L-FUNCTIONS OF SECOND-ORDER CUSP FORMS

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1. Introduction

The study of second-order modular forms has been initiated in connection with percolation theory ([KZ]) and Eisenstein series formed with modular symbols (cf. [CDO]). More recently, second-order modular forms have appeared in research on converse theorems.

Specifically, the pursuit of converse theorems for L-functions requiring the minimum number of twists possible has been a long-standing project of great interest. One of the approaches, due to B. Conrey and D. Farmer, has been successful in small levels (cf. [CF]). It transpires that, for the extension of this approach to higher levels, it is necessary to study a kind of second-order modular form that involves two groups. In particular, proving that, in some cases, there are no such functions (besides the usual modular forms) is enough to prove a converse theorem without twists for some levels (cf. [F]).

Motivated by this relation between such forms and converse theorems of L-functions and by the success of L-functions in the study of usual modular forms, in this paper we initiate a study of L-functions of second-order modular forms.

In Section 2, we first define and classify the holomorphic second-order modular forms. Although the structure of general second-order modular forms has already been determined in [CDO], a separate discussion is necessary here mainly because we require precise information about growth in the sequel. Moreover, for our investigations on converse theorems mentioned above, we are interested in holomorphic second-order modular forms that are not invariant under all parabolics.

In section 3 we see that the L-function of a second-order modular form satisfies the usual functional equation. We did not find a functional equation for the L-function of a second-order modular form with Fourier coefficients twisted by a Dirichlet character. Instead, we used the classification theorem to define two twisting operators which do yield a functional equation (theorem 11).

Given that we do not have a functional equation of the classical type, we should not expect a converse theorem for second order modular forms. Nevertheless, we managed

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to apply Razar's method to obtain a criterion for functions satisfying certain 4-term functional equations to be L-functions of second-order modular forms (theorem 14). The paper ends with a discussion of the effect of the periodicity on functions satisfying this criterion and on second-order modular forms in general.

We do believe that the twisted L-function of a second-order modular form should have a functional equation of the usual type. It seems likely that it will require two Dirichlet characters and be a 4-term functional equation similar to the one in theorem 14. In future work we hope to find it along with its converse theorem.

The eventual goal is to extend these results to more general cases and, especially, to cases related to the converse theorem. For this reason, we have tried to minimize the dependence of our proofs on the specific features of the functions in [CDO].

2. The space of holomorphic second-order cusp forms

The definitions of the holomorphic automorphic forms under study are now given. We try to make the conditions and definitions for these spaces as flexible as possible. Some of this material is standard (see for example [Og], [Iw]) but is included for comparison with the corresponding facts for second-order modular forms.

Let Γ be a Fuchsian group of the first kind with parabolic elements and of genus g. Fix a fundamental domain $\mathfrak{F}$ for $\Gamma\backslash\mathfrak{h}$. We assume its boundary is a polygon and label the inequivalent cusps with Gothic letters such as $\mathfrak{a}, \mathfrak{b}$. The corresponding scaling matrices $\sigma_\mathfrak{a}, \sigma_\mathfrak{b}$ in $\mathrm{SL}_2(\mathbb{R})$ map the neighborhood of each cusp to the upper part of the vertical strip of width one. This means that $\sigma_\mathfrak{a}^{-1}\Gamma_\mathfrak{a}\sigma_\mathfrak{a} = \Gamma_\infty$ for

$$\begin{aligned}\Gamma_\mathfrak{a} &= \{\gamma \in \Gamma \mid \gamma\mathfrak{a} = \mathfrak{a}\},\\ \Gamma_\infty &= \{\pm\left[\begin{smallmatrix}1 & n\\ 0 & 1\end{smallmatrix}\right] \mid n \in \mathbb{Z}\},\end{aligned}$$

where Γ_∞ is not necessarily in Γ and ∞ may not be a cusp of $\mathfrak{F}$.

For every even $k \in \mathbb{Z}$, we define an action of $\mathrm{GL}_2(\mathbb{R})^+$ on the space of functions on the upper-half plane $\mathfrak{h}$, setting

$$(f|_k\gamma)(z) := f(\gamma z)(cz + d)^{-k}(\det(\gamma))^{k/2}$$

for all $f : \mathfrak{h} \to \mathbb{C}$, $z = x + iy \in \mathfrak{h}$ and $\gamma = \left[\begin{smallmatrix}* & *\\ c & d\end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{R})^+$. We extend the action to $\mathbb{C}[\mathrm{GL}_2(\mathbb{R})^+]$ by linearity.

Definition of $S_k(\Gamma)$. Let k be a positive even integer. Define $S_k(\Gamma)$ to be the $\mathbb{C}$-vector space of functions f such that

A1. $f : \mathfrak{h} \to \mathbb{C}$ is holomorphic,
A2. $f|_k(\gamma - 1) = 0$ for all γ in Γ,
A3. ("vanishing at the cusps")$(f|_k\sigma_\mathfrak{a})(z) \ll e^{-cy}$ as $y \to \infty$ uniformly in x for a constant $c > 0$ and an implied constant both depending on Γ and $\mathfrak{F}$.

We call the elements of this space the holomorphic weight k cusp forms. The growth condition **A3** is simple and natural but it is often useful to know the growth of f in the entire upper half plane without refering to the cusps.

Proposition 1. *Suppose f is holomorphic on $\mathfrak{h}$ and that $y^r|f(z)| \ll 1$ for some $r \geqslant k/2$. If $(f|_k\sigma_{\mathfrak{a}})(z+1) = (f|_k\sigma_{\mathfrak{a}})(z)$ for some cusp $\mathfrak{a}$ then*

$$(f|_k\sigma_{\mathfrak{a}})(z) = \sum_{n=0}^{\infty} b_{\mathfrak{a}}(n)e(nz) \tag{1}$$

where $b_{\mathfrak{a}}(n) \ll n^r$ for $n \geqslant 1$. If $r < k$ then $b_{\mathfrak{a}}(0) = 0$.

Proof: If f is holomorphic then so is $f|_k\sigma_{\mathfrak{a}}$ and, if it is periodic, it must have the Fourier expansion (1) since any terms $e(nz)$ with $n < 0$ would violate $y^r|f(z)| \ll 1$. For $n \geqslant 1$ we have

$$\begin{aligned} b_{\mathfrak{a}}(n) &= e^{-2\pi} \int_0^1 (f|_k\sigma_{\mathfrak{a}})(x+i/n)e^{-2\pi inx}\, dx \\ &\ll n^{k/2} \int_0^1 \mathrm{Im}(\sigma_{\mathfrak{a}}(x+i/n))^{k/2} |f(\sigma_{\mathfrak{a}}(x+i/n))|\, dx \\ &\ll n^{k/2} \mathrm{Im}(\sigma_{\mathfrak{a}}(x+i/n))^{k/2-r}. \end{aligned}$$

Now

$$\mathrm{Im}\big(\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right)(x+i/n)\big)^{-1} = n((cx+d)^2 + c^2/n^2) \ll n$$

for $x \in [0,1]$ and the implied constant depending on c, d. Hence $b_{\mathfrak{a}}(n) \ll n^r$.

Also

$$\begin{aligned} b_{\mathfrak{a}}(0) &= \int_0^1 (f|_k\sigma_{\mathfrak{a}})(x+iy)\, dx \\ &\leqslant \int_0^1 y^{-k/2} \mathrm{Im}(\sigma_{\mathfrak{a}}(x+iy))^{k/2} |f(\sigma_{\mathfrak{a}}(x+iy))|\, dx. \end{aligned}$$

As $y \to \infty$ we have $1/y \asymp \mathrm{Im}(\sigma_{\mathfrak{a}}(x+iy))$ if $\sigma_{\mathfrak{a}}$ is not upper triangular. Therefore, as $y \to \infty$,

$$\begin{aligned} b_{\mathfrak{a}}(0) &\ll \int_0^1 \mathrm{Im}(\sigma_{\mathfrak{a}}(x+iy))^{k} |f(\sigma_{\mathfrak{a}}(x+iy))|\, dx \\ &\ll \int_0^1 y^{r-k} \mathrm{Im}(\sigma_{\mathfrak{a}}(x+iy))^{r} |f(\sigma_{\mathfrak{a}}(x+iy))|\, dx \\ &\ll y^{r-k}. \end{aligned}$$

If $\sigma_{\mathfrak{a}}$ is upper triangular then $y \asymp \mathrm{Im}(\sigma_{\mathfrak{a}}(x+iy))$ as $y \to \infty$ and

$$\begin{aligned} b_{\mathfrak{a}}(0) &\ll \int_0^1 |f(\sigma_{\mathfrak{a}}(x+iy))|\, dx \\ &\ll y^{-r}. \end{aligned}$$

Either way $b_{\mathfrak{a}}(0) = 0$, completing the proof. $\square$

We may replace **A3** by

A3.1. $y^{k/2}|f(z)| \ll 1$ for all z in $\mathfrak{h}$.

Lemma 2. *We have $f \in S_k(\Gamma)$ if and only if f satisfies* **A1**, **A2** *and* **A3.1**.

Proof: Any f in $S_k(\Gamma)$ has exponential decay at each cusp and hence $y^{k/2}|f(z)|$ is bounded on $\mathfrak{F}$. Therefore it is bounded on all of $\mathfrak{h}$ since $y^{k/2}|f(z)|$ has weight 0.

In the other direction, suppose f satisfies **A1**, **A2** and **A3.1**. By proposition 1, f has the Fourier expansion (1) at any cusp $\mathfrak{a}$ with Fourier coefficients $b_{\mathfrak{a}}(n) \ll n^{k/2}$ and $b_{\mathfrak{a}}(0) = 0$. Therefore f has exponentional decay at any cusp. $\square$

In the proof we showed Hardy's 'trivial' bound of $n^{k/2}$ for the nth Fourier coefficient of f in $S_k(\Gamma)$.

Definition of $S_k^2(\Gamma)$. We define the space $S_k^2(\Gamma)$ to consist of functions f such that

- **B1.** $f : \mathfrak{h} \to \mathbb{C}$ is holomorphic,
- **B2.** $f|_k(\gamma - 1) \in S_k(\Gamma)$ for all γ in Γ,
- **B3.** ("vanishing at the cusps") $(f|_k\sigma_{\mathfrak{a}})(z) \ll e^{-cy}$ as $y \to \infty$ uniformly in x for a constant $c > 0$ and an implied constant both depending on Γ and $\mathfrak{F}$,
- **B4.** $f|_k(\pi - 1) = 0$ for all parabolic π in Γ.

This is the space of holomorphic, weight k, (parabolic) second-order cusp forms. It is similar to $S_k(\Gamma)$, the only difference being the transformation rule **B2**.

We recall the definitions of the functions Λ_i in [CDO]. For each of the $2g$ hyperbolic generators of Γ (labeled γ_i) we may define

$$\Lambda_i(z) = \int_{z_0}^{z} g_i(w)\, dw + \overline{\int_{z_0}^{z} h_i(w)\, dw}$$

for g_i and h_i in $S_2(\Gamma)$ and z_0 an arbitrary fixed element of $\mathfrak{h}$ (usually taken to be the imaginary number i) to satisfy

$$\Lambda_i(\gamma_j z) - \Lambda_i(z) = \delta_{ij}$$

and also

$$\Lambda_i(\gamma z) - \Lambda_i(z) = 0$$

for the parabolic and elliptic generators γ of Γ. For convenience we also set $\Lambda_0 \equiv 1$.

We need the following result.

Lemma 3. *For $1 \leqslant i \leqslant 2g$, all z in $\mathfrak{h}$, all $\gamma \in \Gamma$ and any cusp $\mathfrak{a}$ we have*

$$\Lambda_i(z) \ll |\log \mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)| + 1.$$

with the implied constant depending on γ, Γ and $\mathfrak{F}$.

Proof: We have

$$\begin{aligned}\Lambda_i(\gamma^{-1}\sigma_{\mathfrak{a}} z) &= \int_{z_0}^{\gamma^{-1}\sigma_{\mathfrak{a}} z} g_i(w)\,dw + \overline{\int_{z_0}^{\gamma^{-1}\sigma_{\mathfrak{a}} z} h_i(w)\,dw} \\ &= \int_{\sigma_{\mathfrak{a}}^{-1}\gamma z_0}^{z} (g_i|_2\sigma_{\mathfrak{a}})(w)\,dw + \overline{\int_{\sigma_{\mathfrak{a}}^{-1}\gamma z_0}^{z} (h_i|_2\sigma_{\mathfrak{a}})(w)\,dw}.\end{aligned}$$

Note that $g_i|_2\sigma_{\mathfrak{a}}$ and $h_i|_2\sigma_{\mathfrak{a}}$ are elements of $S_2(\sigma_{\mathfrak{a}}^{-1}\Gamma\sigma_{\mathfrak{a}})$ and by **A3.1** satisfy $(g_i|_2\sigma_{\mathfrak{a}})(w)$, $(h_i|_2\sigma_{\mathfrak{a}})(w) \ll \mathrm{Im}(w)^{-1}$ for all w in $\mathfrak{h}$. Also, by the Fourier expansion (1),we have

$$\int_{z}^{z+1} (g_i|_2\sigma_{\mathfrak{a}})(w)\,dw = 0$$

and the same for $h_i|_2\sigma_{\mathfrak{a}}$ so that

$$\Lambda_i(\gamma^{-1}\sigma_{\mathfrak{a}} z) \ll \int_{\sigma_{\mathfrak{a}}^{-1}\gamma z_0}^{iy} \mathrm{Im}(w)^{-1}\,dw \ll |\log y| + 1.$$

This completes the proof. $\square$

Theorem 4. *We have f in $S_k^2(\Gamma)$ if and only if $f : \mathfrak{h} \to \mathbb{C}$ is holomorphic and may be written as*

$$f = \sum_{i=0}^{2g} f_i \Lambda_i$$

where f_i is in $S_k(\Gamma)$ for $i > 0$ and f_0 is a smooth function on $\mathfrak{h}$ of weight k that satisfies **A2** *and* **A3**, *i.e. $(f_0|_k\sigma_{\mathfrak{a}})(z) \ll e^{-cy}$ as $y \to \infty$ uniformly in x for some $c > 0$. Also, for fixed Λ_i, the functions f_i are uniquely defined by f.*

Proof: In one direction, if $f = \sum_{i=0}^{2g} f_i\Lambda_i$ then $f|_k(\gamma_i - 1) = f_i$ for all hyperbolic generators γ_i. Also $f|_k(\gamma - 1) = 0$ for γ a parabolic or elliptic generator. Conditions **B2** and **B4** now hold since they are true for the generators of the group. To verify **B3** we see that, by lemma 3, f will have exponential decay at the cusps if each f_i does.

In the other direction, given any $f \in S_k^2(\Gamma)$ set $f_i = f|_k(\gamma_i - 1)$ for $1 \leqslant i \leqslant 2g$ and $f_0 = f - \sum_{i=1}^{2g} f_i\Lambda_i$. It is clear that f_0 is smooth, has weight k and has exponential decay at each cusp.

Finally, that the functions f_i are uniquely determined by f is clear. $\square$

A weaker condition than **B2** is:

B2.1. $f|_k(\gamma - 1)(\delta - 1) = 0$ for all γ, δ in Γ.

The combination **B1**, **B2.1**, **B3** and **B4** does not give $S_k^2(\Gamma)$. We need to strengthen **B3** in this case to have exponential decay on all the images of $\mathfrak{F}$ under the group action. Note that if $\mathfrak{a}$ is a cusp of $\mathfrak{F}$ then $\gamma\mathfrak{a}$ will be a cusp of $\gamma\mathfrak{F}$ and $\sigma_{\gamma\mathfrak{a}} = \gamma\sigma_{\mathfrak{a}}$ since $\sigma_{\gamma\mathfrak{a}}\infty = \gamma\sigma_{\mathfrak{a}}\infty = \gamma\mathfrak{a}$ and $\sigma_{\gamma\mathfrak{a}}^{-1}\Gamma_{\gamma\mathfrak{a}}\sigma_{\gamma\mathfrak{a}} = \Gamma_\infty$. Therefore exponential decay for all images of $\mathfrak{F}$ means the following:

B3'. for all $\gamma \in \Gamma$ we have $(f|_k(\gamma\sigma_{\mathfrak{a}}))(z) \ll e^{-cy}$ as $y \to \infty$, uniformly in x with c and the implied constant depending on γ, Γ and $\mathfrak{F}$.

It is easy to see that we then have

Lemma 5. *$f \in S_k^2(\Gamma)$ if and only if f satisfies* **B1***,* **B2.1***,* **B3'** *and* **B4***.*

This was the definition of $S_k^2(\Gamma)$ given in [CDO]. The analog of **A3.1** is

B3.1. $y^{k/2}(|\log \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)| + 1)^{-1}|f(z)| \ll 1$ for all z in $\mathfrak{h}$, all γ in Γ and any cusp $\mathfrak{a}$ with the implied constant depending on γ, Γ and $\mathfrak{F}$.

Lemma 6. *We have $f \in S_k^2(\Gamma)$ if and only if f satisfies* **B1***,* **B2.1***,* **B3.1** *and* **B4***.*

Proof: For $f \in S_k^2(\Gamma)$, **B2.1** is clearly true and we need only check that **B3.1** holds. By theorem 4 and lemma 3

$$\frac{y^{k/2}}{|\log \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)| + 1}|f(z)| \leqslant \frac{y^{k/2}|f_0(z)|}{|\log \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)| + 1} + \sum_{i=1}^{2g} y^{k/2}|f_i(z)|\frac{|\Lambda_i(z)|}{|\log \operatorname{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z)| + 1}$$
$$\ll 1.$$

Conversely, suppose f satisfies **B1**, **B3.1** and **B4**. Proposition 1 can be adapted to show that f satisfies (1) for $b_{\mathfrak{a}}(n) \ll n^{k/2}\log n$ and $b_{\mathfrak{a}}(0) = 0$. Thus f has exponential decay on the cusps of $\mathfrak{F}$: $(f|_k\sigma_{\mathfrak{a}})(z) \ll e^{-2\pi y}$ as $y \to \infty$ uniformly in x. Replace $\sigma_{\mathfrak{a}}$ by $\sigma_{\gamma\mathfrak{a}} = \gamma\sigma_{\mathfrak{a}}$ in the above proof to show that it also has exponential decay on the cusps of $\gamma\mathfrak{F}$. Thus the difference $(f|_k\gamma)(z) - f(z)$ has exponential decay at the cusps and with **B2.1** it has weight k. Conditions **B2** and **B3** (or alternatively **B3'**) now follow. □

The proof of lemma 6 also gives

Lemma 7 ('Trivial' bound). *The nth Fourier coefficient of a holomorphic second-order cusp form of weight k is $\ll n^{k/2}\log n$.*

In view of the 'twist-less' converse theorem we seek, we are also interested in the larger space of second-order cusp forms that do not necessarily satisfy **B4**, i.e. for parabolic elements π we may not have $f|_k(\pi - 1) = 0$.

Definition of $PS_k^2(\Gamma)$. Define this space to be all functions satisfying

B1. $f : \mathfrak{h} \to \mathbb{C}$ is holomorphic,

B2.1. $f|_k(\gamma - 1)(\delta - 1) = 0$ for all γ, δ in Γ,

B3*. ("vanishing at the cusps") for all $\gamma \in \Gamma$ we have $(f|_k(\gamma\sigma_{\mathfrak{a}}))(z) \ll e^{-cy}(1 + |x|)$ as $y \to \infty$, with c and the implied constant depending on γ, Γ and $\mathfrak{F}$.

To formulate theorem 9 (the analog of the classification theorem 4) we need to define the space of all modular forms, not necessarily with exponential decay at the cusps.

Definition of $M_k(\Gamma)$. Let $M_k(\Gamma)$ denote functions satisfying the conditions **A1**, **A2** and **C3** where

C3. $(f|_k\sigma_{\mathfrak{a}})(z) \ll 1$ as $y \to \infty$ uniformly on $\mathfrak{h}$.

With proposition 1 we may check that **C3** can be replaced by the equivalent condition

C3.1. $y^k|f(z)| \ll 1$ for all z in $\mathfrak{h}$.

Recall that the group Γ is generated by $2g$ hyperbolic elements γ_i, m parabolic elements π_j and a number of elliptic elements. One of the parabolic generators may be expressed in terms of the other generators. For $2g+1 \leqslant i \leqslant 2g+m-1$ we may define functions Λ_i which satisfy

$$\Lambda_i(\pi_j z) - \Lambda_i(z) = \delta_{(i-2g)j}$$

and

$$\Lambda_i(\gamma z) - \Lambda_i(z) = 0$$

for all other non-parabolic generators γ of Γ. By the Eichler-Shimura isomorphism, see [CDO] for example, for $2g+1 \leqslant i \leqslant 2g+m-1$ we may write

$$\Lambda_i(z) = \int_{z_0}^{z} g_i(w)\, dw + \overline{\int_{z_0}^{z} h_i(w)\, dw}$$

with $g_i \in M_2(\Gamma)$, $h_i \in S_2(\Gamma)$ and fixed z_0 in $\mathfrak{h}$. Now, as in the proof of lemma 3, for $g_i \in M_2(\Gamma)$ we see (with C3.1) that

$$\int_{z_0}^{\gamma^{-1}\sigma_\alpha z} g_i(w)dw \ll \frac{1}{y} + \frac{|x|}{y^2} + 1.$$

Therefore,

$$\Lambda_i(\gamma^{-1}\sigma_\alpha z) \ll \frac{1}{y} + \frac{|x|}{y^2} + |\log y| + 1$$

and the analog of lemma 3 is

Lemma 8. *For $2g+1 \leqslant i \leqslant 2g+m-1$, all $z \in \mathfrak{h}$, $\gamma \in \Gamma$ and any cusp α we have*

$$\Lambda_i(z) \ll \frac{1}{Im(\sigma_\alpha^{-1}\gamma z)} + \frac{|Re(\sigma_\alpha^{-1}\gamma z)|}{Im(\sigma_\alpha^{-1}\gamma z)^2} + |\log Im(\sigma_\alpha^{-1}\gamma z)| + 1$$

with the implied constant depending on γ, Γ and $\mathfrak{F}$.

In a way similar to the proof of theorem 4, we can then use lemma 8 to show

Theorem 9. *We have f in $PS^2_k(\Gamma)$ if and only if $f : \mathfrak{h} \to \mathbb{C}$ is holomorphic and may be written as*

$$f = \sum_{i=0}^{2g+m-1} f_i \Lambda_i$$

where f_i is in $S_k(\Gamma)$ for $i > 0$ and f_0 is a smooth function on $\mathfrak{h}$ of weight k that satisfies **A2** *and* **A3**, *i.e. $(f_0|_k\sigma_{\mathfrak{a}})(z) \ll e^{-cy}$ as $y \to \infty$ uniformly in x for some $c > 0$. Also, for fixed Λ_i, the functions f_i are uniquely defined by f.*

3. Functional equations

We now specialize to the case $\Gamma = \Gamma_0(N)$, for a fixed positive integer N and write $S_k(N)$ for $S_k(\Gamma_0(N))$ etc. Set $W_N := \left[\begin{smallmatrix} 0 & -1 \\ N & 0 \end{smallmatrix}\right]$ and define $\hat{f} := f|_k W_N$ for f a weight k, first or second order modular form.

Proposition 10. *If $f \in S_k(N)$ and $F \in S^2_k(N)$ then $\hat{f} \in S_k(N)$ and $\hat{F} \in S^2_k(N)$.*

Proof: Since W_N normalizes $\Gamma_0(N)$, $\hat{f}$ satisfies A2. It also satisfies A3.1 because

$$y^{\frac{k}{2}}|\hat{f}(z)| = y^{\frac{k}{2}}N^{-\frac{k}{2}}|z|^{-k}|f(\frac{-1}{Nz})| \ll y^{\frac{k}{2}}N^{-\frac{k}{2}}|z|^{-k}\mathrm{Im}(\frac{-1}{Nz})^{-\frac{k}{2}} \ll 1.$$

Therefore, by lemma 2, $\hat{f} \in S_k(N)$.

In a similar way, if $F \in S^2_k(N)$, then $\hat{F}$ satisfies B1, B2.1 and B4. On the other hand,

$$\begin{aligned} &y^{\frac{k}{2}}(|\log(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))| + 1)^{-1}|\hat{F}(z)| = y^{\frac{k}{2}}(|\log(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))| + 1)^{-1}N^{-\frac{k}{2}}|z|^{-k}|F(\frac{-1}{Nz})| \\ &\ll (|\log(\mathrm{Im}(\sigma_{\mathfrak{a}}^{-1}\gamma z))| + 1)^{-1}(|\log(\mathrm{Im}(\sigma_{\mathfrak{b}}^{-1}\delta W_N z))| + 1) \end{aligned}$$

for every cusp $\mathfrak{b}$ and each $\delta \in \Gamma_0(N)$. The final step is to choose $\mathfrak{b}$ and δ so that $\sigma_{\mathfrak{a}}^{-1}\gamma = \sigma_{\mathfrak{b}}^{-1}\delta W_N$: Since $\gamma^{-1}\mathfrak{a}$ and $W_N\gamma^{-1}\mathfrak{a}$ are also cusps of $\Gamma_0(N)$ we must have $\mathfrak{b} = \delta W_N\gamma^{-1}\mathfrak{a}$ for a $\mathfrak{b}$ in the set of inequivalent cusps and a $\delta \in \Gamma_0(N)$. Because of the relation $\sigma_{\tau\mathfrak{a}} = \tau\sigma_{\mathfrak{a}}$, this implies that $\sigma_{\mathfrak{a}}^{-1}\gamma = \sigma_{\mathfrak{b}}^{-1}\delta W_N$. $\square$

If $F(z) = \sum_{n=1}^{\infty} a_n e^{2\pi nz} \in S^2_k(N)$, then its L-function is defined by

$$L(s, F) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = (2\pi)^s\Gamma(s)^{-1}\Lambda(s, F)$$

for $\Lambda(s, F) = \int_0^{\infty} F(iy)y^{s-1}dy$. Since ∞ and 0 are cusps of $\Gamma_0(N)$, where F has exponential decay, we see that $L(s, F)$ has a meromorphic continuation to $s \in \mathbb{C}$. The functional equation

$$i^{-k}N^{\frac{k}{2}-s}\Lambda(k - s, F) = \Lambda(s, \hat{F})$$

follows by a simple change of variables in the above integral just as in the case of L-functions of first order cusp forms.

In order to get a functional equation for the L-function of F twisted by a Dirichlet character we may define two twisting operators as follows.

First of all, for any function g on $\mathfrak{h}$ and a Dirichlet character χ mod N, we set

$$g_\chi(z) = \sum_{0<m<N} \chi(m) g\left(\frac{z+m}{N}\right)$$

Here, as in all the sums appearing in the sequel, the sum ranges only over integers that are relative prime to the modulus. Set $T = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$. It is known (cf. [R]) that f is $\Gamma_0(N)$-invariant if and only if it is periodic with period 1 and

$$f_\chi|_k T = \chi(-1) f_{\bar{\chi}}. \tag{2}$$

For $F \in PS_k^2(N)$ decomposed as in theorem 9, we define

$$F^\chi = \sum_{i=1}^{2g+m-1} (f_i)_\chi \Lambda_i + (f_0)_\chi$$

and its "contragredient"

$$\check{F}^\chi(z) = \sum_{i=1}^{2g+m-1} (f_i)_\chi(z) \left(\int_i^z (g_i|_2 T)(w) dw + \overline{\int_i^z (h_i|_2 T)(w) dw} \right) + (f_0)_\chi(z).$$

We also set

$$L(s, F, \chi) := (2\pi)^s \Gamma(s)^{-1} \Lambda(s, F, \chi) = (2\pi)^s \Gamma(s)^{-1} \int_0^\infty F^\chi(iy) y^{s-1} dy \quad \text{and}$$
$$L^*(s, F, \chi) := (2\pi)^s \Gamma(s)^{-1} \Lambda^*(s, F, \chi) = (2\pi)^s \Gamma(s)^{-1} \int_0^\infty \check{F}^\chi(iy) y^{s-1} dy.$$

Theorem 11. *For $F \in PS_k^2(N)$ the following functional equation holds:*

$$\Lambda(k-s, F, \chi) = i^k \chi(-1) \Lambda^*(s, F, \bar{\chi}).$$

Proof. Apply T to F^χ to get

$$\begin{aligned} F^\chi|_k T &= \sum_{i=0}^{2g+m-1} ((f_i)_\chi|_k T)(z) \Lambda_i(Tz) \\ &= \sum_{i=1}^{2g+m-1} ((f_i)_\chi|_k T)(z) \left(\int_i^{Tz} g_i(w) dw + \overline{\int_i^{Tz} h_i(w) dw} \right) + (f_0)_\chi|_k T. \end{aligned}$$

Since T fixes i, we can make the change of variables $w \to Tw$ in the integrals. This, in combination with (2), gives

$$\chi(-1)\Big(\sum_{i=1}^{2g+m-1}(f_i)_{\bar\chi}(z)\int_i^z (g_i|_2T)(w)dw + \overline{\int_i^z (h_i|_2T)(w)dw} + (f_0)_{\bar\chi}(z)\Big) = \chi(-1)\check{F}^{\bar\chi}.$$

It should be noted that, although f_0 is not necessarily holomorphic, (2) applies to it too because the derivation of (2) depends only on algebraic manipulations on $\mathbb{Z}[GL_2(\mathbb{R})]$. Applying the Mellin transform to this equality we obtain

$$i^{-k}\int_0^\infty F^\chi(\frac{i}{y})y^{s-k}\frac{dy}{y} = \chi(-1)\int_0^\infty \check{F}^{\bar\chi}(iy)y^s\frac{dy}{y}.$$

The change of variables $y \to \frac{1}{y}$ in the first integral implies the functional equation.

We cannot use this theorem directly in order to obtain a meaningful converse theorem for second-order modular forms. The reason is that, in this approach, we must be able to isolate the functions f_i from the given function F for the above functional equation to even be set up. The functions f_i are $F|_k(\gamma_i - 1)$ for certain elements γ_i of Γ and in the next section we give a criterion for a function to be in PS_k^2 which is based on each $F|_k(\gamma_i - 1)$ separately.

4. A converse theorem

For $c, c_1 \in \{1, \dots, N\}$, let ψ be a Dirichlet character mod(Nc_1) and let χ, ω be Dirichlet characters mod(Nc). Set

$$F_{\chi,\psi,\omega} := \sum_{\substack{0<m,b<Nc\\0<a<Nc_1}} \psi(a)\chi(m)\omega(b)F\Big|_k\begin{bmatrix} c & ac-bc_1+mc_1\\ 0 & Ncc_1\end{bmatrix} \quad \text{and}$$

$$F^{\chi,\psi,\omega} := \sum_{\substack{0<m,b<Nc\\0<a<Nc_1}} \psi(a)\chi(m)\omega(b)F\Big|_k\begin{bmatrix} mc & m(ac-bc_1)-c_1\\ Nc^2 & Nc(ac-bc_1)\end{bmatrix}.$$

We need to fix a set of generators for $\Gamma_0(N)$. To this end we use the following

Lemma 12. [R] *(i) Let c be a positive integer. For each $0 < a < Nc$ ($(a, Nc) = 1$) choose one matrix $V_a = \left[\begin{smallmatrix} a & b_a\\ Nc & d_a\end{smallmatrix}\right] \in \Gamma_0(N)$ such that $-Nc < d_a < 0$. If S_c denotes the set of all such matrices then $\bigcup_{c=1}^N S_c \cup \{\left[\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right]\} \cup \{\left[\begin{smallmatrix} -1 & 0\\ 0 & -1\end{smallmatrix}\right]\}$ generates $\Gamma_0(N)$.*
(ii) If $N = p^r$ (p prime), then $\Gamma_0(N)$ is generated by $\{\left[\begin{smallmatrix} -1 & 0\\ 0 & -1\end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1\\ 0 & 1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} a & b\\ N & d\end{smallmatrix}\right] \in \Gamma_0(N)$, as a ranges over a system of residues mod N prime to N.

We will also need a lemma from [Fl]. Since it has not been published and since our statement is somewhat more general than that in [Fl], we give a proof here.

Lemma 13. [Fl] *Set* $T = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. *Let* χ *be a character* mod(Nc) *and let* F *be a function on* $\mathfrak{h}$. *Then,*

$$F_\chi|_k T = \chi(-1)F_{\bar{\chi}} + \chi(-1)q_\chi$$

where $q_\chi = \sum_{0<a<Nc} \bar{\chi}(d_a) q_{V_a}\left(\frac{z-d_a}{Nc}\right)$ *and* $q_{V_a} = F|_k V_a - F$.

Proof. We have

$$V_a\left(\frac{z-d_a}{Nc}\right) = \frac{-z^{-1}+a}{Nc} \quad \text{and} \quad Nc\left(\frac{z-d_a}{Nc}\right) + d_a = z.$$

Therefore,

$$\begin{aligned}(F_\chi|_k T)(z) = z^{-k}F_\chi(-z^{-1}) &= \sum_{0<a<Nc} \chi(a) z^{-k} F\left(\frac{-z^{-1}+a}{Nc}\right) \\ &= \sum_{0<a<Nc} \chi(a)(F|_k V_a)\left(\frac{z-d_a}{Nc}\right) \\ &= \chi(-1) \sum_{0<a<Nc} \bar{\chi}(-d_a)(F|_k V_a)\left(\frac{z-d_a}{Nc}\right). \end{aligned} \tag{3}$$

Here we use the fact that $ad_a - Nb_a c = 1$ and therefore $\chi(a)\chi(d_a) = 1$.

On the other hand, $-d_a$ ranges over the elements of $\{1, \ldots, Nc\}$ prime to Nc, as a ranges over the same set. Hence,

$$F_{\bar{\chi}}(z) = \sum_{0<a<Nc} \bar{\chi}(-d_a) F\left(\frac{z-d_a}{Nc}\right).$$

On subtraction from (3) we obtain

$$(F_\chi|_k T)(z) - \chi(-1)F_{\bar{\chi}}(z) = \chi(-1) \sum_{0<a<Nc} \bar{\chi}(-d_a) q_{V_a}\left(\frac{z-d_a}{Nc}\right). \quad \square$$

Theorem 14. *Let* F *be a holomorphic function on* $\mathfrak{h}$ *such that, for all* $\gamma \in \Gamma_0(N)$, $(f|_k(\gamma\sigma_{\mathfrak{a}}))(z) \ll e^{-cy}(1+|x|)$ *as* $y \to \infty$, *with* c *and the implied constant depending on* γ, Γ *and* $\mathfrak{F}$. *Suppose that for all Dirichlet characters* χ, ω mod(Nc), ψ mod(Nc_1) $(c, c_1 \in \{1, \ldots, N\})$ $F_{\chi,\psi,\omega}(iy)$ *and* $F^{\chi,\psi,\omega}(iy)$ *decay exponentially as* $y \to \infty$ *and* $y \to 0$. *Set*

$$\Phi^1(s, \chi, \psi, \omega) := \int_0^\infty F_{\chi,\psi,\omega}(iy) y^{s-1} dy \quad \text{and} \quad \Phi^2(s, \chi, \psi, \omega) := \int_0^\infty F^{\chi,\psi,\omega}(iy) y^{s-1} dy.$$

If $F|_k(\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]-1)(\gamma-1)=F|_k(\gamma-1)(\left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right]-1)=0$ *and the functional equation*

$$\begin{aligned}&\chi(-1)i^{-k}\Phi^2(k-s,\chi,\psi,\omega)-i^k\Phi^1(k-s,\bar{\chi},\psi,\omega)\\&\qquad=\psi(-1)\chi(-1)\Phi^2(s,\chi,\bar{\psi},\omega)-\Phi^1(s,\bar{\chi},\bar{\psi},\omega)\end{aligned}$$

is true, then $F\in PS_k^2(N)$.

Proof. We see that Φ^1 and Φ^2 converge to analytic functions of s for all $s\in\mathbb{C}$. It follows by the Mellin inversion formula that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi^1(s,\chi,\psi,\omega)y^{-s}ds&=F_{\chi,\psi,\omega}(iy),\\ \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\Phi^2(s,\chi,\psi,\omega)y^{-s}ds&=F^{\chi,\psi,\omega}(iy).\end{aligned}$$

are valid for every $\sigma=\text{Re}s\in\mathbb{R}$. Replace s by $k-s$ and y by $\frac{1}{y}$ in the above to see that the functional equation assumed in the theorem implies that

$$(iy)^{-k}\chi(-1)F^{\chi,\psi,\omega}(\frac{-1}{iy})=(iy)^{-k}F_{\bar{\chi},\psi,\omega}(\frac{-1}{iy})+\psi(-1)\chi(-1)F^{\chi,\bar{\psi},\omega}(iy)-F_{\bar{\chi},\bar{\psi},\omega}(iy).$$

Since all the functions involved are analytic, this equality is true on the entire upper-half plane and we can rewrite it in the form

$$\begin{aligned}&\left[\chi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\chi(m)F\Big|_k\begin{bmatrix}mc & m(ac-bc_1)-c_1\\ Nc^2 & Nc(ac-bc_1)\end{bmatrix}\right]\Big|_kT\\&-\left[\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\bar{\chi}(m)F\Big|_k\begin{bmatrix}c & ac-bc_1+mc_1\\ 0 & Ncc_1\end{bmatrix}\right]\Big|_kT\\&-\chi(-1)\psi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)\sum_{0<m<Nc}\chi(m)F\Big|_k\begin{bmatrix}mc & m(ac-bc_1)-c_1\\ Nc^2 & Nc(ac-bc_1)\end{bmatrix}\\&+\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)\sum_{0<m<Nc}\bar{\chi}(m)F\Big|_k\begin{bmatrix}c & ac-bc_1+mc_1\\ 0 & Ncc_1\end{bmatrix}=0\qquad\text{or}\end{aligned}$$

$$\chi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\chi(m)F\Big|_k\begin{bmatrix}1&m\\0&Nc\end{bmatrix}T\begin{bmatrix}c&ac-bc_1\\0&c_1\end{bmatrix}T$$

$$-\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\bar{\chi}(m)F\Big|_k\begin{bmatrix}1&m\\0&Nc\end{bmatrix}\begin{bmatrix}c&ac-bc_1\\0&c_1\end{bmatrix}T$$

$$-\chi(-1)\psi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)\sum_{0<m<Nc}\chi(m)F\Big|_k\begin{bmatrix}1&m\\0&Nc\end{bmatrix}T\begin{bmatrix}c&ac-bc_1\\0&c_1\end{bmatrix}$$

$$+\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)\sum_{0<m<Nc}\bar{\chi}(m)F\Big|_k\begin{bmatrix}1&m\\0&Nc\end{bmatrix}\begin{bmatrix}c&ac-bc_1\\0&c_1\end{bmatrix}=0. \tag{4}$$

We can further use the definition of F_χ (of Section 3) to write the last equality in the form:

$$\chi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)F_\chi\Big|_kT\begin{bmatrix}c&ac-bc_1\\0&c_1\end{bmatrix}T$$

$$=\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)F_{\bar{\chi}}\Big|_k\begin{bmatrix}c&ac-bc_1\\0&c_1\end{bmatrix}T$$

$$-\chi(-1)\psi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)F_\chi\Big|_kT\begin{bmatrix}c&ac-bc_1\\0&c_1\end{bmatrix}$$

$$+\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)F_{\bar{\chi}}\Big|_k\begin{bmatrix}c&ac-bc_1\\0&c_1\end{bmatrix}=0\qquad\text{or}$$

$$\chi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\Big(F_\chi|_kT-\chi(-1)F_{\bar{\chi}}\Big)\Big|_k\begin{bmatrix}Nc&Nac-Nbc_1\\0&Nc_1\end{bmatrix}T$$

$$=\chi(-1)\psi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\bar{\psi}(a)\Big(F_\chi|_kT-\chi(-1)F_{\bar{\chi}}\Big)\Big|_k\begin{bmatrix}Nc&Nac-Nbc_1\\0&Nc_1\end{bmatrix}.$$

Since $\left[\begin{smallmatrix}Nc&Nac-Nbc_1\\0&Nc_1\end{smallmatrix}\right]=\left[\begin{smallmatrix}Nc&-b\\0&1\end{smallmatrix}\right]\left[\begin{smallmatrix}1&d\\0&Nc_1\end{smallmatrix}\right]$ we have

$$\sum_{0<b<Nc}\omega(b)\sum_{0<a<Nc_1}\psi(a)\Big(\Big(F_\chi|_kT-\chi(-1)F_{\bar{\chi}}\Big)\Big|_k\left[\begin{smallmatrix}Nc&-b\\0&1\end{smallmatrix}\right]\Big)\left[\begin{smallmatrix}1&a\\0&Nc_1\end{smallmatrix}\right]T$$

$$=\psi(-1)\sum_{0<b<Nc}\omega(b)\sum_{0<a<Nc_1}\bar{\psi}(a)\Big(\Big(F_\chi|_kT-\chi(-1)F_{\bar{\chi}}\Big)\Big|_k\left[\begin{smallmatrix}Nc&-b\\0&1\end{smallmatrix}\right]\Big)\left[\begin{smallmatrix}1&a\\0&Nc_1\end{smallmatrix}\right].$$

Or, by the defining formula for the twist of a function,

$$\sum_{0<b<Nc} \omega(b)\Big(\Big(F_\chi|_k T - \chi(-1)F_{\bar\chi}\Big)\Big|_k \left[\begin{smallmatrix} Nc & -b \\ 0 & 1\end{smallmatrix}\right]\Big)_\psi\Big|_k T$$
$$= \psi(-1)\sum_{0<b<Nc} \omega(b)\Big(\Big(F_\chi|_k T - \chi(-1)F_{\bar\chi}\Big)\Big|_k \left[\begin{smallmatrix} Nc & -b \\ 0 & 1\end{smallmatrix}\right]\Big)_{\bar\psi}.$$

By character summation (over characters ω mod(Nc)) we obtain

$$\Big(\Big(F_\chi|_k T - \chi(-1)F_{\bar\chi}\Big)\Big|_k \left[\begin{smallmatrix} Nc & -b \\ 0 & 1\end{smallmatrix}\right]\Big)_\psi\Big|_k T$$
$$= \psi(-1)\Big(\Big(F_\chi|_k T - \chi(-1)F_{\bar\chi}\Big)\Big|_k \left[\begin{smallmatrix} Nc & -b \\ 0 & 1\end{smallmatrix}\right]\Big)_{\bar\psi}$$

for all $b \in \{1, \dots, Nc\}$ with $((b, Nc) = 1)$.

Now, for $a = 1, \dots, Nc$, prime to Nc and $b = -d_a > 0$, this implies

$$\Big(\sum_{\chi \pmod{Nc}} \chi(d_a)\Big(F_\chi|_k T - \chi(-1)F_{\bar\chi}\Big)\Big|_k \left[\begin{smallmatrix} Nc & d_a \\ 0 & 1\end{smallmatrix}\right]\Big)_\psi\Big|_k T$$
$$= \psi(-1)\Big(\sum_{\chi \pmod{Nc}} \chi(d_a)\Big(F_\chi|_k T - \chi(-1)F_{\bar\chi}\Big)\Big|_k \left[\begin{smallmatrix} Nc & d_a \\ 0 & 1\end{smallmatrix}\right]\Big)_{\bar\psi}.$$

By the usual character summation argument, lemma 13 then implies that the sum inside the parentheses equals $\phi(Nc)q_{V_a}$, where ϕ denotes Euler's function. So the last equality can be rewritten as

$$(q_{V_a})_\psi|_k T = \psi(-1)(q_{V_a})_{\bar\psi},$$

for all $V_a \in S_c$.

From (2) (together with our assumption on the periodicity of $F|_k(\gamma-1)$'s) we can then deduce that $q_{V_a} = F|_k V_a - F$ is invariant under $\Gamma_0(N)$. Therefore, $F|_k(V_a-1)\gamma = F|_k(V_a-1)$ for all $\gamma \in \Gamma_0(N)$. On the other hand, we have also assumed that $F|_k(\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1\end{smallmatrix}\right]-1)(\gamma-1) = 0$ for all $\gamma \in \Gamma_0(N)$. Now, if $F|_k(\gamma_1-1)(\gamma-1) = 0$ and $F|_k(\gamma_2-1)(\gamma-1) = 0$ for all $\gamma \in \Gamma_0(N)$ then $F|_k(\gamma_1\gamma_2-1)(\gamma-1) = 0$ because $(\gamma_1\gamma_2-1)(\gamma-1) = (\gamma_1-1)(\gamma_2\gamma-1) - (\gamma_1-1)(\gamma_2-1) + (\gamma_2-1)(\gamma-1)$. According to lemma 12($i$), $\Gamma_0(N)$ is generated by the V_a's, the translations and $\left[\begin{smallmatrix} -1 & 0 \\ 0 & -1\end{smallmatrix}\right]$. Therefore, F satisfies **B2.1** and thus by definition, $F \in PS_k^2(N)$. $\square$

The following corollary of the proof allows us to distinguish the case that F is a "trivial" second-order modular form, that is, a usual cusp form.

Proposition 15. *If in the statement of theorem 14, the left-hand side of the functional equation vanishes then F is a usual cusp form.*

Proof. We can repeat the first steps of the proof of theorem 14. Our assumption implies that (4) can be read as

$$\chi(-1)\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\chi(m)F\Big|_k\begin{bmatrix}1 & m\\0 & Nc\end{bmatrix}T\begin{bmatrix}c & ac-bc_1\\0 & c_1\end{bmatrix}T$$

$$-\sum_{\substack{0<b<Nc\\0<a<Nc_1}}\omega(b)\psi(a)\sum_{0<m<Nc}\bar{\chi}(m)F\Big|_k\begin{bmatrix}1 & m\\0 & Nc\end{bmatrix}\begin{bmatrix}c & ac-bc_1\\0 & c_1\end{bmatrix}T$$

A character summation over $\omega \bmod(Nc)$ and $\psi \bmod(Nc_1)$ together with the definition of F_χ implies that $F_\chi=\chi(-1)F_{\bar{\chi}}$ for all $\chi \bmod(Nc)$ and, according to [R], $F\in S_k(N)$. $\square$

5. Periodicity

Let $S=\begin{bmatrix}1&1\\0&1\end{bmatrix}$. In theorem 14 we had to include the assumption $F|_k(\gamma-1)(S-1)=F|_k(S-1)(\gamma-1)=0$ for all $\gamma\in\Gamma_0(N)$. The second equality is clearly satisfied if F has period 1. In this section we examine how the imposition of this stronger assumption of periodicity affects F.

Indeed, suppose that $F|_k(S-1)=0$ and that $F|_k(\gamma-1)(S-1)=0$ for all $\gamma\in\Gamma_0(N)$. Then $F|_k\gamma S=F|_k\gamma$, i.e. F is invariant under the group $\tilde{\Gamma}_0(N)$ generated by the $\gamma S\gamma^{-1}$'s ($\gamma\in\Gamma_0(N)$). It is reasonable to ask whether this invariance implies the modularity of F, thus making the remaining assumptions of the theorem redundant. We will show that, for $N\geq 4$, this is far from being the case.

Specifically, set

$$\tilde{\Gamma}_1(N)=\langle\gamma^{-1}S\gamma\mid\gamma\in\Gamma_1(N)\rangle.$$

As usual, we identify the groups with their images in $\mathrm{PSL}(2,\mathbb{Z})$.

Theorem 16. *$\tilde{\Gamma}_0(N)$ has infinite index in $SL(2,\mathbb{Z})$ for $N\geq 4$.*

Proof: It is well-known that, for $N\geq 4$, $\Gamma_1(N)$ is free and its rank equals

$$r:=1+\frac{N^2}{12}\prod_{p|N}(1-\frac{1}{p^2}). \tag{5}$$

Next note that $\Gamma_1(N)\trianglelefteq\Gamma_0(N)$ and $|\Gamma_0(N):\Gamma_1(N)|=\frac{1}{2}\phi(N)=:f$, say. Let $g_1=1,g_2,\ldots,g_f$ be a set of coset representatives of $\Gamma_1(N)$ in $\Gamma_0(N)$.

Observe that

$$\tilde{\Gamma}_0(N)=\langle g_i^{-1}\gamma^{-1}S\gamma g_i\mid\gamma\in\Gamma_1(N),1\leq i\leq f\rangle$$
$$=\langle g_i^{-1}\tilde{\Gamma}_1(N)g_i|1\leq i\leq f\rangle.$$

Set $\Delta_i(N) = g_i^{-1}\tilde{\Gamma}_1(N)g_i$. Because $\Delta_1(N) = \tilde{\Gamma}_1(N) \trianglelefteq \Gamma_1(N) \trianglelefteq \Gamma_0(N)$ then $\Delta_i(N) \trianglelefteq \Gamma_1(N)$ for each i, and therefore

$$\tilde{\Gamma}_0(N) = \Delta_1(N)\ldots\Delta_f(N) \leq \Gamma_1(N). \tag{6}$$

Set $S_i = g_i^{-1}Sg_i$. We then also have

$$\Delta_i(N) = \langle \gamma^{-1}S_i\gamma \mid \gamma \in \Gamma_1(N)\rangle = \langle S_i[S_i,\gamma] \mid \gamma \in \Gamma_1(N)\rangle \tag{7}$$

where we use standard notation $[x, y] = x^{-1}y^{-1}xy$ for elements x, y in a group.

Now let A be the abelianization of the group $\Gamma_1(N)$, i.e. the quotient of $\Gamma_1(N)$ by its commutator subgroup. Because $\Gamma_1(N)$ is free then

$$A \cong Z^r$$

where r is given by (5). It follows from the second equality in (7) that the image of each $\Delta_i(N)$ in A is cyclic, being generated by the image of S_i. Hence by (6), the image of $\tilde{\Gamma}_0(N)$ in A has rank no greater than f. If we can show that

$$r > f$$

then it follows immediately that the Theorem holds. But this is a triviality: it says that

$$1 + \frac{N^2}{12}\prod_{p|N}(1-\frac{1}{p^2}) > \frac{1}{2}\phi(N) = \frac{N}{2}\prod_{p|N}(1-\frac{1}{p}),$$

that is

$$\frac{2}{\phi(N)} + \frac{N}{6}\prod_{p|N}(1+\frac{1}{p}) > 1,$$

which is obvious. $\square$

On the other hand, we have

Proposition 17. *For* $1 \leq N \leq 3$, $\tilde{\Gamma}(N) = \Gamma_0(N^2)$.

Proof: $N = 1$. We want to prove that $\Gamma_0(1) = \Gamma(1)$ can be generated by $\gamma S\gamma^{-1}$ ($\gamma \in \Gamma(1)$). A simple check shows that $T = -S^2P$, where $P = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} S \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$. Since $S^2 \in \tilde{\Gamma}(1)$, this settles the case $N = 1$.

$N = 2$. $\Gamma_0(4)$ (or, more precisely, its projection onto $\mathrm{PSL}_2(\mathbb{Z})$) is generated by S, $P_1 = \begin{bmatrix} -1 & 0 \\ 4 & -1 \end{bmatrix}$ and $P_2 = \begin{bmatrix} 1 & -1 \\ 4 & -3 \end{bmatrix}$. This can be seen by lemma 12(ii). However, $P_1P_2 = -S^{-1}$. Thus, since $S^{-1} \in \tilde{\Gamma}(2)$ (obviously), it suffices to prove that $P_2 \in \tilde{\Gamma}(2)$. Indeed,

$$P_2 = -\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} S \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}^{-1}$$

and our result follows for $N = 2$.

$N = 3$. By lemma 12(ii), $\Gamma_0(9)$ is generated by S, $P_1 = \begin{bmatrix} -1 & 0 \\ 9 & -1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 2 & -1 \\ 9 & -4 \end{bmatrix}$, $P_3 = \begin{bmatrix} 5 & -4 \\ 9 & -7 \end{bmatrix}$, $P_4 = \begin{bmatrix} 7 & -4 \\ 9 & -5 \end{bmatrix}$, $P_5 = \begin{bmatrix} 4 & -1 \\ 9 & -2 \end{bmatrix}$ and $P_6 = \begin{bmatrix} 1 & 0 \\ 9 & 1 \end{bmatrix}$. Since $P_4 = -P_3^{-1}$, $P_5 = -P_2^{-1}$, $P_6 = -P_1^{-1}$ and $P_1P_2P_3 = S^{-1}$ it is sufficient to show that P_2 and P_3 are in $\tilde{\Gamma}(3)$. Indeed,

$$P_2 = -\begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix} S \begin{bmatrix} 1 & -1 \\ 3 & -2 \end{bmatrix}^{-1} \qquad \text{and} \quad P_3 = -\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} S \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1}. \quad \square$$

We also observe that the invariance under $\Gamma_0(N^2)$ implied by proposition 17 (when $N = 1, 2, 3$) for functions satisfying the assumptions of Theorem 14, in fact implies modularity for $\Gamma_0(N)$. This is a consequence (with $\Gamma_1 = \Gamma_0(N)$, $\Gamma_2 = \Gamma_0(N^2)$) of

Proposition 18. *If F satisfies* **B2.1** *for Γ_1 and is invariant under a group Γ_2 with $[\Gamma_1 : \Gamma_2] < \infty$, then F is invariant under Γ_1.*

Proof: Since Γ_2 contains a subgroup of finite index which is normal in Γ_1, we can assume, without loss of generality, that Γ_2 is normal in Γ_1. Let $\mu = [\Gamma_1 : \Gamma_2]$. Then, for $\gamma \in \Gamma_1$, $\gamma^\mu \in \Gamma_2$. Thus, $F|_k\gamma^\mu - F = 0$. On the other hand, $F|_k\gamma - F$ is invariant under Γ_1, therefore we have:

$$0 = F|_k(\gamma^\mu - 1) = F|_k(\gamma - 1)(\gamma^{\mu-1} + \cdots + 1) = \mu F|_k(\gamma - 1)$$

for all $\gamma \in \Gamma_1$. $\square$

We should finally remark that the discussion of this paragraph applies more generally to all periodic second-order modular forms and therefore it can be made independently of theorem 14. This is a consequence of

Proposition 19. *Every periodic second-order modular form G is $\tilde{\Gamma}_0(N)$-invariant.*

Proof: Let F be a periodic second-order modular form. For all $\gamma, \delta, \epsilon \in \Gamma_0(N)$ we have:

$$F|_k(\gamma\delta\epsilon - \gamma - \delta - \epsilon + 2) = F|_k((\gamma\delta - 1)(\epsilon - 1) + (\gamma - 1)(\delta - 1)) = 0.$$

For $\delta = S$ and $\epsilon = \gamma^{-1}$ this gives $F|_k(\gamma S\gamma^{-1} - \gamma - S - \gamma^{-1} + 2) = 0$. This, in turn, in combination with $F|_k(\gamma + \gamma^{-1} - 2) = F|_k(\gamma - 1)(\gamma^{-1} - 1) = 0$ implies $F|_k(\gamma S\gamma^{-1} - 1) = 0$ for all $\gamma \in \Gamma_0(N)$. $\square$

Therefore, we can deduce from propositions 17 and 18 that, for $N = 1, 2, 3$, if $F \in S_k^2(N)$ and $F(z + 1) = F(z)$ then $F \in S_k(N)$.

References

[CDO] G. Chinta, N. Diamantis, C. O'Sullivan, *Second order modular forms*, Acta Arithmetica **103** (2002), 209–223.

[CF] B. Conrey, D. Farmer, *An extension of Hecke's converse theorem*, Internat. Math. Res. Notices **9** (1995), 445–463.

[F] D. Farmer, *Converse theorems and second order modular forms*, AMS sectional meeting talk, Salt Lake City (2002).

[Fl] K. Flood, *Two Correspondence Theorems for Modular Integrals*, Dissertation (Temple University) (1993).

[Iw] Iwaniec, H., *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics, **17** (1997).

[KZ] P. Kleban, D. Zagier, *Crossing probabilities and modular forms*, preprint arXiv: http://xxx.lanl.gov/abs/math-ph/0209023.

[Og] A. Ogg, *Modular forms and Dirichlet series*, W. A. Benjamin, Inc. (1969).

[R] M. Razar, *Modular forms for* $\Gamma_0(N)$ *and Dirichlet series*, Transactions of the AMS **231** (1977), 489-495.