Review of Chap. 9 Relations

Definitions we need:

- A relation from A to B is a subset of $A \times B$.
- A relation on A is a relation from A to A.
- A relation on a set A can be
  - reflexive
  - symmetric
  - antisymmetric
  - transitive
- A relation on a set A is an equivalence relation if it is reflexive, symmetric and transitive.

  (links elements that are the same in some way and makes a partition of A.)
- A relation on a set A is a partial order if it is reflexive, antisymmetric and transitive.

  In a partial order, a partially ordered set A is called a poset. Elements can be

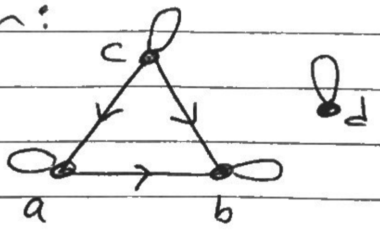
  - maximal
  - minimal
  - greatest
  - least.

Example ① Let $A = \{3, 7\}$, $B = \{p, q, r, s\}$. Give an example of a relation from A to B with 5 ordered pairs.

Answer: $\{(3,p), (3,r), (7,p), (7,q), (7,r)\}$.

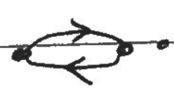
Example ② Let $R = \{(a,a), (b,b), (c,c), (d,d), (a,b), (c,a), (c,b)\}$, be a relation on $A = \{a,b,c,d\}$. Draw the digraph for R (directed graph) and also give its matrix representation.

Solution:



digraph

$$\begin{array}{c|cccc} & a & b & c & d \\ \hline a & & & & \\ b & & & & \\ c & & & & \\ d & & & & \end{array} \quad \begin{matrix} a \\ b \\ c \\ d \end{matrix}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

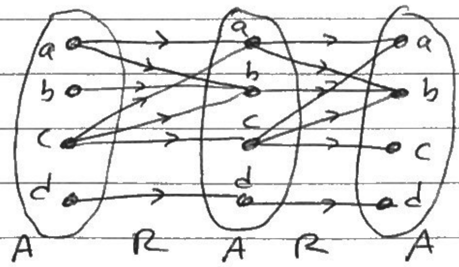
We see R is reflexive, not symmetric because missing (b,a) for example. It is antisymmetric because never have 

It is also transitive  (can't see any problems on digraph).

So R in example ② is a partial order.

Example (3) For R in example 2, compute the composition $R^2 = R \circ R$.

Solution: Best way to do this



for example, arrows connect c on left to b on right

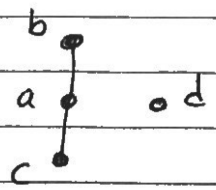
$$R^2 = \{(a,a), (a,b), (b,b), (c,a), (c,b), (c,c), (d,d)\}$$

We see that $R^2 = R$ in this case. If

$$\boxed{R^2 \subseteq R}$$

(If and only if)

then R is transitive. This confirms that R is transitive.



Hasse diagram for R

Here b, d maximal, c, d minimal, no greatest, no least elements.

R is not a <u>total order</u> because it has elements that are not <u>comparable</u> (for example c and d.

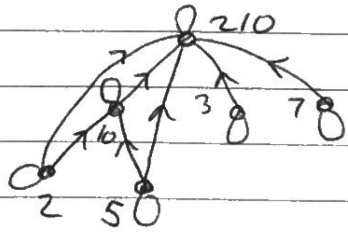
Example (4) draw the Hesse diagram for

$$(\{2,3,5,7,10,210\}, \mid).$$

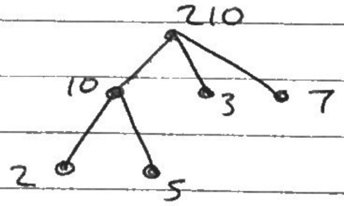
Solution: The relation here "$\mid$" is divisibility.

For example $2 \mid 10$ is true but $2 \mid 7$ false.

Digraph is



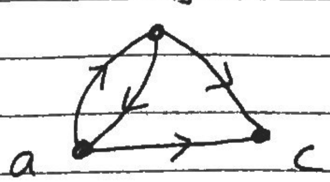
So Hasse diagram is



maximal: 210
minimal: 2, 3, 5, 7
greatest: 210
least: none

Example (5) Let $P = \{(a,b), (a,c), (b,a), (b,c)\}$ be a relation on $S = \{a,b,c\}$.
Draw the digraph for P, give its matrix, and decide if it is reflexive, symmetric, or antisymmetric.

Solution:



digraph

$$M_P = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

matrix

ρ is

- not reflexive, eg. (a,a) missing
- not symmetric, eg. (c,a) missing
- not antisymmetric, because has (a,b) and (b,a).

Example ⑥

Compute $M_\rho \cdot M_\rho$, the matrix product and $M_\rho \odot M_\rho$, the Boolean product.

Since $M_\rho \odot M_\rho = M_{\rho^2}$, use this to check if $\rho^2 \subseteq \rho$ and if ρ is transitive.

Solution:
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = M_\rho \cdot M_\rho$$

Replace any numbers bigger than 1 in by 1 to get $M_\rho \odot M_\rho$, so

$$M_\rho \odot M_\rho = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = M_{\rho^2}$$

We see this M_{ρ^2} has some 1s that M_ρ didn't have so $\rho^2 \not\subseteq \rho$ and ρ is not transitive.

(for example $(a,a) \in \rho^2$ and we need (a,a) for ρ to be transitive since $(a,b) \in \rho$ and $(b,a) \in \rho$.)

Example ⑦ Let $S = \{1,2,3,4,5\}$ and let

$$R = \{(a,b) \mid a \equiv b \mod 3\}$$

be a relation on S. Write out R as a list of ordered pairs. Then give the partition of S made by the distinct equivalence classes.

Solution: Remember $a \equiv b \mod 3$ means

3 divides the difference $a-b$. So for example $(5,2) \in R$ because $3 \mid 5-2$. Also $(2,5) \in R$ and $(2,2)$ because $3 \mid 0$.

$$R = \{(1,1),(2,2),(3,3),(4,4),(5,5),\\ (1,4),(4,1),(2,5),(5,2)\}.$$

$[1] = [1]_R$ means everything in S that is related to 1.

$$\left.\begin{aligned} [1] &= \{1,4\} \\ [2] &= \{2,5\} \\ [3] &= \{3\} \\ [4] &= \{1,4\} \\ [5] &= \{2,5\} \end{aligned}\right\}$$

these are all the equivalence classes.

Can ignore the repeats $[4]$, $[5]$ and get the partition