# New York State Mathematics Association of Two-Year Colleges

## Math League Contest ~ Spring 2015

Directions: You have one hour to take this test. Scrap paper is allowed. The use of calculators is NOT permitted, as well as computers, books, math tables, and notes of any kind. You are not expected to answer all the questions. However, do not spend too much time on any one problem. Four points are awarded for each correct answer, one point is deducted for each incorrect answer, and no points are awarded/deducted for blank responses. There is no partial credit. Unless otherwise indicated, answers must given in *exact* form, i.e. in terms of fractions, radicals, $\pi$, etc. NOTE: NOTA = None Of These Answers.

1. The *mod* function is defined as: $m \bmod n =$ "the remainder when $m$ is divided by $n$," for integers $m$ and $n$. For example: $8 \bmod 5 = 3$, $8 \bmod 2 = 0$, and $8 \bmod 9 = 8$. If $(x \bmod y) + (y \bmod x) = 1$, with $x$ and $y$ positive integers, then which of the following must be true?

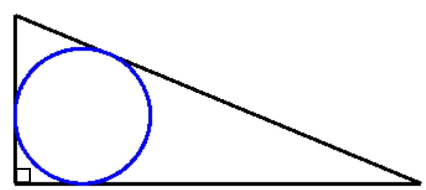
   I. $x = 1$ or $y = 1$ II. $|x - y| = 1$ III. $x \neq y$

   A) I only B) I and III only C) II only D) II and III only

2. How many ways can we obtain \$20.15 using *only* quarters and dimes?

3. The hyperbolic cosine function is defined as: $\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$. Thus, the inverse hyperbolic cosine function, $\cosh^{-1}(x)$, for $x \geq 0$ is

   A) $2\ln\left(e^x + e^{-x}\right)$ B) $2\ln\left(\frac{1+e^x}{1+e^{-x}}\right)$ C) $\ln\left(x - \sqrt{x^2-1}\right)$ D) $\ln\left(x + \sqrt{x^2-1}\right)$



4. What is the length of the radius of a circle that is inscribed in a 5-12-13 right triangle?

5. What is $2015^{2016} \bmod 9$? See problem 1 for the definition of *mod*.

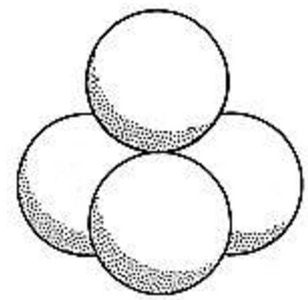
6. The sum of all solutions to the equation: $x|x| - 6x + 7 = 0$, where $|x|$ is the absolute value of $x$, is

   A) $-1$ B) $0$ C) $6$ D) $7$

7. The floor function, $\lfloor x \rfloor$, is defined as *the greatest integer less than or equal to x*. For example, $\lfloor 1.9 \rfloor = 1$, $\lfloor 2.1 \rfloor = 2$, $\lfloor -2.1 \rfloor = -3$ and $\lfloor 7 \rfloor = 7$. What is the numerical value of $\sum_{n=1}^{50} \lfloor \sqrt{n} \rfloor$?

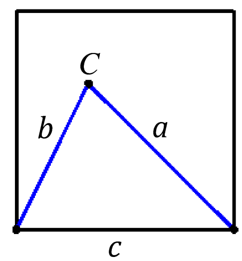
Note: $\sum_{n=1}^{50} \lfloor \sqrt{n} \rfloor = \lfloor \sqrt{1} \rfloor + \lfloor \sqrt{2} \rfloor + \ldots + \lfloor \sqrt{50} \rfloor$

8. Three mutually tangent spheres of radius 1 rest on a horizontal plane. A fourth sphere, also with a radius of length 1, rests on top of the other three so it is tangent to each of them (as shown). What is the distance from the plane to the top of the fourth sphere?



A) $2+\sqrt{2}$ B) $2+\frac{2}{3}\sqrt{6}$ C) $2+\sqrt{3}$ D) $3+\frac{1}{2}\sqrt{3}$

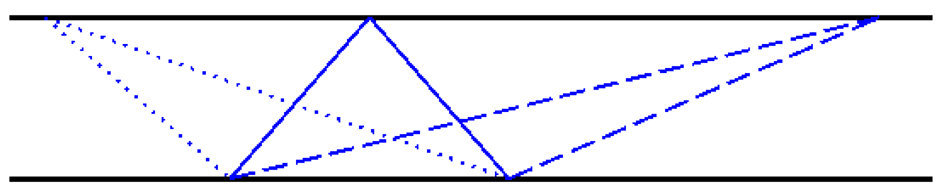
9. A point, $C$, is selected at random inside a square, we then form a triangle with sides $a$, $b$, and $c$ using the bottom corners for the other two points (as shown). What is the probability the triangle is obtuse?




A) $\frac{1}{4}$ B) $\frac{\sqrt{2}}{4}$ C) $\frac{\pi}{8}$ D) $\frac{1}{2}$

10. The accompanying diagram shows three different triangles with a common base drawn between two parallel lines. Let $T_1$ represent the triangle with the dotted lines, $T_2$ represent the triangle with the solid lines, and $T_3$ represent the triangle with the dashed lines. Which triangle has the <u>smallest</u> area?

A) $T_1$
B) $T_3$
C) The areas are all the same.
D) There is not enough information.



11. What real value for $b$ solves this equation $\dfrac{\log_{b-1}(b)}{\log_{b+1}(b)} = 2$?

12. The mathematician Augustus De Morgan lived his entire life in the 1800's. As a young man, when asked his age, he responded "I will be $x$ years old in year $x^2$." How old was he when that occurred?

13. Two people are standing at the base of a tall building. Person 1 walks due east, while Person 2 walks, at a quicker pace, due north. After a while they both stop. Person 1 needs to tilt his head up 45° (from horizontal) to see the top of the building, while Person 2 needs to tilt her head up only 30°. If the distance between the two people is 2015 feet, then how tall (in feet) is the building? Neglect the height of the people, and assume (of course) the building is perpendicular to the level ground.

14. Which two functions when graphed will intersect at *exactly* three points?

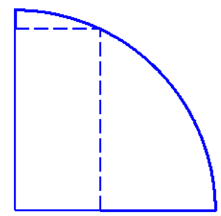
A) $y = 3^x$ and $y = x^6$  B) $y = 3^x$ and $y = \log_3(x)$

C) $y = 3^x$ and $y = \tan(x)$  D) NOTA

15. A line is drawn from point A: (1, 2) to point B: (3, 4), then a triangle is formed by the addition of another point, C, so that the sum of the slopes of the three sides is one. Which angle(s) *could* be a right angle?

A) $\angle$A or $\angle$B only  B) $\angle$C only  C) $\angle$A, $\angle$B or $\angle$C  D) none of them

16. A rectangle is inscribed in a quarter-circle of radius 6, as shown, so that the sum of the width and height is 8. What is the area of the rectangle?



17. If $\cos(x) + 2\sin(x) = 2$ with $\cos(x) \neq 0$, then what is $\tan(x)$?

A) $\frac{2}{3}$  B) $\frac{3}{4}$  C) $\frac{2}{\sqrt{5}}$  D) $\frac{\sqrt{5}}{2}$

18. If $x^2 + y^2 = 10x - 4y - 29$, and $x$ and $y$ are both real numbers, then what is $x + y$?

A) 1  B) 2  C) 3  D) There are infinitely many possible values.

19. The infinite sum $\sum_{n=1}^{\infty} \frac{n}{3^n} = \frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \ldots + \frac{n}{3^n} + \ldots$ converges to which value?

A) $\frac{3}{4}$  B) $\frac{7}{9}$  C) $\frac{8}{9}$  D) 1

20. There are three boxes, one contains only apples, one contains only oranges, and one contains both apples and oranges. The boxes have been incorrectly labeled such that no label correctly identifies the actual contents of the box it labels. Opening just one box, and without looking in the box, you take out one piece of fruit (and without feeling any other fruit). By looking at the fruit you just selected, you are immediately able to, and without any doubt, correctly re-label all the boxes. From which box must you have chosen the fruit?

A) The one labeled *APPLES*.
B) The one labeled *ORANGES*.
C) The one labeled *APPLES & ORANGES*.
D) It does not matter, it is impossible to be absolutely certain.

# New York State Mathematics Association of Two-Year Colleges

## Math League Contest ~ Spring 2015 ~ Solutions

**1.** Let's consider the three cases: ① $x = y$, ② $x < y$, and ③ $x > y$.

① If $x = y$, then $(x \bmod x) + (x \bmod x) = 0 + 0 = 0$. Thus, choice III must be true.

② If $x < y$, then $(x \bmod y) + (y \bmod x) = x + (y \bmod x)$. Since $x$ and $y$ are positive integers, this last sum can be 1 only if $x = 1$ (which makes $y \bmod x = 0$).

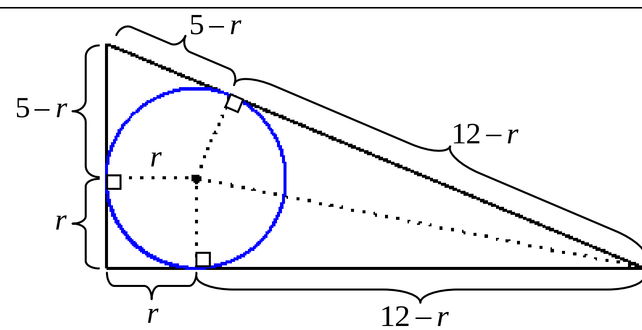
③ If $x > y$, this is the same as case ② with $x$ and $y$ switched, making $y = 1$.

Thus, choices I and III must be true. $\boxed{\text{Answer: B}}$

**2.** Let $x$ = the number of quarters used, and $y$ = the number of dimes used. Thus, $25x + 10y = 2015$, with $x$ and $y$ non-negative integers. Solving for $y$ gives: $y = \frac{403-5x}{2} = \frac{402+1-5x}{2} = 201 + \frac{1-5x}{2}$. Since $y$ must be an integer, $\frac{1-5x}{2}$ must be a whole number. Hence, $1-5x$ must be even, and $x$ must be odd. Therefore, $x$ may be 1, 3, 5, ..., 79 (81 quarters is \$20.25, too much!). Hence, there are 40 ways to obtain \$20.15 using only quarters and dimes. $\boxed{\text{Answer: 40}}$

**3.** Solving for $f^{-1}(x)$: $x = \frac{1}{2}\left(e^y + e^{-y}\right)$ and solve for $y$. $e^y + e^{-y} = 2x \Rightarrow e^{2y} + 1 = 2xe^y$, after multiplying by $e^y$, which gives $e^{2y} - 2xe^y + 1 = 0$ or $(e^y)^2 - 2x(e^y) + 1 = 0$. Using the quadratic formula to solve for $e^y$ yields: $e^y = \dfrac{2x \pm \sqrt{4x^2 - 4}}{2} = x \pm \sqrt{x^2 - 1} \Rightarrow y = \ln\left(x \pm \sqrt{x^2-1}\right)$. We take the positive square root, since $\ln\left(x - \sqrt{x^2-1}\right) \le 0$. $\boxed{\text{Answer: D}}$

**4.** Letting $r$ be the radius of the inscribed circle, gives the following diagram. Since the hypotenuse has a length of 13, $(5-r) + (12-r) = 13 \Rightarrow r = 2$. $\boxed{\text{Answer: 2}}$




**5.** The closest multiple of 9 to 2015 is 2016, thus if we write as $2015 = 2016 - 1$, $2015^{2016} = (2016-1)^{2016}$. Expanding this binomial would give 2017 terms, the first 2016 of them would have 2016 as a factor. The last would be $(-1)^{2016} = 1$. Thus, $2015^{2016}$ is 1 more than a multiple of 2016. Since 2016 is a multiple of 9, $2015^{2016}$ is also 1 more than a multiple of 9. Hence, the remainder is 1. $\boxed{\text{Answer: 1}}$

**6.** $x|x| - 6x + 7 = 0$ has two cases: ① $x \ge 0$ and ② $x < 0$.

① $x \ge 0 \Rightarrow |x| = x \Rightarrow x|x| - 6x + 7 = 0 \Rightarrow x^2 - 6x + 7 = 0$, which gives $x = 3 \pm \sqrt{2}$.

② $x < 0 \Rightarrow |x| = -x \Rightarrow x|x| - 6x + 7 = 0 \Rightarrow -x^2 - 6x + 7 = 0$, which gives $x = -7, 1$. However, $x < 0$, so $x = -7$ is the only solution in this case. The sum is $-7 + 3 + \sqrt{2} + 3 - \sqrt{2} = -1$. $\boxed{\text{Answer: A}}$

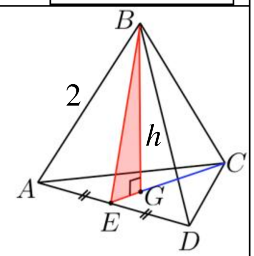
7. $\lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+\ldots+\lfloor\sqrt{50}\rfloor=$

$\lfloor\sqrt{1}\rfloor+\lfloor\sqrt{2}\rfloor+\lfloor\sqrt{3}\rfloor+\lfloor\sqrt{4}\rfloor+\lfloor\sqrt{5}\rfloor+\ldots+\lfloor\sqrt{8}\rfloor+\lfloor\sqrt{9}\rfloor+\lfloor\sqrt{10}\rfloor+\ldots+\lfloor\sqrt{15}\rfloor+\lfloor\sqrt{16}\rfloor+\lfloor\sqrt{17}\rfloor+\ldots$

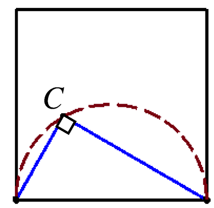
$+\lfloor\sqrt{24}\rfloor+\lfloor\sqrt{25}\rfloor+\lfloor\sqrt{26}\rfloor+\ldots+\lfloor\sqrt{35}\rfloor+\lfloor\sqrt{36}\rfloor+\lfloor\sqrt{37}\rfloor+\ldots+\lfloor\sqrt{48}\rfloor+\lfloor\sqrt{49}\rfloor+\lfloor\sqrt{50}\rfloor$

$=\underbrace{1+1+1}_{3}+\underbrace{2+2+\ldots+2}_{5}+\underbrace{3+3+\ldots+3}_{7}+\underbrace{4+4+\ldots+4}_{9}+\underbrace{5+5+\ldots+5}_{11}+\underbrace{6+6+\ldots+6}_{13}+7+7$

$=3\cdot1+5\cdot2+7\cdot3+9\cdot4+11\cdot5+13\cdot6+2\cdot7=3+10+21+36+55+78+14=217$ $\boxed{\text{Answer: } 217}$

8. 


The centers of the four spheres forms a regular tetrahedron, i.e. a solid with four equilateral triangular faces. Let $h$ be the height of this tetrahedron, then the height we seek (the distance from the plane to the top of the 4th sphere) is $h+2$. The edges have a length of 2, since they are formed by the radii of adjacent spheres. The highlighted triangle, shown, has height $h$ with hypotenuse $BE=\sqrt{3}$, since it is the height of an equilateral face, and base $EG=\frac{1}{3}\sqrt{3}$, since the medians of a triangle meet $\frac{1}{3}$ of the way from each base. Thus, the Pythagorean theorem gives $\left(\frac{1}{3}\sqrt{3}\right)^2+h^2=\sqrt{3}^2$, or $h^2=\frac{8}{3}$. Thus, $h=\sqrt{\frac{8}{3}}=\frac{2}{3}\sqrt{6}$. Hence, the height is $2+\frac{2}{3}\sqrt{6}$. $\boxed{\text{Answer: B}}$

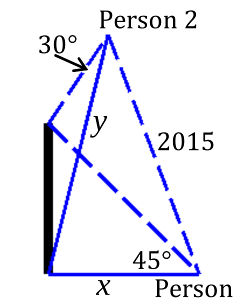
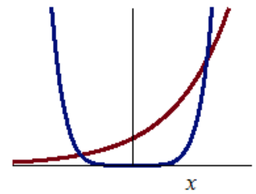
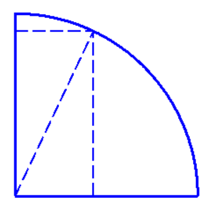
9. 


We know that the triangle formed with point $C$ on the semi-circle makes $\angle C$ a right angle. Hence, we need point $C$ to be on the interior of the semi-circle to form an obtuse triangle. Without loss of generality, let the square have side length 2, thus the semi-circle has radius 1. The area of the square is 4, while the area of the semi-circle is $\frac{1}{2}\pi\cdot1^2=\frac{1}{2}\pi$. Therefore, the probability of obtaining an obtuse triangle is $\frac{\frac{1}{2}\pi}{4}=\frac{\pi}{8}$. $\boxed{\text{Answer: C}}$

10. The area of a triangle is determined by the length of its base and height. They all have the same base, and since height is the perpendicular distance from the base to the highest point, they also all have the same height (because the lines are parallel). Hence, all areas are the same. $\boxed{\text{Answer: C}}$

11. Since $\dfrac{\log_{b-1}(b)}{\log_{b+1}(b)}=\dfrac{\left(\frac{\ln(b)}{\ln(b-1)}\right)}{\left(\frac{\ln(b)}{\ln(b+1)}\right)}=\dfrac{\ln(b)}{\ln(b-1)}\cdot\dfrac{\ln(b+1)}{\ln(b)}=\dfrac{\ln(b+1)}{\ln(b-1)}=\log_{b-1}(b+1)$, we can now write the equation as: $\log_{b-1}(b+1)=2 \Rightarrow (b-1)^2=b+1 \Rightarrow b^2-2b+1=b+1 \Rightarrow b^2-3b=0$. Solving for $b$ yields: $b(b-3)=0 \Rightarrow b=0$ or $b=3$. Since $b>1$, $b=3$. $\boxed{\text{Answer: } 3}$

12. Let $y=$ the year in which De Morgan was born. Thus, the relationship is expressed as: $x^2=y+x$. Solving $x^2-x-y=0$ for $x$ using the quadratic formula gives: $x=\frac{1+\sqrt{1+4y}}{2}$, taking the positive square root since $x$ cannot be negative. Thus, $1+4y$ must be a perfect square $\Rightarrow 1+4y=m^2$, or $4y=m^2-1=(m-1)(m+1)$ for some integer $m$. Since $(m-1)(m+1)$ is a multiple of 4, $m-1$ and $m+1$ must both be even, and hence, consecutive even numbers. Also, $y=\frac{(m-1)(m+1)}{4}=\frac{m-1}{2}\cdot\frac{m+1}{2}$, which indicates that De Morgan's year of birth must be the product of consecutive integers. Now we just need to determine what year(s) in the 1800's is the product of two consecutive integers. Noting that $40\cdot41=1640$, we continue: $41\cdot42=1722$, $42\cdot43=1806$, $43\cdot44=1892$, and $44\cdot45=1980$. The only two years that fall in the 1800's is 1806 and 1892. If $y=1806$, then $m=85$ and $x=43$. If $y=1892$, then $m=87$ and $x=44$, making De Morgan live into the 1900's. Thus, De Morgan was 43 years old in 1849 (which is $43^2$). $\boxed{\text{Answer: } 43}$

| | |
|---|---|
| 13. | Let = the height of the building, $x$ = the distance Person 1 is from the building, and $y$ = the distance Person 2 is from the building, as shown in the diagram (not drawn to scale). The Pythagorean Theorem yields: $x^2+y^2=2015^2$. Using the relationships for the sides of a 30°-60°-90° right triangle and a 45°-45°-90°, we have: $x=$ and $y=\sqrt{3}$ . Thus, $^2+\left(\sqrt{3}\ \right)^2=2015^2 \Rightarrow 4\ ^2=2015^2$ $\Rightarrow 2\ =2015$. This gives $=\frac{2015}{2}=1007.5$ feet. $\boxed{\text{Answer: 1007.5}}$   |
| 14. | A rough sketch of $y=3^x$ and $y=x^6$ gives the impression that there are only two points of intersection. However, exponential growth *always*, eventually, dominates polynomial (or power) growth. Hence, there must be another point of intersection, beyond which $3^x>x^6$ for all greater *x*-values (which turns out to be about 14.67, the other two *x*-values are about $-0.86$ and 1.26). $\boxed{\text{Answer: A}}$   Choice B: The curves do not intersect; Choice C: The curves intersect at infinitely many points. |
| 15. | The slope from A: $(1,2)$ to B: $(3,4)$ is 1. If a right angle is formed at either A or B, then the slope of the joining side would be $-1$ (i.e. the negative reciprocal of 1). Adding those slopes give zero. Hence, the 3rd side would have to have a slope of 1 to make the sum of all slopes 1. However, that would make it parallel to the line joining A and B, which is impossible! If the right angle is at point C, then the sum of the slopes of the two new legs (from A to C and B to C) must be zero. Suppose the slope from A to C is *m*, then the slope from B to C would be $-\frac{1}{m}$, which means $-\frac{1}{m}+m=0$. whose only solution is $m=1$, which is impossible (as explained above). $\boxed{\text{Answer: D}}$ |
| 16. | Let $x$ = the width of the rectangle, and $y$ = the height of the rectangle. Thus, $x+y=8$ and $x^2+y^2=6^2$, since the diagonal of the rectangle is a radius of the circle. Substituting $y=8-x$ into the other equation, gives $x^2+(8-x)^2=36$, which yields $x^2-8x+14=0$. The quadratic formula gives $x=4\pm\sqrt{2}$. If we take $x=4-\sqrt{2}$, then $y=4+\sqrt{2}$, and the area is $\left(4-\sqrt{2}\right)\left(4+\sqrt{2}\right)=16-2=14$. If we take $x=4+\sqrt{2}$, then $y=4-\sqrt{2}$, and the area is still 14. $\boxed{\text{Answer: 14}}$  |
| 17. | Dividing $\cos(x)+2\sin(x)=2$ by $\cos(x)$, gives: $1+2\tan(x)=2\sec(x)$. Now square both sides to get: $1+4\tan(x)+4\tan^2(x)=4\sec^2(x)$. Using the identity $\sec^2(x)=1+\tan^2(x)$, gives us $1+4\tan(x)+4\tan^2(x)=4\left(1+\tan^2(x)\right)$, which simplifies to $\tan(x)=\frac{3}{4}$. $\boxed{\text{Answer: B}}$ |

| | |
|---|---|
| 18. | $x^2+y^2=10x-4y-29 \Rightarrow x^2-10x+\left(y^2+4y+29\right)=0$, solving for $x$ in terms of $y$ with the quadratic formula gives: $x=\dfrac{10\pm\sqrt{100-4\left(y^2+4y+29\right)}}{2}=\dfrac{10\pm\sqrt{-4y^2-16y-16}}{2}=\dfrac{10\pm\sqrt{-4(y+2)^2}}{2}$.<br>In order for $x$ to be a real number, $-4(y+2)^2=0$ or $y=-2$. Thus, $x=\frac{10\pm 0}{2}=5$ and $x+y=3$.<br>OR ... Complete-the-Square for the $x$-variables and the $y$-variables:<br>$x^2-10x+y^2+4y=-29 \Rightarrow x^2-10x+(-5)^2+y^2+4y+2^2=-29+(-5)^2+2^2$<br>$\Rightarrow (x-5)^2+(y+2)^2=0 \Rightarrow x=5$ and $y=-2 \Rightarrow x+y=3$ Answer: C |
| 19. | Let ① $S=\frac{1}{3}+\frac{2}{3^2}+\frac{3}{3^3}+\frac{4}{3^4}+\ldots+\frac{n-1}{3^{n-1}}+\frac{n}{3^n}+\frac{n+1}{3^{n+1}}+\ldots$. Multiplying by 3 gives:<br>② $3S=1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots+\frac{n-1}{3^{n-2}}+\frac{n}{3^{n-1}}+\frac{n+1}{3^n}+\ldots$. Now subtracting ① from ② gives:<br>③ $2S=1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots+\frac{1}{3^{n-2}}+\frac{1}{3^{n-1}}+\frac{1}{3^n}+\ldots$, which is a geometric series!<br>Using the formula for an infinite geometric series, $\displaystyle\sum_{n=0}^{\infty} r^n=\frac{1}{1-r}$ for $-1<r<1$, with $r=\frac{1}{3}$ gives us:<br>$2S=\dfrac{1}{1-\frac{1}{3}} \Rightarrow 2S=\dfrac{3}{2} \Rightarrow S=\dfrac{3}{4}$. Answer: A<br>In case we do not know the formula for an infinite geometric series, we can repeat the steps above to solve for $S$. |
| 20. | Select a fruit from the box labeled *APPLES & ORANGES*. If it is an apple, then we know that box is all apples (since it is incorrectly labeled and cannot contain both apples and oranges). Similarly, if it is an orange, then we know that box is all oranges. Once that box has correctly been labeled, then the other boxes can easily be labeled by default. Answer: C |