Homework #6 Solutions

(1) $f(n)$ defined recursively with $f(0)=2$
and $f(n+1) = f(n)^2 - 3$

So $f(0+1) = f(0)^2 - 3$ when $n=0$
means $f(1) = 2^2 - 3 = 1$

$$f(2) = f(1)^2 - 3 = 1^2 - 3 = -2$$

$$f(3) = f(2)^2 - 3 = (-2)^2 - 3 = 1$$

$$f(4) = f(3)^2 - 3 = 1^2 - 3 = -2$$

This shows $\boxed{f(4) = -2}$

(2) Sequence b_n with $b_0 = 1$, $b_1 = 3$
and $b_{n+1} = b_n - 2b_{n-1}$

$$b_2 = b_1 - 2b_0 = 3 - 2(1) = 1$$

$$b_3 = b_2 - 2b_1 = 1 - 2(3) = -5$$

$$b_4 = b_3 - 2b_2 = -5 - 2(1) = -7$$

$$b_5 = b_4 - 2b_3 = -7 - 2(-5)$$
$$= -7 + 10 = 3$$

Then $b_5 = 3$ as we wanted to show.

(3) (a) $a_n = 6n$ for $n \geq 1$

Compare a_{n+1} and a_n: $a_{n+1} = 6(n+1) = 6n+6$
$a_n = 6n$

We can define a_n recursively by

Basis step: $a_1 = 6$
Recursive step: $a_{n+1} = a_n + 6$

(c) $a_n = 10^n$ for $n \geq 1$

Compare a_{n+1} and a_n: $a_{n+1} = 10^{n+1} = 10^n \cdot 10$
$a_n = 10^n$

So write
Basis step: $a_1 = 10$
Recursive step: $a_{n+1} = 10\,a_n$

(4) Remember the Fibonacci numbers, given by $f_0 = 0$, $f_1 = 1$, $f_{n+1} = f_n + f_{n-1}$

so $f_2 = 1$, $f_3 = 2$, $f_4 = 3$, ...

Here we want to prove the following:

(A) $P(n)$ says $f_1 + f_3 + \cdots + f_{2n-1} = f_{2n}$

(B) Basis step - check $P(1)$. It says $f_1 = f_2$.
True because $f_1 = 1$, $f_2 = 1$.

(C) Inductive step - assume $P(k)$ is true

$f_1 + f_3 + \cdots + f_{2k-1} = f_{2k}$ true add f_{2k+1} to both sides

so $f_1 + f_3 + \cdots + f_{2k-1} + \underline{f_{2k+1}} = f_{2k} + \underline{f_{2k+1}}$ is true

right side equals f_{2k+2} by Fibonacci definition

so $f_1 + f_3 + \cdots + f_{2k-1} + f_{2k+1} = f_{2k+2}$

same as $f_1 + f_3 + \cdots + f_{2(k+1)-1} = f_{2(k+1)}$

so we have proved $P(k+1)$ is true.

(D) By mathematical induction $P(n)$ is true for all $n \geq 1$.

(5) $\Sigma = \{8\}$ and Σ^* is defined recursively with λ the empty string in Σ^* and if $w \in \Sigma^*$ and $x \in \Sigma$ then $wx \in \Sigma^*$.

$w = \lambda$, $x = 8$ means $wx = \lambda 8 = 8$ in Σ^*

$w = 8$, $x = 8$ means $wx = 88$ in Σ^*

$w = 88$, $x = 8$ means $wx = 888$ in Σ^*

$\vdots$

8888, 8888888 in Σ^* etc.

Σ^* just contains strings of 8s.

(6) T defined recursively with $(0,0) \in T$

and if $(a,b) \in T$ then $(a+1, b-2) \in T$ (A)
and $(a-1, b+1) \in T$ (B)

(a) Starting with $(0,0)$ get

$$\begin{tikzcd}
 & & (0,0) \arrow[dl, "A"'] \arrow[dr, "B"] & & \\
 & (1,-2) \arrow[dl, "A"'] \arrow[dr, "B"] & & (-1,1) \arrow[dl, "A"'] \arrow[dr, "B"] & \\
(2,-4) & & (0,-1) \arrow[dl, "A"'] & & (-2,2) \\
 & (1,-3) \arrow[dr, "B"] & & & \\
 & & (0,-2) \arrow[dl, "A"'] & & \\
 & (1,-4) \arrow[dr, "B"] & & & \\
 & & (0,-3) & &
\end{tikzcd}$$

(b) Playing around we see that $(0,-3) \in T$.

(7) Set U defined recursively with
Basis step: $(0,0) \in U$
Recursive step: $(a,b) \in U$ implies
$(a+1, b-1) \in U$ and $(a-4, b+4) \in U$.

We'll use structural induction to prove that if $(a,b) \in U$ then $a+b=0$.

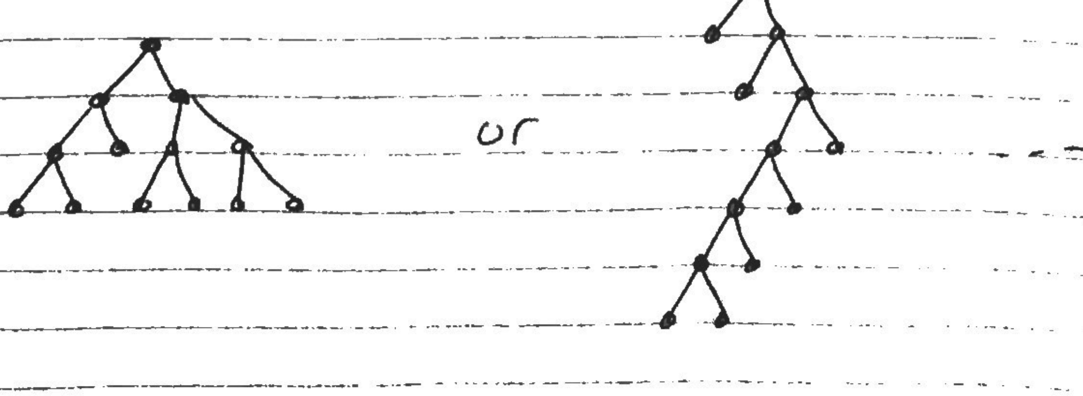
First, check the basis object which is $(0,0)$. Yes $0+0=0$ so true for basis object.

Second, suppose $(a,b) \in U$ with $a+b=0$. What about the new elements from the recursive step $(a+1, b-1)$ and $(a-4, b+4)$?

If $a+b=0$ then $(a+1)+(b-1)=0$
and $(a-4)+(b+4)=0$ too.

So the new elements have the property we want. By structural induction, every element $(a,b) \in U$ has $a+b=0$.

(8) Examples of full binary trees with 13 vertices are



(9) Function $g(n)$ defined recursively

Basis step: $g(0) = 1$

Recursive step: $g(n+1) = 2g(n) + n - 1$

Prove that $g(n) = 2^n - n$ for $n \geq 0$.

We'll use the four steps of usual induction.

(A) $P(n)$ says $g(n) = 2^n - n$

(B) Basis step: $P(0)$ says $g(0) = 2^0 - 0$

$1 = 1 - 0$ ✓ true

(C) Inductive step: Assume $P(k)$ is true

So $g(k) = 2^k - k$ is true

By the recursive step for g we know

$$g(k+1) = 2\underbrace{g(k)}_{=2^k - k \text{ by } P(k)} + k - 1$$

So
$$g(k+1) = 2(2^k - k) + k - 1$$
$$= 2^{k+1} - 2k + k - 1$$
$$= 2^{k+1} - k - 1 = 2^{k+1} - (k+1)$$

This shows $P(k+1)$ is true and completes the inductive step.

(D) By induction $g(n) = 2^n - n$ for all $n \geq 0$.