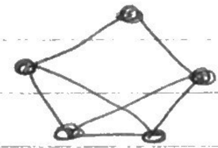
Review of Chapter 10 Graphs.

Graphs are made up of vertices and edges:



example of a *simple graph*

A *multigraph* allows 

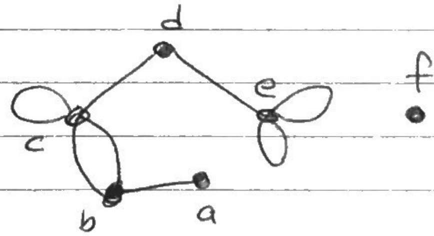
A *pseudograph* also allows loops 

In a *directed* graph every edge has an arrow 

The *degree* of a vertex in a graph is the number of edges that have it as an endpoint (where loops add 2 to the degree).

Example (1)

G



For the graph G find the degree of every vertex and the total degree.

Solution:

$\deg(a) = 1$
$\deg(b) = 3$
$\deg(c) = 5$
$\deg(d) = 2$
$\deg(e) = 5$
$\deg(f) = 0$

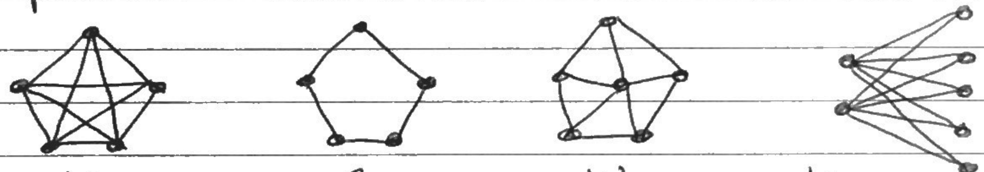
total degree 16

The *Handshaking Theorem* says that the total degree equals twice the number of edges in any pseudograph.

Know how to draw graphs from the different families

- K_n complete graphs (n vertices)
- C_n cycle graphs (n vertices)
- W_n wheel graphs ($n+1$ vertices)
- Q_n cube graphs (2^n vertices)
- $K_{n,m}$ complete bipartite ($n+m$ vertices).

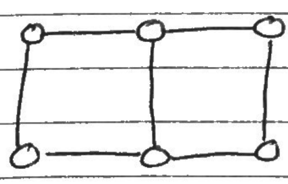
examples



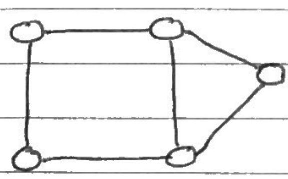
K_5 C_5 W_5 $K_{2,5}$

A graph is bipartite if its vertices can be made into two groups so that every edge has its endpoints in different groups.

A good way to check if a graph is bipartite is to see if you can color its vertices with 2 (or 1) colors so that edges only connect different color vertices.



bipartite

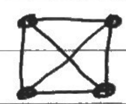


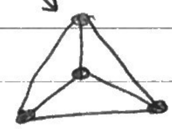
not bipartite.

Graph isomorphism

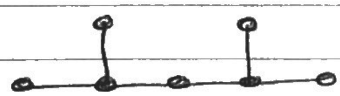
Two graphs are isomorphic if they are really the same – though maybe drawn differently.

For example K_4 and W_3 are isomorphic





If two graphs have different numbers of edges, vertices or different degree lists then they are not isomorphic. You have to be careful though.



G

H

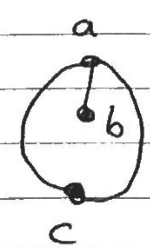
degree lists 1,1,1,1,2,3,3 1,1,1,1,2,3,3

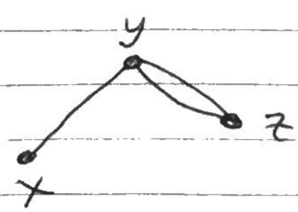
G and H here are not isomorphic.

H has two adjacent vertices of degree 3 and G does not.

Example (2) Show that these graphs are isomorphic

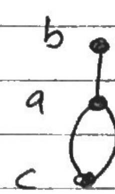
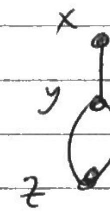
G_1





G_2

Solution: If we draw them a bit differently we can see what matches

G_1   G_2

We want
$$\left.\begin{array}{l} f(a) = y \\ f(b) = x \\ f(c) = z \end{array}\right\}$$
f connects matching vertices

$$A_{G_1} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \end{array} \qquad A_{G_2} = \begin{array}{c c} & \begin{array}{ccc} y & x & z \end{array} \\ \begin{array}{c} y \\ x \\ z \end{array} & \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \end{array}$$

same

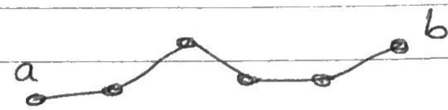
This proves that G_1 and G_2 are isomorphic.

Another application of adjacency matrices is in counting paths. For example the n^{th} power of A_{G_1} counts paths of length n in G_1 above.

$$A_{G_1}^2 = A_{G_1} A_{G_1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} = \begin{array}{c c} & \begin{array}{ccc} a & b & c \end{array} \\ \begin{array}{c} a \\ b \\ c \end{array} & \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \end{array}$$

So there are 5 paths of length 2 from a to a and 2 paths of length 2 from b to c. Can you find them?

Paths and circuits

A path looks like this: 

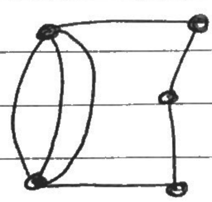
path of length 5 from a to b

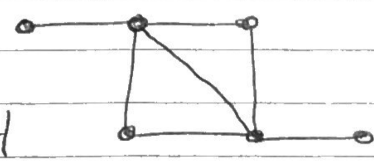
It is simple if it does not use an edge more than once. It is a circuit if it finishes where it starts.

A graph is connected if there is a path between any two vertices. Is ex 1 connected?

An Euler path uses every edge in a graph exactly once. An Euler circuit is a circuit using every edge once.

Theorem A A connected multigraph has an Euler circuit if and only if every vertex has even degree.

G 

H 

Theorem B A connected multigraph has an Euler path if and only if at most two vertices have odd degree. The path must start and finish at the odd degree vertices.

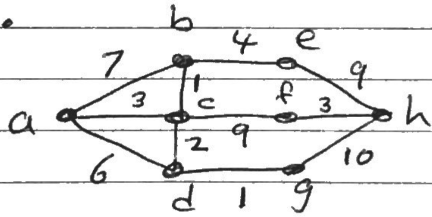
See if you can find Euler circuits and paths for G, H above.

A Hamilton path passes through every vertex exactly once.

Shortest paths

If every edge is given a number then we call this a weighted graph. Finding the path with the smallest sum of weights between two vertices is an important problem.

Example (3) Find the shortest path from a to h.



Can you see the path of length 15?

Dijkstra's Algorithm gives an efficient method for computers to find shortest paths. Review this in the notes.

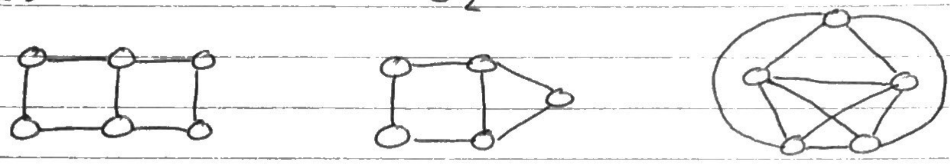
Graph coloring

A coloring of a graph means giving every vertex a color so that adjacent vertices have different colors.

The chromatic number of a graph G has notation $\chi(G)$ and means the smallest number of colors needed to color G.

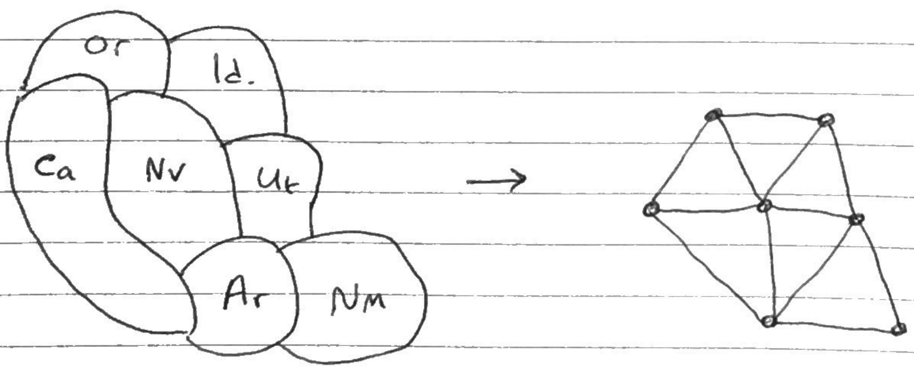
Examples

G_1 G_2 G_3



$\chi(G_1) = 2$ $\chi(G_2) = 3$ $\chi(G_3) = 5$

We can use graph coloring to color maps as well. Make the <u>dual graph</u> of a map by giving each region a vertex and two regions with a common border get an edge. Now coloring the dual graph corresponds to coloring the map so regions with a common border have different colors.



Map dual graph

$\chi = 4$

both need 4 colors.