, using Bruhat's double coset decomposition, we have that

$$E^*(z,s) = \phi^*(s) y^{1-s} + \sum_{n \neq 0} \phi^*(n,s) W_s(nz), \tag{0.2}$$

$$\text{with} \quad \phi^*(s) = \sqrt{\pi} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{c \in C} \frac{S^*(0,0,f;c)}{c^{2s}}, \tag{0.3}$$

$$\phi^*(n,s) = \frac{\pi^s}{\Gamma(s)} |n|^{s-1} \sum_{c \in C} \frac{S^*(n,0,f;c)}{c^{2s}}, \tag{0.4}$$

where $\phi^*(s)$ and $\phi^*(n,s)$ are meromorphic for $s \in \mathbb{C}$ by Theorem 0.1 since they occur as Fourier coefficients of $E^*(z,s)$. The conventional functions $E(z,s), \phi(s)$ and $\phi(n,s)$ are defined similarly - just replace each occurance of the modular symbol by 1. Now we are in a position to state the functional equation

Theorem 0.2. *$E^*(z,s)$ satisfies,*

$$\phi(s) E^*(z, 1-s) = E^*(z,s) - \phi^*(s) \phi(1-s) E(z,s)$$

for all $s \in \mathbb{C}$ and

$$\phi(s) \phi^*(1-s) = -\phi^*(s) \phi(1-s)$$

where $\phi^*(s)$ is defined by (0.3) for Re$(s) > 2$.

These results are first steps in developing the theory of these series. As discussed in [Go2], estimating sums of the form

$$\sum w(\gamma) \langle \gamma, f \rangle \tag{0.5}$$

with weighting factor w and f in $S_2(\Gamma)$ allows us to uncover information on Goldfeld's conjecture, see [Go3], which is equivalent to Szpiro's conjecture on elliptic curves and implies a version of the abc conjecture.

In a further paper I will describe the behaviour of E^* on the critical line $\mathrm{Re}(s) = 1/2$. It has infinitely many simple poles on this line at points corresponding to the discrete spectrum of the Laplacian $\triangle$ on the space $L^2(\Gamma\backslash\mathbb{H})$. The residues of these poles are Maass forms. These ideas should lead to precise estimates for series such as (0.5).

The method we use to prove Theorem 0.1 generalizes naturally to more general series of the form

$$E^{m,n}(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \langle \gamma, f \rangle^m \overline{\langle \gamma, g \rangle}^n \mathrm{Im}(\gamma z)^s, \tag{0.6}$$

for example, with f, g in $S_2(\Gamma)$. It can be shown using the same ideas and induction on $m+n$ that $E^{m,n}(z,s)$ has a meromorphic continuation to the entire s-plane. It also satisfies a (complicated) functional equation relating values at s and $1-s$. This allows us to consider higher moments of (0.5).

Acknowledgement: This work was done as part of my Ph.D. thesis under the guidance of Dorian Goldfeld. I would like to express my deep thanks to him for introducing me to such a stimulating area and for his continuing help and support.

1. Preliminaries

More generally let $\Gamma \subset PSL_2(\mathbb{R})$, (or $-I \in \Gamma \subset SL_2(\mathbb{R})$), be any fixed non co-compact Fuchsian group of the first kind. Γ has a finite number of inequivalent cusps. By conjugation we may assume ∞ is a cusp and that its stability group is Γ_∞ as defined earlier. We shall denote by $\mathbb{F}$ the *Ford fundamental domain* for Γ. This consists of the closure of all points z in $\mathbb{H}$ exterior to the isometric circles of elements of Γ and contained in the vertical strip $\{z \in \mathbb{H} : 0 \le \mathrm{Re}(z) \le 1\}$. See [Ka], Chapter 3. The finite set $\mathbb{F} \cap \mathbb{R}$ gives us inequivalent representatives for the remaining cusps. For each cusp $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$, ... introduce scaling matrices $\sigma_\mathfrak{a}, \sigma_\mathfrak{b}$, ... These are elements of $SL_2(\mathbb{R})$ that satisfy

$$\sigma_\mathfrak{a}\infty = \mathfrak{a} \quad \text{and} \quad \sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \Gamma_\infty.$$

It shall be useful to break up $\mathbb{F}$ into cuspidal zones labelled $\mathbb{F}_\mathfrak{a}(Y)$ for each cusp $\mathfrak{a}$ and a central, compact region $\mathbb{F}(Y)$:

$$\begin{aligned} \mathbb{F}_\infty(Y) &= \{z \in \mathbb{F} : \mathrm{Im}\, z > Y\}, \\ \mathbb{F}_\mathfrak{a}(Y) &= \sigma_\mathfrak{a}\mathbb{F}_\infty(Y), \\ \mathbb{F}(Y) &= \mathbb{F} - \bigcup_\mathfrak{a} \mathbb{F}_\mathfrak{a}(Y) \end{aligned}$$

where the union is taken over all cusps.

Note that each scaling matrix $\sigma_\mathfrak{a}$ is determined up to translation on the right, so they can be chosen to ensure that $\mathbb{F}_\mathfrak{a}(Y) \subset \mathbb{F}$ for large Y. The *invariant height* of z will be denoted $y(z)$,

$$y(z) = \max \ \max \{\mathrm{Im}\, \sigma_\mathfrak{b}^{-1}\gamma z\} \tag{1.0}$$

and we see that

$$\mathbb{F}(Y) = \{z \in \mathbb{F} : y(z) \leq Y\}.$$

In other words, the closer z is to a cusp, the larger the height.

Define $j(\gamma, z) = \gamma_c z + \gamma_d$. Let χ be a one dimensional unitary representation of Γ, singular at every cusp, i.e. with

$$\chi\left(\sigma_{\mathfrak{a}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \sigma_{\mathfrak{a}}^{-1}\right) = 1 \quad \text{for each } \mathfrak{a}.$$

Let $f(z)$ be a fixed holomorphic cusp form of weight 2 for Γ. Define the generalized Eisenstein series

$$E^*_{\mathfrak{a}}(z, s, \chi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \chi(\gamma) \langle \gamma, f \rangle \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^s, \quad z \in \mathbb{H}.$$

As a concrete example, think of $E^*_\infty(z, s) = E^*_\infty(z, s, 1)$ for $\Gamma = \Gamma_0(N)$. It equals

$$\sum_{c,d} \left\langle \begin{pmatrix} * & * \\ c & d \end{pmatrix}, f \right\rangle \frac{y^s}{|cz + d|^{2s}}$$

where $f \in S_2(\Gamma_0(N))$ and the sum is over all coprime integers c, d, with $0 < c$ and $N \mid c$.

These series transform with a shift: for all $\gamma \in \Gamma$

$$E^*_{\mathfrak{a}}(\gamma z, s, \chi) = \overline{\chi}(\gamma) E^*_{\mathfrak{a}}(z, s, \chi) - \overline{\chi}(\gamma) \langle \gamma, f \rangle E_{\mathfrak{a}}(z, s, \chi).$$

$E^*_{\mathfrak{a}}(z, s, \chi)$ is absolutely convergent for $\operatorname{Re}(s) > 2$ and holomorphic in s there. This can be seen with the following result.

Lemma 1.1. *For all $\gamma \in \Gamma$, $z \in \mathbb{H}$ and any cusp $\mathfrak{a}$*

$$\langle \gamma, f \rangle \ll \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} \gamma z)^{-1} + \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1} z)^{-1},$$

where the implied constant depends only on Γ and f.

Proof:

$$\begin{aligned} \langle \gamma, f \rangle &= -\, 2\pi i \int_z^{\gamma z} f(z')\, dz' \\ &= -\, 2\pi i \int_{\sigma_{\mathfrak{a}}^{-1} z}^{\sigma_{\mathfrak{a}}^{-1} \gamma z} g(z')\, dz' \end{aligned}$$

where $g(z') = \frac{f(\sigma_{\mathfrak{a}} z')}{j(\sigma_{\mathfrak{a}}, z')^2}$ is a cusp form of weight 2 for $\sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{a}}$. We have the Fourier expansion

$$g(z') = \sum^{\infty} b_n e^{2\pi i n z'}$$

and by an elementary result [Sh], Lemma 3.62, $b_n \ll n$. Therefore

$$\begin{aligned}
\langle \gamma, f \rangle &= -2\pi i \int_{\sigma_{\mathfrak{a}}^{-1} z}^{\sigma_{\mathfrak{a}}^{-1}\gamma z} \left[\sum_{n=1}^{\infty} b_n e^{2\pi i n z'}\right] dz' \\
&= -2\pi i \sum_{n=1}^{\infty} b_n \int_{\sigma_{\mathfrak{a}}^{-1} z}^{\sigma_{\mathfrak{a}}^{-1}\gamma z} e^{2\pi i n z'} \, dz' \\
&= -\sum_{n=1}^{\infty} \frac{b_n}{n} \left(e^{2\pi i n \sigma_{\mathfrak{a}}^{-1}\gamma z} - e^{2\pi i n \sigma_{\mathfrak{a}}^{-1} z}\right) \\
&\ll \sum_{n=1}^{\infty} \left(e^{-2\pi n \,\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)} + e^{-2\pi n \,\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1} z)}\right).
\end{aligned}$$

Using $\frac{e^{-x}}{1-e^{-x}} \le \frac{1}{x}$ completes the proof. ■

Remark. In particular, for $\mathfrak{a} = \infty$ and $z = -\frac{\gamma_d}{\gamma_c} + \frac{i}{|\gamma_c|}$ we obtain the bound $\langle \gamma, f \rangle \ll |\gamma_c|$. This can be improved to $|\gamma_c|^{\frac{1}{2}+\epsilon}$ for congruence groups, see [Go].

We know that the usual Eisenstein series

$$E_{\mathfrak{a}}(z, s, \chi) = \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \chi(\gamma) \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s$$

is absolutely convergent (and uniformly convergent on compacta) for $\mathrm{Re}(s) > 1$. Thus Lemma 1.1 gives us immediately

Corollary 1.2. *The modified Eisenstein series $E^*_{\mathfrak{a}}(z, s, \chi)$ converges absolutely and uniformly on compacta for $Re(s) > 2$.*

Next, to describe the Fourier expansion of $E^*_{\mathfrak{a}}(z, s, \chi)$, we need the Kloosterman sum

$$S^*_{\mathfrak{a}\mathfrak{b}}(m, n, \chi, f; c) = \sum_{\substack{\gamma \in \Gamma_\infty \backslash \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/\Gamma_\infty \\ \gamma_c = c}} \chi(\sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}) \langle \sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}, f \rangle \, e^{2\pi i (n\frac{\gamma_a}{c} + m\frac{\gamma_d}{c})}$$

defined for $c \in C_{\mathfrak{a}\mathfrak{b}} = \left\{ c > 0 : \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}} \right\}$.

For $\mathrm{Re}(s) > 2$ we can write the Fourier expansion of $E^*_{\mathfrak{a}}(z, s, \chi)$ explicitly

(1.1)
$$E^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) = \phi^*_{\mathfrak{a}\mathfrak{b}}(s, \chi) y^{1-s} + \sum_{n \ne 0} \phi^*_{\mathfrak{a}\mathfrak{b}}(n, s, \chi) W_s(nz),$$

(1.2) with
$$\phi^*_{\mathfrak{a}\mathfrak{b}}(s, \chi) = \sqrt{\pi}\frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \sum_{c \in C_{\mathfrak{a}\mathfrak{b}}} \frac{S^*_{\mathfrak{a}\mathfrak{b}}(0, 0, \chi, f; c)}{c^{2s}},$$

(1.3)
$$\phi^*_{\mathfrak{a}\mathfrak{b}}(n, s, \chi) = \frac{\pi^s}{\Gamma(s)} |n|^{s-1} \sum \frac{S^*_{\mathfrak{a}\mathfrak{b}}(n, 0, \chi, f; c)}{c^{2s}}.$$

By comparision,

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) = \delta_{\mathfrak{a}\mathfrak{b}} y^s + \phi_{\mathfrak{a}\mathfrak{b}}(s,\chi) y^{1-s} + \sum_{n \neq 0} \phi_{\mathfrak{a}\mathfrak{b}}(n,s,\chi) W_s(nz) \tag{1.4}$$

where, as before, $\phi_{\mathfrak{a}\mathfrak{b}}$ is obtained from $\phi^*_{\mathfrak{a}\mathfrak{b}}$ by replacing the modular symbols in $S^*_{\mathfrak{a}\mathfrak{b}}$ with 1s. The analytic continuation of this series, for general Γ, was first demonstrated by Selberg, see [Se], Theorem 7.3 for example. The following statement is based on [Iw], Prop 6.1:

Theorem 1.3. *Given $c > 1$ define the vertical strip*

$$\mathbb{S} = \{s \in \mathbb{C} : 1 - c \leq \mathrm{Re}(s) \leq c\}.$$

Then there exist functions $A_{\mathfrak{a}}(s,\chi) \not\equiv 0$ on $\mathbb{S}$ and $A_{\mathfrak{a}}(z,s,\chi)$ on $\mathbb{H} \times \mathbb{S}$ such that the following hold:

(i) *$A_{\mathfrak{a}}(s,\chi)$ and $A_{\mathfrak{a}}(z,s,\chi)$ are analytic in s of order ≤ 8,*
(ii) *$A_{\mathfrak{a}}(s,\chi) E_{\mathfrak{a}}(z,s,\chi) = A_{\mathfrak{a}}(z,s,\chi)$ for $1 < \mathrm{Re}(s) \leq c$,*
(iii) *$(\triangle_z + s(1-s)) A_{\mathfrak{a}}(z,s,\chi) = 0$,*
(iv) *$A_{\mathfrak{a}}(\gamma z,s,\chi) = \overline{\chi}(\gamma) A_{\mathfrak{a}}(z,s,\chi)$ for all $\gamma \in \Gamma$,*
(v) *$A_{\mathfrak{a}}(z,s,\chi)$ is real analytic in (z,s),*
(vi) *$A_{\mathfrak{a}}(z,s,\chi) \ll e^{\eta y(z)}$ for any $\eta > 0$,*

where the implied constant depends on η, s and Γ.

So we see that (ii) allows us to represent $E_{\mathfrak{a}}(z,s,\chi)$ as the quotient of analytic functions $\frac{A_{\mathfrak{a}}(z,s,\chi)}{A_{\mathfrak{a}}(s,\chi)}$ on an arbitrarily wide strip $\mathbb{S}$. Our goal is to develop similar results for the new series E^*.

2. Constructing an Integral Equation

We begin with some growth estimates that we shall need to establish equation (2.1), the main result of this section.

Lemma 2.1. *For $z \in \mathbb{H}$ and $\sigma = Re(s) > 1$ we have*

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) \ll \frac{1}{y^\sigma} + y^\sigma$$

the implied constant depending on σ and Γ.

Proof: First, for $y \geq 1$, using the Fourier expansion (1.4) we see that

$$E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) \ll y^\sigma.$$

For $z = x + iy$ define $\hat{z} = x + i$. Then it's easy to show that

$$\mathrm{Im}(z)\mathrm{Im}(\gamma z) \leq \mathrm{Im}(\gamma \hat{z})$$

when $y \leq 1$ and $\gamma \in SL_2(\mathbb{R})$. Thus, for $y \leq 1$,

$$|E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) y^\sigma| \leq E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} \hat{z}, \sigma) \ll 1. \quad \blacksquare$$

Lemma 2.2. *For $z \in \mathbb{H}$ and $\sigma = Re(s) > 2$ we have*

$$E^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi) \ll \frac{1}{y^{\sigma+1}} + y^{\sigma+1}$$

the implied constant depending on σ, Γ and f.

Proof: We'll use the result of Lemma 1.1:

$$\langle \gamma, f \rangle \ll \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}} z)^{-1} + \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\sigma_{\mathfrak{b}} z)^{-1}.$$

$$\begin{aligned}
|E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, s, \chi)| &\leq \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} |\langle \gamma, f \rangle| \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}} z)^{\sigma} \\
&\ll \sum_{\gamma \in \Gamma_{\mathfrak{a}} \backslash \Gamma} \left(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}} z)^{\sigma-1} + \frac{\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma\sigma_{\mathfrak{b}} z)^{\sigma}}{\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\sigma_{\mathfrak{b}} z)} \right) \\
&= E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, \sigma - 1) + \frac{E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z, \sigma)}{\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\sigma_{\mathfrak{b}} z)}
\end{aligned}$$

Finally with $\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\sigma_{\mathfrak{b}} z)^{-1} \ll y^{-1} + y$ one obtains the lemma. ■

The Laplace operator on the hyperbolic plane is given by

$$\triangle_z = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

If $\rho(z, w)$ is the hyperbolic distance between two points z and w in $\mathbb{H}$ we define

$$u(z, w) = \frac{1}{2}\big(\cosh \rho(z, w) - 1\big),$$

and we can derive the simple formula

$$u(z, w) = \frac{|z - w|^2}{4 \,\mathrm{Im}\, z \,\mathrm{Im}\, w}.$$

This metric is more convenient to work with. We need the Green function

$$G_a(u) = \frac{1}{4\pi} \int_0^1 (t(1-t))^{a-1}(t+u)^{-a} \, dt$$

for $u, a > 0$. This function is discussed in [Iw], §1.7. It is smooth except for a logarithmic singularity at 0,

$$G_a(u) = \frac{-1}{4\pi} \log(u) + O(1) \quad \text{as} \quad u \to 0$$

and we have easily that $G_a(u) \leq u^{-a}$ for $a \geq 1$. The analytic continuation of $E^*_{\mathfrak{a}}(z, s, \chi)$ will depend on the integral equation we establish next.

Theorem 2.3. *If $\theta(z) : \mathbb{H} \to \mathbb{C}$ is an eigenfunction of $\triangle$ with eigenvalue λ that satisfies $\theta(z) \ll y^\sigma + y^{-\sigma}$ for $\sigma > 0$ then, when $a > \sigma + 1$,*

$$\frac{-\theta(w)}{\lambda + a(1-a)} = \int_{\mathbb{H}} G_a(u(w,z))\theta(z)\, d\mu z$$

where from now on $d\mu z$ will mean the hyperbolic invariant measure $dxdy/y^2$.

Proof: The condition $a > \sigma + 1$ ensures the integral is absolutely convergent; use the bounds

$$\frac{G_a(u(z,w))}{y^2} \ll \begin{cases} \frac{1}{|z|^{a+2}} & y \geq 1 \\ \frac{y^{a-2}}{|z|^{2a}+1} & y < 1 \end{cases}$$

assuming that z is not close to w, the implied constant depending on w. Thus

$$\begin{aligned} \big(\lambda + a(1-a)\big) \int_{\mathbb{H}} & G_a(u(w,z))\theta(z)\, d\mu z \\ &= \int_{\mathbb{H}} G_a(u(w,z))\big(\lambda + a(1-a)\big)\theta(z)\, d\mu z \\ &= \int_{\mathbb{H}} G_a(u(w,z))\big(\triangle_z + a(1-a)\big)\theta(z)\, d\mu z. \end{aligned}$$

As in the last part of [Iw], Theorem 1.17 we divide $\mathbb{H}$ into a disc U and its complement V. The integral over U disappears as we let its radius go to zero. Applying Green's formula to V and using the equation

$$\big(\triangle_z + a(1-a)\big)G_a(u(z,w)) = 0$$

proves that

$$-\theta(w) = \int_{\mathbb{H}} G_a(u(w,z))\big(\triangle_z + a(1-a)\big)\theta(z)\, d\mu z. \qquad \blacksquare$$

In fact a stronger result (which we won't use) may be proven.

Theorem 2.4. *If $\theta(z) : \mathbb{H} \to \mathbb{C}$ satisfies $\theta(z) \ll y^\sigma + y^{-\sigma}$ for $\sigma > 0$ and has partial derivatives up to order 2 with similar bounds then, when $\sigma \ll a$,*

$$-\big(\triangle_w + a(1-a)\big) \int_{\mathbb{H}} G_a(u(w,z))\theta(z)\, d\mu z = \theta(w).$$

Now we can exploit the fact that $E^*_{\mathfrak{a}}(z,s,\chi)$ is an eigenfunction of the Laplacian,

$$\triangle_z E^*_{\mathfrak{a}}(z,s,\chi) = s(s-1)E^*_{\mathfrak{a}}(z,s,\chi),$$

along with Lemma 2.2 and Theorem 2.3 to write

$$\frac{-E^*_{\mathfrak{a}}(z,s,\chi)}{(a(1-a) - s(1-s))} = \int_{\mathbb{H}} G_a(u(z,z'))E^*_{\mathfrak{a}}(z',s,\chi)\, d\mu z'$$

for $2 < \mathrm{Re}(s) < a-1$. The next Lemma will be used to break this integral up into pieces.

Lemma 2.5. *Let Γ be a discrete subgroup of $SL_2(\mathbb{R})$ with fundamental domain $\mathbb{F}$. If f is any integrable function on $\mathbb{H}$ with $\int_{\mathbb{H}} |f(z)|\, d\mu z < \infty$ then*

$$\int_{\mathbb{H}} f(z)\, d\mu z = \int_{\mathbb{F}} \sum_{\gamma \in \Gamma} f(\gamma z)\, d\mu z.$$

Proof: Use Lebesgue's theorems on monotone and dominated convergence. ■

Since

$$\int_{\mathbb{H}} |G_a(u(z,z'))E^*_{\mathfrak{a}}(z',s,\chi)|\; d\mu z' < \infty$$

for $\mathrm{Re}(s) < a-1$

$$\begin{aligned}\frac{-E^*_{\mathfrak{a}}(z,s,\chi)}{(a(1-a)-s(1-s))} &= \int_{\mathbb{F}} \sum_{\gamma\in\Gamma} G_a(u(z,\gamma z'))E^*_{\mathfrak{a}}(\gamma z',s,\chi)\, d\mu z' \\ &= \int_{\mathbb{F}} \sum_{\gamma} G_a(u(z,\gamma z'))\overline{\chi}(\gamma)\big(E^*_{\mathfrak{a}}(z',s,\chi) - \langle \gamma, f\rangle E_{\mathfrak{a}}(z',s,\chi)\big)\, d\mu z'.\end{aligned}$$

Therefore, for $2 < \mathrm{Re}(s) < a-1$ we get

$$\begin{aligned}\frac{-E^*_{\mathfrak{a}}(z,s,\chi)}{(a(1-a)-s(1-s))} &= \int_{\mathbb{F}} G_a(z,z',\chi)E^*_{\mathfrak{a}}(z',s,\chi)\, d\mu z' \\ &\quad + \int_{\mathbb{F}} G^*_a(z,z',\chi)E_{\mathfrak{a}}(z',s,\chi)\, d\mu z'\end{aligned} \tag{2.1}$$

where, for $z \not\equiv z' (\mathrm{mod}\ \Gamma)$

$$\begin{aligned}G_a(z,z',\chi) &= \sum_{\gamma} \overline{\chi}(\gamma) G_a(u(z,\gamma z')) \\ &= \sum_{\gamma} \chi(\gamma) G_a(u(\gamma z, z')), \\ G^*_a(z,z',\chi) &= -\sum_{\gamma} \overline{\chi}(\gamma)\langle \gamma, f\rangle G_a(u(z,\gamma z')) \\ &= \sum_{\gamma} \chi(\gamma)\langle \gamma, f\rangle G_a(u(\gamma z, z')).\end{aligned}$$

Proposition 2.6. *Let $\Gamma_z = \{\gamma \in \Gamma : \gamma z = z\}$. Then, for $a > 1$, the function $G_a(z,z',\chi) : \mathbb{H}\times\mathbb{H} \to \mathbb{C}$ is continuous except on the diagonal $z \equiv z' (\mathrm{mod}\ \Gamma)$ where we have, for some fixed $\gamma' \in \Gamma$,*

$$G_a(z,z',\chi) = \frac{-1}{2\pi} \log|\gamma' z - z'|\, \overline{\chi}(\gamma') \sum_{\gamma \in \Gamma_z} \chi(\gamma) + O(1), \qquad \text{as } z' \to \gamma' z.$$

For $a > 2$ the function $G^*_a(z,z',\chi) : \mathbb{H}\times\mathbb{H} \to \mathbb{C}$ is continuous except at pairs (z,z') where we have $z' = \gamma' z$ and $\langle \gamma', f\rangle \neq 0$, in which case

$$G^*_a(z,z',\chi) = \frac{1}{2\pi} \log|\gamma' z - z'|\, \overline{\chi}(\gamma')\langle \gamma', f\rangle \sum \chi(\gamma) + O(1), \qquad \text{as } z' \to \gamma' z.$$

Proof: Start with any $z_0, z_0' \in \mathbb{H}$. Let $\mathbb{D}_r(z) = \{w \in \mathbb{H} : u(z,w) < r\}$ denote an open hyperbolic ball. By [Sh], Prop 1.7 we can choose $\varepsilon > 0$ such that if $u(\gamma z_0, z_0) < 2\varepsilon$ for any $\gamma \in \Gamma$ then in fact $\gamma z_0 = z_0$ (and the same for z_0'). Fix an ordering of $\Gamma = \{\gamma_i\}_{i \in \mathbb{N}}$ with $u(\gamma_i z_0, z_0') \leq u(\gamma_{i+1} z_0, z_0')$. From [Iw], Lemma 2.11 there exists a C so that

$$\#\{\gamma \in \Gamma : u(\gamma z_0, z_0') \leq R\} \leq C(R+1). \tag{2.2}$$

Let Φ be the finite set $\{\gamma_1, \gamma_2, \dots, \gamma_N\} \subset \Gamma$ with $u(\gamma_i z_0, z_0') < 2\varepsilon$. Consider the series

$$\sum_{i=N+1}^{\infty} |G_a(u(\gamma_i z_0, z_0'))| \leq \sum_{i=N+1}^{\infty} \frac{1}{u(\gamma_i z_0, z_0')^a} \quad \text{for } a \geq 1.$$

It is absolutely convergent for $a > 1$ by (2.2). Now for any $z \in \mathbb{D}_{\varepsilon/2}(z_0)$, $z' \in \mathbb{D}_{\varepsilon/2}(z_0')$ we have

$$\begin{aligned} G_a(z, z', \chi) &= \sum_{i=1}^{N} G_a(u(\gamma_i z, z')) + \sum_{i=N+1}^{\infty} G_a(u(\gamma_i z, z')) \\ &= S_1(z,z') + S_2(z,z'), \quad \text{say.} \end{aligned}$$

The terms in S_2 are continuous and converge uniformly because

$$\begin{aligned} |G_a(u(\gamma_i z, z'))| \leq u(\gamma_i z, z')^{-a} &\leq (u(\gamma_i z_0, z_0') - \varepsilon)^{-a} \\ &\leq 2^a u(\gamma_i z_0, z_0')^{-a} \end{aligned}$$

so $S_2(z,z')$ is continuous at (z_0, z_0'). S_1 can have a discontinuity at (z_0, z_0') only if there is some $\gamma' \in \Gamma$ with $\gamma' z_0 = z_0'$. In that case $\gamma' \Gamma_{z_0} = \Phi$. So

$$\begin{aligned} S_1(z_0, z') &= \sum_{\gamma \in \gamma' \Gamma_{z0}} \chi(\gamma) G_a(u(\gamma z_0, z')) \\ &= \sum_{\gamma \in \gamma' \Gamma_{z0}} \chi(\gamma) \frac{-1}{4\pi} \log|u(\gamma' z_0, z')| + O(1) \quad \text{as } z' \to z_0' = \gamma' z_0 \\ &= \frac{-1}{2\pi} \log|\gamma' z_0 - z')| \sum_{\gamma \in \gamma' \Gamma_{z0}} \chi(\gamma) + O(1) \quad \text{as } z' \to z_0' = \gamma' z_0 \end{aligned}$$

as required. The result for G^* is proven similarly. ■

3. Applying Fredholm Theory

We must carry out some modifications to equation (2.1) before we can use the next result.

Theorem 3.1 (Fredholm). *Assume $\int_{\mathbb{F}} d\mu z' = V < \infty$ and that $K(z,z')$ is bounded and integrable on $\mathbb{F} \times \mathbb{F}$, then for all $\lambda \in \mathbb{C}$ there exist $D(\lambda)$ and $D_\lambda(z,z')$*

entire in λ with the following property. If $f(z)$ is any bounded integrable function on $\mathbb{F}$ and if $g(z)$ (defined on $\mathbb{F}$) satisfies

$$g(z) = f(z) + \lambda \int_{\mathbb{F}} K(z,z')g(z')\,d\mu z'$$

then $g(z)$ is uniquely determined and given by the formula

$$g(z) = f(z) + \frac{\lambda}{D(\lambda)} \int_{\mathbb{F}} D_\lambda(z,z')f(z')\,d\mu z'$$

when $D(\lambda) \neq 0$.

Proof: See [Iw] A.4. ■

Our situation is slightly complicated by dependence on a parameter s that will be contained in a compact set $\mathbb{S} \subset \mathbb{C}$. $K(z,z') = K_s(z,z')$ so $D(\lambda)$ and $D_\lambda(z,z')$ vary with s. Also $f(z) = f(z,s)$ and $\lambda = \lambda(s)$, where f, λ and K are analytic functions of s on $\mathbb{S}$. If $g(z,s)$ is analytic in some neighborhood $\mathbb{S}' \subset \mathbb{S}$ and satisfies

$$g(z,s) = f(z,s) + \lambda \int_{\mathbb{F}} K_s(z,z')g(z',s)\,d\mu z' \quad \forall s \in \mathbb{S}' \tag{3.1}$$

then by the theorem

$$g(z,s) = f(z,s) + \frac{\lambda}{D(\lambda)} \int_{\mathbb{F}} D_\lambda(z,z')f(z',s)\,d\mu z'$$

for all $s \in \mathbb{S}'$, where $D(\lambda) \neq 0$. (We have assumed that $K_s(z,z')$ and $f(z,s)$ are uniformly bounded in $\mathbb{S}$.) But the right side of this last equation will be meromorphic in the larger domain $\mathbb{S}$. This is how we shall achieve the analytic continuation.

To get (2.1) into the right form we first eliminate the singularity at $z = z'$ of $G_a(z,z',\chi)$ by taking the difference

$$G_{ab}(z,z',\chi) = G_a(z,z',\chi) - G_b(z,z',\chi), \qquad b < a$$

and similarly set $G^*_{ab} = G^*_a - G^*_b$. With Lemma 2.6 we can see that these new series are continuous on $\mathbb{H} \times \mathbb{H}$. We obtain

$$\begin{aligned} E^*_{\mathfrak{a}}(z,s,\chi)\nu_{ab}(s) = &\int_{\mathbb{F}} G_{ab}(z,z',\chi)E^*_{\mathfrak{a}}(z',s,\chi)\,d\mu z' \\ &+ \int_{\mathbb{F}} G^*_{ab}(z,z',\chi)E_{\mathfrak{a}}(z',s,\chi)\,d\mu z', \end{aligned} \tag{3.2}$$

when $\nu_{ab}(s) = (a(1-a) - s(1-s))^{-1} - (b(1-b) - s(1-s))^{-1}$ and $2 < \operatorname{Re}(s) < b-1$. The kernel $G_{ab}(z,z',\chi)$ can be estimated in cuspidal zones using its Fourier expansion, see [Iw], Lemma 5.4:

$$\begin{aligned} G_{ab}(\sigma_{\mathfrak{a}} z, \sigma_{\mathfrak{b}} z', \chi) &- (2a-1)^{-1}(y')^{1-a}E_{\mathfrak{b}}(\sigma_{\mathfrak{a}} z, a, \chi) \\ &+ (2b-1)^{-1}(y')^{1-b}E_{\mathfrak{b}}(\sigma_{\mathfrak{a}} z, b, \chi) \ll e^{-2\pi(y'-y)} \\ &\text{for } y' > y > Y \end{aligned}$$

Following [Iw] we replace $G_{ab}(z,z',\chi)$ by a truncated version

$$G^Y_{ab}(z,z',\chi) : \mathbb{H}\times\mathbb{F}\to\mathbb{C}$$

which is bounded in z'. If $z'\in\mathbb{F}(Y)$ then set $G^Y_{ab}(z,z',\chi)=G_{ab}(z,z',\chi)$ otherwise if $z'\in\mathbb{F}_{\mathfrak{b}}(Y)$ then

$$\begin{aligned}G_{ab}(z,z',\chi) = G^Y_{ab}(z,z',\chi) &+ (2a-1)^{-1}\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}z')^{1-a}E_{\mathfrak{b}}(z,a,\chi)\\ &-(2b-1)^{-1}\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}z')^{1-a}E_{\mathfrak{b}}(z,b,\chi).\end{aligned}\tag{3.3}$$

Consequently

$$G^Y_{ab}(\sigma_{\mathfrak{a}}z,\sigma_{\mathfrak{b}}z',\chi)\ll e^{-2\pi(y'-y)} \qquad \text{for}\quad y'>y>Y$$

and, (using the symmetry $G_{ab}(z,z',\chi)=G_{ab}(z',z,\overline{\chi})$)

$$\begin{aligned}&G^Y_{ab}(\sigma_{\mathfrak{a}}z,\sigma_{\mathfrak{b}}z',\chi) + [\text{ small things }]\\ &\qquad -(2a-1)^{-1}(y')^{1-a}E_{\mathfrak{b}}(\sigma_{\mathfrak{a}}z,a,\chi)\ll e^{-2\pi(y-y')} \quad \text{for}\quad y>y'>Y\end{aligned}$$

implying that

$$G^Y_{ab}(\sigma_{\mathfrak{a}}z,\sigma_{\mathfrak{b}}z',\chi)\ll y^a e^{-2\pi\max\{y'-y,0\}} \quad \text{for}\quad y,y'>Y.\tag{3.4}$$

To see how this changes (3.2) we compute

$$\begin{aligned}\int_{\mathbb{F}_{\mathfrak{b}}(Y)} G_{ab}(z,z',\chi)E^*_{\mathfrak{a}}(z',s,\chi)\,d\mu z' &= \int_{\mathbb{F}_{\mathfrak{b}}(Y)} G^Y_{ab}(z,z',\chi)E^*_{\mathfrak{a}}(z',s,\chi)\,d\mu z'\\ &+(2a-1)^{-1}E_{\mathfrak{b}}(z,a,\chi)\int_{\mathbb{F}_{\mathfrak{b}}(Y)}\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}z')^{1-a}E^*_{\mathfrak{a}}(z',s,\chi)\,d\mu z'\\ &-(2b-1)^{-1}E_{\mathfrak{b}}(z,b,\chi)\int_{\mathbb{F}_{\mathfrak{b}}(Y)}\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}z')^{1-b}E^*_{\mathfrak{a}}(z',s,\chi)\,d\mu z'.\end{aligned}$$

Now

$$\begin{aligned}\int_{\mathbb{F}_{\mathfrak{b}}(Y)}\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}z)^{1-a}E^*_{\mathfrak{a}}(z,s,\chi)\,d\mu z &= \int_0^1\int_Y^\infty \mathrm{Im}(z)^{1-a}E^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s,\chi)\,d\mu z\\ &= \int_0^1\int_Y^\infty y^{1-a}(\phi^*_{\mathfrak{a}\mathfrak{b}}(s,\chi)y^{1-s}+\cdots)\frac{dxdy}{y^2}\\ &=\phi^*_{\mathfrak{a}\mathfrak{b}}(s,\chi)\int_Y^\infty y^{-a-s}\,dy\\ &=\phi^*_{\mathfrak{a}\mathfrak{b}}(s,\chi)\frac{Y^{1-a-s}}{a+s-1}.\end{aligned}$$

So

$$\begin{aligned}-\nu_{ab}(s)E^*_{\mathfrak{a}}(z,s,\chi) = \int_{\mathbb{F}} & G^Y_{ab}(z,z',\chi)E^*_{\mathfrak{a}}(z',s,\chi)\,d\mu z'\\ &+\frac{Y^{1-a-s}}{(2a-1)(a+s-1)}\sum_{\mathfrak{b}\in\text{cusps}}\phi^*_{\mathfrak{a}\mathfrak{b}}(s,\chi)E_{\mathfrak{b}}(z,a,\chi)\\ &-\frac{Y^{1-b-s}}{(2b-1)(b+s-1)}\sum_{\mathfrak{b}\in\text{cusps}}\phi^*_{\mathfrak{a}\mathfrak{b}}(s,\chi)E_{\mathfrak{b}}(z,b,\chi)\\ &+\int G^*_{ab}(z,z',\chi)E_{\mathfrak{a}}(z',s,\chi)\,d\mu z',\end{aligned}$$

and we'll denote the right-hand side of the last equality $R(Y)$. Next, to remove the appearances of $\phi^*_{\mathfrak{ab}}(s)$, we take a linear combination of the last equation with Y replaced by Y, $2Y$ and $4Y$. To be precise we use

$$\begin{aligned} R(Y) - 2^{s-1}(2^a{+}2^b)R(2Y) + 2^{2s-2+a+b}R(4Y) \\ &= (2^{s-1+a}-1)(2^{s-1+b}-1)\big(-\nu_{ab}(s)E^*_{\mathfrak{a}}(z,s,\chi)\big) \\ &= \int_{\mathbb{F}} G(z,z',\chi)E^*_{\mathfrak{a}}(z',s,\chi)\,d\mu z' \\ &\quad + (2^{s-1+a}-1)(2^{s-1+b}-1)\int_{\mathbb{F}} G^*_{ab}(z,z',\chi)E_{\mathfrak{a}}(z',s,\chi)\,d\mu z' \end{aligned}$$

where $G(z,z',\chi) = \left(G^Y_{ab} - 2^{s-1}(2^a+2^b)G^{2Y}_{ab} + 2^{2s-2+a+b}G^{4Y}_{ab}\right)(z,z',\chi)$ and Y is a fixed large number. This implies that

$$\begin{aligned} (2^{s-1+a}-1)(2^{s-1+b}-1)\nu_{ab}(s)E^*_{\mathfrak{a}}(z,s,\chi) \\ &= \frac{-1}{\nu_{ab}(s)}\int_{\mathbb{F}} \frac{G(z,z',\chi)}{(2^{s-1+a}-1)(2^{s-1+b}-1)} \\ &\qquad \times (2^{s-1+a}-1)(2^{s-1+b}-1)\nu_{ab}(s)E^*_{\mathfrak{a}}(z',s,\chi)\,d\mu z' \\ &\qquad + (2^{s-1+a}-1)(2^{s-1+b}-1)\int_{\mathbb{F}} G^*_{ab}(z,z',\chi)E_{\mathfrak{a}}(z',s,\chi)\,d\mu z'. \end{aligned}$$

This can be rewritten neatly as

$$h(z,s) = f(z,s) + \lambda(s)\int_{\mathbb{F}} H_s(z,z')h(z',s)\,d\mu z' \tag{3.5}$$

when $3 < \mathrm{Re}(s)+1 < b < a$ with, (suppressing dependence on χ, a, b)

$$\begin{aligned} h(z,s) &= (2^{s-1+a}-1)(2^{s-1+b}-1)\nu_{ab}(s)E^*_{\mathfrak{a}}(z,s,\chi), \\ \lambda(s) &= \frac{-1}{\nu_{ab}(s)} = \frac{(a-s)(a+s-1)(b-s)(b+s-1)}{(b-a)(a+b-1)}, \\ H_s(z,z') &= \frac{G(z,z',\chi)}{(2^{s-1+a}-1)(2^{s-1+b}-1)}, \\ f(z,s) &= (2^{s-1+a}-1)(2^{s-1+b}-1)\int_{\mathbb{F}} G^*_{ab}(z,z',\chi)E_{\mathfrak{a}}(z',s,\chi)\,d\mu z'. \end{aligned}$$

Compare (3.5) with equation (3.1). To control the growth of H in the z variable we use Selberg's trick. Multiply through by $\eta(z) = e^{-\eta y(z)}$ with $0 < \eta < 2\pi$ where $y(z)$ is the invariant height of z. It is not hard to see that $\eta(z)$ is continuous. Our restriction on η ensures that the term $\left(\eta(z)\eta(z')^{-1}H_s(z,z')\right)$ appearing in (3.6) below is bounded, (see inequality (3.4)).

Define

$$\mathbb{B}_r = \{s \in \mathbb{C} : |s| \le r\}.$$

Then Theorem 1.3 tells us that $E_{\mathfrak{a}}(z,s,\chi)$ is meromorphic for $s \in \mathbb{C}$ and can be written

$$E_{\mathfrak{a}}(z,s,\chi) = \frac{A_{\mathfrak{a}}(z,s,\chi)}{A(s,\chi)}$$

with both $A_{\mathfrak{a}}(z,s,\chi)$ and $A_{\mathfrak{a}}(s,\chi)$ analytic for s in some ball, $\mathbb{B}_c$. Thus, multiplying (3.5) through by $A_{\mathfrak{a}}(s,\chi)$ should eliminate the poles of $f(z,s)$ for s inside this ball.

Set $\tilde{f}(z,s) = A_{\mathfrak{a}}(s,\chi)f(z,s)$ and $\tilde{h}(z,s) = A_{\mathfrak{a}}(s,\chi)h(z,s)$. Choose $c < b-1$, then for $2 < \mathrm{Re}(s)$ and $s \in \mathbb{B}_c$.

$$\eta(z)\tilde{h}(z,s) = \eta(z)\tilde{f}(z,s) + \lambda \int_{\mathbb{F}} \big(\eta(z)\eta(z')^{-1}H_s(z,z')\big)\big(\eta(z')\tilde{h}(z',s)\big)\, d\mu z'. \tag{3.6}$$

Our plan of attack is to use Fredholm's Theorem to express $\tilde{h}$ as an integral defined on the entire disc $\mathbb{B}_c$ and then to show this integral is in fact meromorphic. Increasing the size of the disc will give the continuation to the whole plane.

Proposition 3.2. *The function $\tilde{f}(z,s)$ is bounded for $(z,s) \in \mathbb{F}\times\mathbb{B}_c$ and continuous.*

This will require several lemmas. To begin with we need to study G^*_{ab} by computing its Fourier expansion. Let $c(\mathfrak{a},\mathfrak{b}) = \min\{C_{\mathfrak{a}\mathfrak{b}}\}$ for $C_{\mathfrak{a}\mathfrak{b}}$ as defined in section 1.

Lemma 3.3. *If $\delta > c(\mathfrak{a},\mathfrak{b})^{-2}$, $yy' > \delta$ and $\epsilon > 0$ then*

$$G^*_{ab}(\sigma_{\mathfrak{a}}z, \sigma_{\mathfrak{b}}z',\chi) \ll (yy')^{1-b} \qquad \text{for} \;\; 2+\epsilon \le b < a$$

with the implied constant depending on δ, ϵ and b, (along with Γ, f).

Proof: By employing techniques described in [Iw], Chapter 5 we find that for $y' > y$ with $yy' > \delta > c(\mathfrak{a},\mathfrak{b})^{-2}$ and $\sigma = \mathrm{Re}(s) > 2$

$$\begin{aligned}(2s-1)G^*_s(\sigma_{\mathfrak{a}}z,\sigma_{\mathfrak{b}}z',\chi) = {} & \phi^*_{\mathfrak{a}\mathfrak{b}}(s,\chi)(yy')^{1-s} \\ & + y^{1-s}\sum_{m\neq 0}\phi^*_{\mathfrak{a}\mathfrak{b}}(m,s,\chi)W_s(mz') \\ & + (y')^{1-s}\sum_{n\neq 0}\psi^*_{\mathfrak{a}\mathfrak{b}}(n,s,\chi)\overline{W}_{\overline{s}}(nz) \\ & + (2s-1)\sum_{mn\neq 0} Z^*_s(m,n,\chi)W_s(mz')\overline{W}_{\overline{s}}(nz),\end{aligned} \tag{3.7}$$

where $\phi^*_{\mathfrak{a}\mathfrak{b}}(m,s,\chi)$ is defined by (1.3) and similarly

$$\psi^*_{\mathfrak{a}\mathfrak{b}}(n,s,\chi) = \frac{\pi^s}{\Gamma(s)}|n|^{s-1}\sum_{c\in C_{\mathfrak{a}\mathfrak{b}}}\frac{S^*_{\mathfrak{a}\mathfrak{b}}(0,n,\chi,f;c)}{c^{2s}}.$$

Also

$$2\sqrt{|mn|}Z^*_s(m,n,\chi) = \sum_{c\in C_{\mathfrak{a}\mathfrak{b}}} c^{-1}S^*_{\mathfrak{a}\mathfrak{b}}(m,n,\chi,f;c)\cdot\begin{cases} J_{2s-1}\left(\dfrac{4\pi}{c}\sqrt{mn}\right) & mn>0 \\ I_{2s-1}\left(\dfrac{4\pi}{c}\sqrt{|mn|}\right) & mn<0\end{cases}$$

where $I_{2s-1}(y)$ and $J_{2s-1}(y)$ are the standard Bessel functions. We must show that the first term on the right of (3.7) dominates.

Choose a constant C such that for all $y \ge 0$ we have $(s=\sigma+it)$,

$$|I_{2s-1}(y)|,\ |J_{2s-1}(y)| \le Cy^{2\sigma-1}e^y.$$

We also require

Lemma 3.4.

$$\sum_{c\in C_{\mathfrak{a}\mathfrak{b}}} c^{-2s} S^*_{\mathfrak{a}\mathfrak{b}}(m,n,\chi,f;c) = O(1) \quad \textit{for} \ \ \sigma \geq 2+\epsilon,$$

the implied constant depending on ϵ, f *and* Γ.

Proof: Consider the series

$$E^+_{\mathfrak{a}}(z,s;f) = \sum_{\gamma\in\Gamma_{\mathfrak{a}}\backslash\Gamma} |\langle\, \gamma, f\,\rangle| \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)^s.$$

This series is absolutely convergent for $\mathrm{Re}(s) > 2$ giving a holomorphic function of s. $E^+_{\mathfrak{a}}(z+1,s;f) = E^+_{\mathfrak{a}}(z,s;f)$ so we can develop its Fourier expansion. We are interested in the constant term:

$$\int_0^1 E^+_{\mathfrak{a}}(x+iy,s;f)\,dx = \sqrt{\pi}y^{1-s}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} \sum_{c\in C_{\mathfrak{a}\mathfrak{b}}} \frac{S^+_{\mathfrak{a}\mathfrak{b}}(0,0,f;c)}{c^{2s}},$$

where

$$S^+_{\mathfrak{a}\mathfrak{b}}(0,0,f;c) = \sum_{\substack{\gamma\in\Gamma_\infty\backslash\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{b}}/\Gamma_\infty \\ \gamma_c = c}} |\langle\, \sigma_{\mathfrak{a}}\gamma\sigma_{\mathfrak{b}}^{-1}, f\,\rangle|.$$

Thus, for $\sigma \geq 2+\epsilon$

$$\sum_{c\in C_{\mathfrak{a}\mathfrak{b}}} \frac{|S^*_{\mathfrak{a}\mathfrak{b}}(m,n,\chi,f;c)|}{c^{2\sigma}} \leq \frac{\Gamma(2+\epsilon)}{\Gamma(2+\epsilon-\frac{1}{2})\sqrt{\pi}} \int_0^1 E^+_{\mathfrak{a}}(x+i,2+\epsilon;f)\,dx,$$

because the left side is a decreasing function of σ. ■

Now we can see that

$$\begin{aligned} \phi^*_{\mathfrak{a}\mathfrak{b}}(m,s,\chi) &\ll |m|^s, \\ \psi^*_{\mathfrak{a}\mathfrak{b}}(n,s,\chi) &\ll |n|^s \\ \text{and} \quad Z^*_s(m,n,\chi) &\ll e^{\frac{4\pi}{c(\mathfrak{a},\mathfrak{b})}\sqrt{|mn|}}. \end{aligned}$$

With the additional estimate $W_s(mz) \ll e^{-2\pi my}$, we have

$$\begin{aligned} &\sum_{m\neq 0} \phi^*_{\mathfrak{a}\mathfrak{b}}(m,s,\chi)W_s(mz') \ll \ e^{-2\pi y'}, \\ &\sum_{n\neq 0} \psi^*_{\mathfrak{a}\mathfrak{b}}(n,s,\chi)\overline{W}_{\overline{s}}(nz) \ll \ e^{-2\pi y}, \\ &\sum_{mn\neq 0} Z^*_s(m,n,\chi)W_s(mz')\overline{W}_{\overline{s}}(nz) \ll \ e^{-2\pi(y+y')}. \end{aligned}$$

Combining our estimates proves that, for $\mathrm{Re}(s) > 2$,

$$G^*_s(\sigma_{\mathfrak{a}}z, \sigma_{\mathfrak{b}}z', \chi) \ll (yy')^{1-s}$$

when $y' > y$ and $yy' > \delta$. For G^*_{ab} we can remove the restriction $y' > y$ by using the fact that $G^*_s(w,w',\chi) = -G^*_s(w',w,\overline{\chi})$ and noting that G^*_{ab} is continuous at $y = y'$. This completes the proof of Lemma 3.3. ■

Lemma 3.5. *The product*

$$G^*_{ab}(w, w', \chi) A_{\mathfrak{a}}(s, \chi) E_{\mathfrak{a}}(w', s, \chi)$$

is bounded for $(w, w', s) \in \mathbb{F} \times \mathbb{F} \times \mathbb{B}_c$ *when* $3 < c + 1 < b < a$.

Proof: First define

$$y_\Gamma = \min_{z \in \mathbb{H}} y(z),$$

where $y(z)$ is the invariant height of z, see (1.0). Clearly $y_\Gamma > 0$ and we can put $T = \delta / y_\Gamma$, (the same δ as in Lemma 3.3). There are three cases:

Case 1: $(w, w', s) \in \mathbb{F}(T) \times \mathbb{F}(T) \times \mathbb{B}_c$.
$G^*_{ab}(w, w', \chi) A_{\mathfrak{a}}(s, \chi) E_{\mathfrak{a}}(w', s, \chi)$ is bounded on this set since the set is compact and this function is continuous on it.

Case 2: $(w, w', s) \in \mathbb{F}_{\mathfrak{b}}(T) \times \mathbb{F} \times \mathbb{B}_c$.
Let $\sigma_{\mathfrak{b}} z = w$. We have that $\mathrm{Im}(z) \geq \delta / y_\Gamma$. For each $w' \in \mathbb{F}$ we can choose a cusp τ such that $\sigma_\tau z' = w'$ and $\mathrm{Im}(z') \geq y_\Gamma$. By Lemma 3.3, $G^*_{ab}(\sigma_{\mathfrak{b}} z, \sigma_\tau z', \chi) \ll (yy')^{1-b}$. As $\sigma_{\mathfrak{b}} z'$ approaches the cusp $\mathfrak{b}$, $E_{\mathfrak{a}}(\sigma_{\mathfrak{b}} z', s, \chi) \approx \delta_{\mathfrak{a}\tau}(y')^s + \phi_{\mathfrak{a}\tau}(s, \chi)(y')^{1-s}$ and the product is bounded. Also $A_{\mathfrak{a}}(s, \chi)$ cancels any poles of $E_{\mathfrak{a}}$ so we're done.

Case 3: $(w, w', s) \in \mathbb{F}(T) \times \mathbb{F}_{\mathfrak{b}}(T) \times \mathbb{B}_c$.
Similar proof to case 2. ■

Therefore $\tilde{f}(z, s)$ is bounded on $\mathbb{F} \times \mathbb{B}_c$ since the volume of $\mathbb{F}$ is finite. Lastly Lebesgue's theorem on dominated convergence can be applied to establish continuity. This completes the proof of Proposition 3.2. ■

We are now in a position to apply Fredholm's theorem to (3.6). The result is:

$$\eta(z)\tilde{h}(z, s) = \eta(z)\tilde{f}(z, s) + \frac{\lambda}{D(\lambda)} \int_{\mathbb{F}} D_\lambda(z, z')\eta(z')\tilde{f}(z', s)\, d\mu z',$$

thus

$$\tilde{h}(z, s) = \tilde{f}(z, s) + \frac{\lambda}{D(\lambda)} \int_{\mathbb{F}} \eta(z)^{-1}\eta(z') D_\lambda(z, z')\tilde{f}(z', s)\, d\mu z' \tag{3.9}$$

for each $s \in \mathbb{B}_c$ such that $D(\lambda) \neq 0$. Recall that $\tilde{h}(z, s)$ is just the product of a meromorphic function and $E^*_{\mathfrak{a}}(z, s, \chi)$. Our final task is to demonstrate that the right-hand side of (3.9) is meromorphic in s. We need a proposition giving criteria for the analyticity of the types of integrals we're encountering. The next result is based on [G], Theorem 8-1-5.

Proposition 3.6. *Let* $\mathbb{F}$ *be a region in* $\mathbb{C}$ *with* $\int_{\mathbb{F}} d\mu z = V < \infty$. *Let* r *and* R *be the radii of closed balls* $\mathbb{B}_r$ *and* $\mathbb{B}_R$ *with* $r < R$. *Assume* $\phi : \mathbb{F} \times \mathbb{B}_R \to \mathbb{C}$ *has the following properties:*

(i) *for fixed* s, $\phi(z, s)$ *is a continuous function of* z *(except perhaps for* z *in a set of measure zero),*
(ii) *for fixed* z, $\phi(z, s)$ *is analytic in* s,
(iii) ϕ *is bounded on* $\mathbb{F} \times \mathbb{B}_R$

Then $\psi(s) = \int_{\mathbb{F}} \phi(z,s)\, d\mu z$ is analytic on $\mathbb{B}_r$ and $\frac{d}{ds}\psi(s) = \int_{\mathbb{F}} \phi_s(z,s)\, d\mu z$ where $\phi_s(z,s) = \frac{d}{ds}\phi(z,s)$.

Remark. The proposition remains true if the condition $V < \infty$ is dropped.

Corollary 3.7. *For each z, $\tilde{f}(z,s)$ is analytic for $s \in \mathbb{B}_c$.*

Proof: Let $\phi(z,s) = G^*_{ab}(w,z,\chi) A_{\mathfrak{a}}(s,\chi) E_{\mathfrak{a}}(z,s,\chi)$. Then G^*_{ab} is continuous in z by Prop. 2.8 and $E_{\mathfrak{a}}$ is continuous in z by Theorem 1.3. So condition (i) holds. We know that $A_{\mathfrak{a}}(s,\chi)E_{\mathfrak{a}}(z,s,\chi) = A_{\mathfrak{a}}(z,s,\chi)$ on $\mathbb{B}_c$ with $A_{\mathfrak{a}}(z,s,\chi)$ analytic in s, (Theorem 1.3), so condition (ii) holds. Finally condition (iii) is Lemma 3.5. ■

It remains to prove that $\int_{\mathbb{F}} \eta(w)^{-1}\eta(z) D_\lambda(w,z) \tilde{f}(z,s)\, d\mu z$ is analytic in s. Since $\eta(z)$ and $\tilde{f}(z,s)$ are both continuous in z, analytic in s and bounded, to apply Prop. 3.6 we need only check that $D_\lambda(w,z)$ has these properties.

$D(\lambda)$ and $D_\lambda(w,z)$ are constructed from the kernel

$$K_s(w,z) = \eta(w)\eta(z)^{-1} H_s(w,z)$$

occuring in equation (3.6) as follows. Define

$$K_s \begin{pmatrix} w_1, \dots, w_n \\ z_1, \dots, z_n \end{pmatrix} = \det\bigl(K_s(w_i, z_j)\bigr),$$

for $1 \le i,j \le n$ and put

$$\begin{aligned} n!\, d_n(s) &= \int \cdots \int K_s \begin{pmatrix} z_1, \dots, z_n \\ z_1, \dots, z_n \end{pmatrix} d\mu z_1 \cdots d\mu z_n, \\ n!\, d_n(w,z,s) &= \int \cdots \int K_s \begin{pmatrix} w, z_1, \dots, z_n \\ z, z_1, \dots, z_n \end{pmatrix} d\mu z_1 \cdots d\mu z_n \end{aligned}$$

where the multiple integrals are over $\mathbb{F}$ and let

$$\begin{aligned} D(\lambda) =& 1 + \sum_{n=1}^{\infty} d_n(s)(-\lambda)^n, \\ D_\lambda(w,z) =& K_s(w,z) + \sum_{n=1}^{\infty} d_n(w,z,s)(-\lambda)^n. \end{aligned}$$

Recall that $K_s(w,z)$ is bounded by K, say, for $(w,z,s) \in \mathbb{F} \times \mathbb{F} \times \mathbb{B}_c$. We use Hadamard's inequality

$$|\det(a_{ij})|^2 \le \prod_i \left(\sum_j |a_{ij}|^2 \right),$$

to bound the coefficients

$$\left| K_s \begin{pmatrix} w_1, \dots, w_n \\ z_1, \dots, z_n \end{pmatrix} \right| \le \left(\sqrt{n} K\right)^n,$$

implying that $n!\,|d_n(s)| \leq \left(\sqrt{n}KV\right)^n$ (where again V is the volume of $\mathbb{F}$). Thus $D(\lambda)$ is analytic in s and $D(\lambda) \ll \exp(3|\lambda|KV)^2$. Similarly we can see that

$$n!\,|d_n(w,z,s)| \leq \left(\sqrt{n+1}K\right)^{n+1}V^n.$$

This inequality shows that the series converges uniformly and is bounded. Each term in the series is analytic in s for $s \in \mathbb{B}_c$ and continuous in z except on the set

$$\bigcup_{\mathfrak{b} \in \text{cusps}} \{\sigma_{\mathfrak{b}}(x+iY), \sigma_{\mathfrak{b}}(x+i2Y), \sigma_{\mathfrak{b}}(x+i4Y) : 0 \leq x \leq 1\}$$

where we truncated G_{ab} in (3.3). Therefore $D(\lambda)\tilde{h}(z,s)$, by the last proposition, is analytic on $\mathbb{B}_c$ for any fixed z.

Theorem 3.8. *Let Γ be a Fuchsian group of the first kind with cusps. For each cusp $\mathfrak{a}$ the Eisenstein series $E^*_{\mathfrak{a}}(z,s)$ has a meromorphic continuation to the entire s-plane. More precisely for any $c > 0$ with $c+1 < b < a$ there exist non-zero functions $A^*_{\mathfrak{a}}(z,s,\chi) : \mathbb{F} \times \mathbb{B}_c \to \mathbb{C}$ and $A^*_{\mathfrak{a}}(s,\chi) : \mathbb{B}_c \to \mathbb{C}$ such that the following hold:*

(i) *$A^*_{\mathfrak{a}}(s,\chi)$ and $A^*_{\mathfrak{a}}(z,s,\chi)$ are analytic in s of order ≤ 8,*
(ii) *$A^*_{\mathfrak{a}}(s,\chi)E^*_{\mathfrak{a}}(z,s,\chi) = A^*_{\mathfrak{a}}(z,s,\chi)$ for $s \in \mathbb{B}_c$ and $2 < \mathrm{Re}(s)$,*
(iii) *$(\triangle_z + s(1-s))A^*_{\mathfrak{a}}(z,s,\chi) = 0$,*
(iv) *$A^*_{\mathfrak{a}}(\gamma z,s,\chi) = \overline{\chi}(\gamma)\left[A^*_{\mathfrak{a}}(z,s,\chi) - \langle \gamma, f\rangle E_{\mathfrak{a}}(z,s,\chi)\right]$ for all $\gamma \in \Gamma$,*
(v) *$A^*_{\mathfrak{a}}(z,s,\chi)$ is real analytic in (z,s),*
(vi) *$A^*_{\mathfrak{a}}(z,s,\chi) \ll e^{\eta y(z)}$ for any $\eta > 0$,*

where the implied constant depends on η, s and of course Γ and f.

Proof: We set

$$A^*_{\mathfrak{a}}(s,\chi) = (2^{s-1+a}-1)(2^{s-1+b}-1)D(\lambda)A_{\mathfrak{a}}(s,\chi)$$
$$\text{and} \qquad A^*_{\mathfrak{a}}(z,s,\chi) = -\lambda D(\lambda)\tilde{h}(z,s)$$

where $\tilde{h}(z,s)$ is given by (3.9). Parts (i) and (ii) have been shown. Parts (iii) and (iv) are true by analytic continuation. Part (vi) can be seen by noting that, for a fixed s, the only unbounded term occuring in $A^*_{\mathfrak{a}}(z,s,\chi)$ is $\eta(z)^{-1}$. Part (v) is harder to prove. It follows from the theory of partial differential equations, in particular the theorem on the analyticity of the solutions of an elliptic operator with dependence on a parameter. To see that $A^*_{\mathfrak{a}}(z,s,\chi)$ is at least continuous in (z,s), use equation (3.9), Proposition 3.2 and that $K_s(z,z')$ is continuous in (z,s).

Finally, by increasing c, the continuation can be extended to the entire complex s-plane. ■

We can deduce a number of results from Theorem 3.8:

- The Fourier coefficients $\phi^*_{\mathfrak{ab}}(s,\chi)$ and $\phi^*_{\mathfrak{ab}}(n,s,\chi)$ can be meromorphically continued to all of $\mathbb{C}$.
- The Fourier expansion (1) is valid for all s with $A^*_{\mathfrak{a}}(s,\chi) \neq 0$.
- We have the bound

$$E^*_{\mathfrak{a}}(\sigma_{\mathfrak{b}}z,s,\chi) = \phi^*_{\mathfrak{ab}}(s,\chi)y^{1-s} + O(e^{-2\pi y})$$

as $y \to \infty$ when $A^*_{\mathfrak{a}}(s,\chi) \neq 0$, the implied constant depending on s.

4. The Functional Equation

Define $\mathcal{E}(z,s,\chi)$ to be the column vector $\big(E_{\mathfrak{a}}(z,s,\chi)\big)$ as $\mathfrak{a}$ varies over all inequivalent cusps. Similarly $\mathcal{E}^*(z,s,\chi) = \big(E^*_{\mathfrak{a}}(z,s,\chi)\big)$ and we have the 'scattering' matrices

$$\Phi(s,\chi) = \big(\phi_{\mathfrak{ab}}(s,\chi)\big), \quad \Phi^*(s,\chi) = \big(\phi^*_{\mathfrak{ab}}(s,\chi)\big).$$

Two more pieces of notation: define $\mathcal{A}(\Gamma\backslash\mathbb{H},\chi)$ to be the set of all functions f on $\mathbb{H}$ satisfying $f(\gamma z) = \chi(\gamma)f(z)$ for all $\gamma \in \Gamma$. The set of such f that also satisfy

$$\big(\triangle + s(1-s)\big)f = 0$$

we'll call $\mathcal{A}_s(\Gamma\backslash\mathbb{H},\chi)$. Now, assuming the meromorphic continuation of $\mathcal{E}$ and $\mathcal{E}^*$ to the entire s-plane, we can show the following

Theorem 4.1. *$\mathcal{E}^*(z,s,\chi)$ satisfies,*

$$\Phi(s,\chi)\mathcal{E}^*(z,1-s,\chi) = \mathcal{E}^*(z,s,\chi) - \Phi^*(s,\chi)\Phi(1-s,\chi)\mathcal{E}(z,s,\chi)$$

for all $s \in \mathbb{C}$ and

$$\Phi(s,\chi)\Phi^*(1-s,\chi) = -\Phi^*(s,\chi)\Phi(1-s,\chi).$$

Proof: For $\mathrm{Re}(s) > 2$ we know that

$$E^*_{\mathfrak{a}}(\gamma z,s,\chi) = \overline{\chi}(\gamma)\big(E^*_{\mathfrak{a}}(z',s,\chi) - <\gamma,f> E_{\mathfrak{a}}(z',s,\chi)\big)$$

and $(\triangle + s(1-s))E^*_{\mathfrak{a}}(z,s,\chi) = 0$, so by analytic continuation they remain valid for all $s \in \mathbb{C}$. Assume, for the remainder, that $\mathrm{Re}(s) > 2$. Consider

$$\mathcal{F}(z,s,\chi) = \mathcal{E}^*(z,s,\chi) - \Phi(s,\chi)\mathcal{E}^*(z,1-s,\chi).$$

We compute

$$\begin{aligned}
\mathcal{F}(\gamma z,s,\chi) &= \mathcal{E}^*(\gamma z,s,\chi) - \Phi(s,\chi)\mathcal{E}^*(\gamma z,1-s,\chi)\\
&= \overline{\chi}(\gamma)\,\Big[\big(\mathcal{E}^*(z,s,\chi) - <\gamma,f> \mathcal{E}(z,s,\chi)\big)\\
&\qquad -\Phi(s,\chi)\big(\mathcal{E}^*(z,1-s,\chi) - <\gamma,f> \mathcal{E}(z,1-s,\chi)\big)\Big]\\
&= \overline{\chi}(\gamma)\,[\mathcal{E}^*(z,s,\chi) - \Phi(s,\chi)\mathcal{E}^*(z,1-s,\chi)\\
&\qquad - <\gamma,f> \big(\mathcal{E}(z,s,\chi) - \Phi(s,\chi)\mathcal{E}(z,1-s,\chi)\big)]\\
&= \overline{\chi}(\gamma)\,[\mathcal{F}(z,s,\chi) - <\gamma,f> \cdot 0] \;=\; \overline{\chi}(\gamma)\mathcal{F}(z,s,\chi).
\end{aligned}$$

(We used the functional equation for $\mathcal{E}(z,s,\chi)$, [Se],Theorem 7.3). Therefore we've shown that

$$\mathcal{F}_{\mathfrak{a}}(z,s,\chi) = E^*_{\mathfrak{a}}(z,s,\chi) - \sum_{\mathfrak{b}} \phi_{\mathfrak{ab}}(s,\chi)E^*_{\mathfrak{b}}(z,1-s,\chi) \in \mathcal{A}(\Gamma\backslash\mathbb{H},\overline{\chi}).$$

Also

$$\big(\triangle + s(1-s)\big)\mathcal{F}_{\mathfrak{a}}(z,s,\chi) = 0 \quad \text{implies that} \quad \mathcal{F}_{\mathfrak{a}}(z,s,\chi) \in \mathcal{A}_s(\Gamma\backslash\mathbb{H},\overline{\chi}).$$

$\mathcal{F}_{\mathfrak{a}}(z,s,\chi) \ll e^{\epsilon y(z)}$ for $\epsilon > 0$ by Theorem 3.8 so, invoking [Se], Lemma 7.1, it must be a linear combination of Eisenstein series,

$$\mathcal{F}_{\mathfrak{a}}(z,s,\chi) = \sum_{\mathfrak{b}} \Upsilon_{\mathfrak{a}\mathfrak{b}}(s,\chi) E_{\mathfrak{a}}(z,s,\chi).$$

Hence

$$\begin{aligned}\left(\phi^*_{\mathfrak{a}\mathfrak{c}}(s,\chi)y^{1-s} + \cdots\right) - \sum_{\mathfrak{b}} \phi_{\mathfrak{a}\mathfrak{b}}(s,\chi)\left(\phi^*_{\mathfrak{b}\mathfrak{c}}(1-s,\chi)y^s + \cdots\right) \\ = \sum_{\mathfrak{b}} \Upsilon_{\mathfrak{a}\mathfrak{b}}(s,\chi)\left(\delta_{\mathfrak{b}\mathfrak{c}} y^s + \phi_{\mathfrak{b}\mathfrak{c}}(s,\chi)y^{1-s} + \cdots\right).\end{aligned}$$

Equating coefficients of y^s and y^{1-s} we get

$$-\sum_{\mathfrak{b}} \phi_{\mathfrak{a}\mathfrak{b}}(s,\chi)\phi^*_{\mathfrak{b}\mathfrak{c}}(1-s,\chi) = \sum_{\mathfrak{b}} \Upsilon_{\mathfrak{a}\mathfrak{b}}(s,\chi)\delta_{\mathfrak{b}\mathfrak{c}} = \Upsilon_{\mathfrak{a}\mathfrak{c}}(s,\chi)$$

$$\text{and} \quad \phi^*_{\mathfrak{a}\mathfrak{c}}(s,\chi) = \sum_{\mathfrak{b}} \Upsilon_{\mathfrak{a}\mathfrak{b}}(s,\chi)\phi_{\mathfrak{b}\mathfrak{c}}(s,\chi).$$

We know, (because $\mathcal{E}(z,s,\chi) = \Phi(s,\chi)\mathcal{E}(z,1-s,\chi)$), that

$$\Phi(s,\chi)\Phi(1-s,\chi) = I.$$

Consequently we arrive at the following formulas

$$\begin{aligned} -\Phi(s,\chi)\Phi^*(1-s,\chi) = \Upsilon(s,\chi) = \Phi^*(s,\chi)\Phi(1-s,\chi), \\ \mathcal{E}^*(z,s,\chi) - \Phi(s,\chi)\mathcal{E}^*(z,1-s,\chi) = \Upsilon(s,\chi)\mathcal{E}(z,s,\chi).\end{aligned}$$

Therefore

$$\Phi(s,\chi)\mathcal{E}^*(z,1-s,\chi) = \mathcal{E}^*(z,s,\chi) - \Phi^*(s,\chi)\Phi(1-s,\chi)\mathcal{E}(z,s,\chi),$$

or perhaps more simply,

$$\mathcal{E}^*(z,1-s,\chi) = \Phi(1-s,\chi)\mathcal{E}^*(z,s,\chi) + \Phi^*(1-s,\chi)\mathcal{E}(z,s,\chi).$$

These equalities are true for $\mathrm{Re}(s) > 2$ but extend to all $s \in \mathbb{C}$ by analytic continuation. ■

The scattering matrix $\Phi^*(s)$ plays an important role in the theory. We already know that $\Phi(s)$ is symmetric.

Proposition 4.2. *For Γ a Fuchsian group of the first kind, $\Gamma \subset PSL_2(\mathbb{R})$ or $-I \in \Gamma \subset SL_2(\mathbb{R})$, the scattering matrix $\Phi^*(s)$ with character $\chi = 1$ is skew symmetric.*

Proof: Recall that $C_{\mathfrak{a}\mathfrak{b}} = \left\{ c > 0 : \begin{pmatrix} * & * \\ c & * \end{pmatrix} \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} \right\}$. If $\gamma \in \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}}$ then $\gamma^{-1} \in \sigma_{\mathfrak{b}}^{-1} \Gamma \sigma_{\mathfrak{a}}$. Also note that $\gamma_c = \left(-I\,\gamma^{-1}\right)_c$ implying that $C_{\mathfrak{a}\mathfrak{b}} = C_{\mathfrak{b}\mathfrak{a}}$. Then, by a simple calculation,

$$\begin{aligned} S^*_{\mathfrak{b}\mathfrak{a}}(0,0;c) &= \sum_{\substack{\delta \in \Gamma_\infty \backslash \sigma_{\mathfrak{b}}^{-1} \Gamma \sigma_{\mathfrak{a}} / \Gamma_\infty \\ \delta_c = c}} \langle\, \sigma_{\mathfrak{b}} \delta \sigma_{\mathfrak{a}}^{-1}, f \,\rangle \\ &= \sum_{\substack{\delta \in \Gamma_\infty \backslash \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} / \Gamma_\infty \\ \delta_c = c}} \langle\, \sigma_{\mathfrak{b}} (-I\,\delta^{-1}) \sigma_{\mathfrak{a}}^{-1}, f \,\rangle \\ &= \sum_{\substack{\delta \in \Gamma_\infty \backslash \sigma_{\mathfrak{a}}^{-1} \Gamma \sigma_{\mathfrak{b}} / \Gamma_\infty \\ \delta_c = c}} -\langle\, \sigma_{\mathfrak{a}} \delta \sigma_{\mathfrak{b}}^{-1}, f \,\rangle \\ &= -\, S^*_{\mathfrak{a}\mathfrak{b}}(0,0;c). \end{aligned}$$

Thus, for $\mathrm{Re}(s) > 2$, $\Phi^*(s) = -{}^t\Phi^*(s)$. The Proposition then follows. ■

Corollary 4.3. *For Γ as above, $\phi^*_{\mathfrak{a}\mathfrak{a}}(s) = 0$, for each cusp $\mathfrak{a}$.*

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