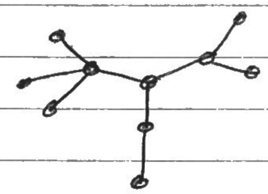
11.1 Trees

Trees are a special type of graph.
Recall the definitions for path, circuit, simple circuit and connected.

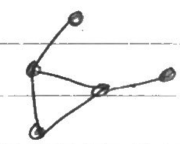
Definition: A tree is a connected graph with no simple circuits.

Examples

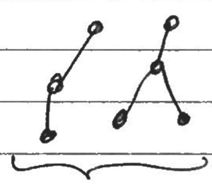


tree

tree



not a tree



not a tree

← this is a collection of 2 trees

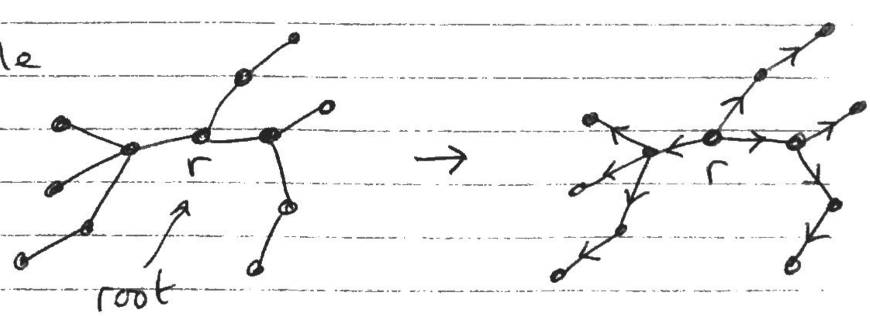
(collections of trees are called forests).

Property of trees: there is a unique simple path between any two vertices.

Definition: A rooted tree is just a tree where one vertex has been labelled the root.

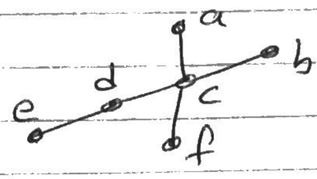
This makes the tree into a directed graph since we can direct every edge away from the root.

For example

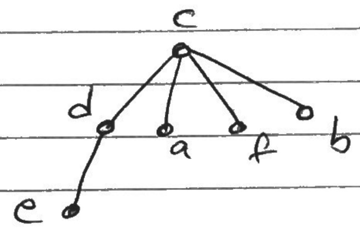


If you choose a different vertex for the root you will get a different directed graph. Usually we draw it so that the root is at the top and all arrows point down.

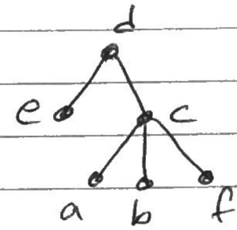
Example



choose c as root get



choose d as root



G

For a rooted tree such as G there is a lot of terminology to describe the vertices. See pages 747-748 of the text for

- parent
- child
- siblings
- ancestors
- descendants
- leaf
- internal vertex
- subtree rooted at a vertex

think of a family tree.

For example, in G, c is the parent of a, b, f and a, b, f are the children of c. Also c and d are the ancestors of a.

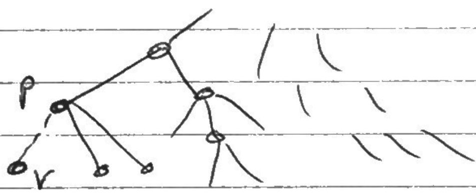
The leaves of G are e, a, b, f and the internal vertices are c, d.

Theorem A tree with $n \geq 1$ vertices has $n-1$ edges.

Proof. Let $P(n)$ be this statement and we'll prove it by induction.

Basis case: $P(1)$ is true   tree = •

Inductive step: Assume true $P(k)$ and use to prove $P(k+1)$. By choosing a root we may assume a tree T with $k+1$ vertices is rooted. It must have a leaf v with parent p

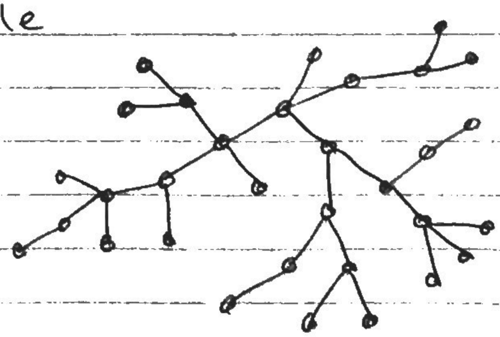


Remove the edge (p,v) and the vertex v. A tree with k vertices remains, so by $P(k)$ it has $k-1$ edges. Therefore T has k edges. This proves the induction step.

So $P(n)$ is true for all $n \geq 1$. □

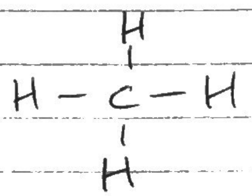
The converse of this theorem is also true: A connected graph with n vertices and $n-1$ edges must be a tree.

Example

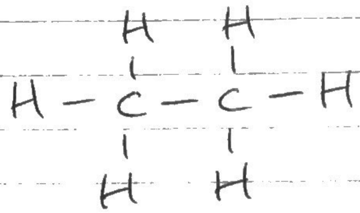


vertices?
edges?

Hydrocarbon models



methanol



Ethanol

These are examples of saturated hydrocarbon molecules C_nH_{2n+2} (see p 750).

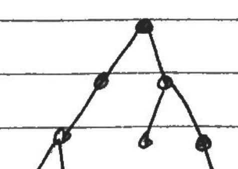
Each carbon is a vertex with degree 4 and each Hydrogen a vertex with degree 1.

So C_nH_{2n+2} has $3n+2$ vertices. Its total degree is $4n + 1(2n+2)$. This is twice the number of edges by the Handshaking theorem, so $3n+1$ edges.

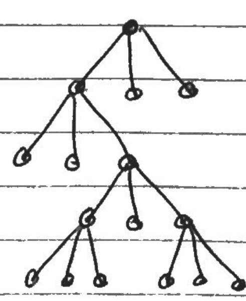
By our previous result C_nH_{2n+2} must be a tree.

Definition: A rooted tree is called an <u>m-ary tree</u> if every internal vertex has at most m children. If they have exactly m children then called a <u>full m-ary tree</u>.

examples



2-ary tree
(binary tree)



a full 3-ary tree

<u>Theorem</u> A full m-ary tree with i internal vertices has a total of $mi+1$ vertices.

<u>Proof</u> Use induction on i. $\square$

<u>2^{nd} Proof</u> Let n be the total number of vertices. Then the number of leaves is $n-i$.

The total degree is $\underbrace{m}_{\text{root}} + \underbrace{(m+1)(i-1)}_{\text{other internal}} + \underbrace{n-i}_{\text{leaves}}$

The tree has $n-1$ edges so

$$2(n-1) = m + (m+1)(i-1) + n - i.$$

Solve for n: $n = mi+1$. $\square$

<u>Example 9 p753</u>

Someone starts a chain email. Each person who receives the letter forwards it to either 4 or 0 new people. Suppose the chain ends when 100 get the letter but don't send it out. How many people did forward it?

Solution. We see the email chain makes a 4-ary tree. Suppose it has n vertices in total with i internal vertices. By the Theorem above $n = 4i+1$. The number of leaves is $n-i$ and this must equal 100, so $n-i=100$.

This gives $100+i = n = 4i+1$

$$99 = 3i, \quad i = 33$$

So 33 people forwarded the email.

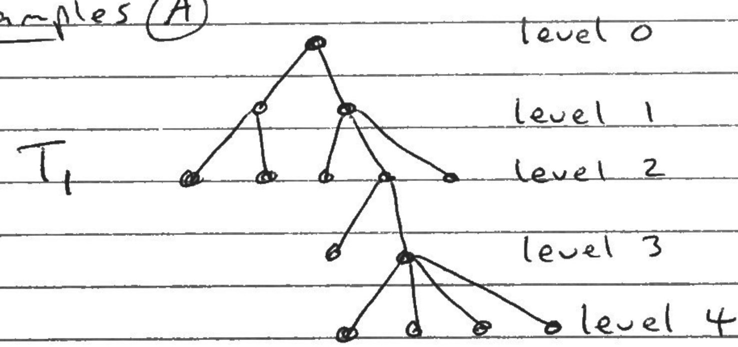
Some last terminology for rooted trees.

The <u>level</u> of a vertex is the length of the path from that vertex to the root.

The <u>height</u> of a rooted tree is the biggest level of any vertex.

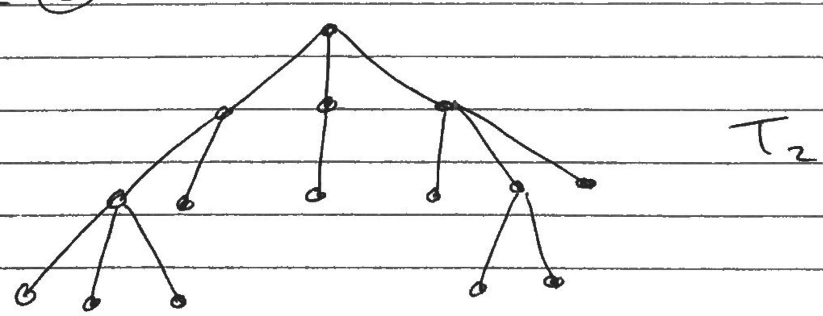
Suppose a rooted tree has height h. Then it is <u>balanced</u> if all leaves are at level h or $h-1$.

<u>Examples</u> (A)



This tree T_1 has height 4 and is not balanced.

Example (B)



This tree T_2 has height 3 and is balanced. It is a 3-ary tree.