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Connectivity and Dimension of the p-locus in moduli space

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Abstract. We construct a finite simplicial complex whose vertices correspond to the (p, n)-gonal loci in the moduli space of compact Riemann surfaces of genus $g > 1$. Here p is a fixed odd prime and n is the genus of the quotient surface resulting from the action of a cyclic group of order p on a surface of genus g. Edges between vertices (p, n) and (p, m) exist whenever there is a surface which is both (p, n)- and (p, m)-gonal. We determine the connected components of the complex, and an upper bound on the dimension of the largest simplex. We show the bound is attained for sufficiently large g. The complex is a first step toward a potential cohomological approach to the singular locus of the moduli space.

1. Introduction

The singular locus in the moduli space $\mathcal{M}_g$, $g > 1$, consists of conformal equivalence classes of compact Riemann surfaces of genus g with a non-trivial automorphism group. It is covered by its p-loci, consisting of surfaces admitting an automorphism of prime order p ([**8**], [**9**]). The covering is finite, since only finitely many primes p act in any given genus [**12**]. By 'covering' we simply mean a decomposition into a non-disjoint union of subsets. A finer (but still finite) covering consists of (p, n)-*gonal* subloci, consisting of surfaces for which the quotient modulo the p action has fixed genus $n \geq 0$.

In this paper, for fixed odd prime p and fixed genus $g > 1$, we construct a finite simplicial complex $\mathcal{G}_g^p$, the *p-local complex*, whose vertices are (p, n)-gonal loci in $\mathcal{M}_g$. An edge (1-simplex) is drawn between two vertices (p, n) and (p, m), $n \neq m$, if and only if the corresponding loci have non-empty intersection, i.e., there exists a surface of genus g which is both (p, n)-gonal and (p, m)-gonal. We show that $\mathcal{G}_g^p$, when non-empty, has at most one non-trivial path-connected component, and possibly some isolated vertices. The non-trivial path-connected component, when it exists, is spanned by a "star-like" tree – any two vertices are joined by a path of length at most two passing through a unique "central" vertex (see Figure 1). We also determine an upper bound on the *geometric dimension* d of $\mathcal{G}_g^p$, which

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is the highest dimension of a simplex; equivalently, d is the largest integer such that a surface of genus g is simultaneously (p, n_1)-, (p, n_2)-, ..., (p, n_{d+1})-gonal, for pairwise distinct $n_1, n_2, \ldots n_{d+1}$. We show that the bound is attained for sufficiently large g. Knowledge of the connectivity and geometric dimension of the p-loci should shed some light on corresponding properties of the full singular locus, a subject of long-standing and still-current interest [**2**], [**16**].

2. Preliminaries

In discussing group actions on compact Riemann surfaces of genus $g > 1$, we use the uniformization approach originating with Klein, Poincaré and Koebe, and reinvigorated by Macbeath in 1961 [**17**], which describes the actions in terms of covering actions by Fuchsian groups on the upper half-plane $\mathcal{H}$ endowed with the hyperbolic metric $ds^2 = (dx^2 + dy^2)/y^2$.

A *Fuchsian group* is a discrete group of orientation-preserving isometries of $\mathcal{H}$, the full group being isomorphic to $\mathrm{PSL}(2, \mathbb{R})$. We consider only co-compact Fuchsian groups, having compact orbit (quotient) space, and henceforth this is what we mean when we use the term Fuchsian group. The orbit space inherits the complex structure from $\mathcal{H}$ and hence becomes a compact Riemann surface. A Fuchsian group Λ has a *signature* of the form $(h; m_1, \ldots, m_r)$ indicating that the orbit genus (genus of $\mathcal{H}/\Lambda$) is $h \geq 0$, and the quotient map $\pi\colon \mathcal{H} \to \mathcal{H}/\Lambda$ branches over r points with ramification indices $m_1, \ldots, m_r > 1$. When $r = 0$, the signature is written $(h; -)$ and Λ is called a *surface group.* The signature determines a presentation for Λ as follows:

$$\left\langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \gamma_1, \ldots, \gamma_r \middle| \gamma_1^{m_1}, \ldots, \gamma_r^{m_r}, \prod_{i=1}^{h} [\alpha_j, \beta_j] \prod_{j=1}^{r} \gamma_i \right\rangle$$

The generators $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ are *hyperbolic* isometries, of infinite order and having no fixed points in $\mathcal{H}$ (but two on the ideal boundary $y = 0$), while $\gamma_1, \ldots, \gamma_r$ are *elliptic* isometries of maximal finite order having unique fixed points in $\mathcal{H}$. Any element of finite order in Λ is conjugate to a power of one of the elliptic generators (compactness of the quotient space rules out parabolic isometries.) Together, $\alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \gamma_1, \ldots, \gamma_r$ comprise a set of *canonical generators* for Λ.

Any compact Riemann surface X of genus $g > 1$ is conformally equivalent to the orbit space $\mathcal{H}/\Gamma$ where Γ is a surface group of genus g. X admits a group G of conformal automorphisms if and only if there is a Fuchsian group Λ, containing Γ as a normal subgroup, such that $G \cong \Lambda/\Gamma$. Equivalently, there exists an epimorphism $\theta : \Lambda \to G$ with Γ as the kernel. Such epimorphisms are called *smooth* or *surface-kernel* to indicate that no element of finite order is mapped to an element of smaller order, or (equivalently), that the kernels are torsion-free. Λ is called the *covering* or *uniformizing* group of the G action. By a famous theorem of Hurwitz [**13**], G is in fact a finite group of order $\leq 84(g-1)$.

The *Riemann-Hurwitz relation* ties together all the topological data associated with a G action on a surface of genus g, as follows. If the covering Fuchsian group has signature $(h; m_1, \ldots, m_r)$, and $|G|$ denotes the order of G, then

$$2g - 2 = |G| \Big[2h - 2 + \sum_{i=1}^{r} \Big(1 - \frac{1}{m_i} \Big) \Big].$$

Since $g > 1$, the rational number in brackets must be positive. In fact this number is proportional to the hyperbolic area of a fundamental region for the action of the covering Fuchsian group on $\mathcal{H}$. Signatures for which this number is 0 or negative define non-Fuchsian groups which, if not trivial, are isomorphic to groups of euclidean or spherical isometries, respectively.

We shall use a theorem due to Accola (Theorem 2.2 below) which yields a remarkably simple characterization of the possible sets of nonnegative integers which can be simultaneous orbit genera for p-actions on a single surface (Lemma 4.5). Accola's theorem treats *groups with a partition*. Such groups possess two or more proper non-trivial subgroups whose union is the whole group and whose pairwise intersections are the trivial group. For example, the elementary abelian p-group $\mathbb{Z}_p^2 = \mathbb{Z}_p \oplus \mathbb{Z}_p = \langle x, y\rangle$ has the partition

$$\langle xy\rangle, \langle xy^2\rangle, \ldots, \langle xy^{p-1}\rangle, \langle x\rangle, \langle y\rangle \tag{1}$$

into $p+1$ proper non-trivial subgroups. This is a special case of a more general result. Let

$$\sigma_N = 1 + p + p^2 + \cdots + p^{N-1}. \tag{2}$$

LEMMA 2.1. *$\mathbb{Z}_p^{e+1}$, $e \geq 1$, admits a partition into σ_{e+1} distinct cyclic subgroups of order p.*

PROOF. By (1), the statement is true for $e = 1$. Proceeding by induction on e, suppose it is true for $e = k \geq 1$. Let g_i, $i = 1, \ldots, \sigma_{k+1}$ be a generator for the ith group in the partition of $\mathbb{Z}_p^{k+1}$. Adjoin a new generator h of order p which commutes with all the g_i, to produce a group isomorphic to $\mathbb{Z}_p^{k+2}$. There are now $p-1$ new subgroups $\langle hg_i\rangle$, $\langle h^2 g_i\rangle$, $\ldots$, $\langle h^{p-1} g_i\rangle$, for each $i = 1, \ldots, \sigma_{k+1}$, in addition to the new subgroup $\langle h\rangle$. It is straightforward to verify that the new and old subgroups taken together form a partition of $\mathbb{Z}_p^{k+2}$. The partition has the original σ_{k+1} subgroups plus $1 + (p-1)\sigma_{k+1}$ new ones. The fact that

$$\sigma_{k+1} + 1 + (p-1)\sigma_{k+1} = \sigma_{k+2}$$

follows from the identity $p\sigma_N + 1 = \sigma_{N+1}$. □

When a group with partition acts on a surface, the total ramification is the sum of the separate ramifications of the induced actions of the subgroups. This is the crucial point in Accola's proof.

THEOREM 2.2 (Accola [**1**]). *Let G be a finite group with partition $\{G_1, \ldots, G_t\}$. If G acts on a compact Riemann surface of genus g with orbit genus m, and the restricted G_i actions have orbit genera m_i, $i = 1, \ldots, t$, then*

$$(t-1)g + |G| \cdot m = \sum_{i=1}^{t} |G_i| \cdot m_i.$$

PROOF. The Riemann-Hurwitz relation for the G_i action has the form

$$2g - 2 = |G_i|(2m_i - 2) + r_i, \tag{3}$$

where $r_i \geq 0$ is the ramification term. If a non-trivial element $g \in G$ fixes a point, it generates a cyclic subgroup which is contained in exactly one of the subgroups

G_i (since the G_i form a partition). Thus, in the Riemann-Hurwitz relation for the full G action,

$$2g-2 = |G|(2m-2)+r, \tag{4}$$

the ramification term r is equal to $\sum_{i=1}^{t} r_i$. After summing (3) for $i = 1, 2, \ldots, t$ we obtain two expressions for r, the second coming from (4). Equating them yields $t(2g-2) - \sum_{i=1}^{t} |G_i|(2m_i-2) = 2g-2-|G|(2m-2)$. The relation in the theorem follows from this and the element count $\sum_{i=1}^{t} |G_i| = |G| + t - 1$. □

3. An upper bound on the geometric dimension of $\mathcal{G}_g^p$

In this section we prove that the geometric dimension of $\mathcal{G}_g^p$ is at most $\sigma_{e+1}-1$, where p^e is the highest power of p dividing $g-1$. This requires a brief excursion into the theory of finite p-groups. We are grateful to Andrei Jaikin for the statements and proofs of Lemmas 3.2 and 3.3.

Let G be a group acting on a surface of genus $g > 1$. Conjugate subgroups of G act with identical signatures since their uniformizing Fuchsian groups are conjugate in $\mathrm{PSL}(2,\mathbb{R})$. If the subgroups in question are cyclic of prime order p, this is equivalent to the quotient genera being equal. Therefore, the largest integer d such that a surface of genus g is simultaneously (p,n_1)-, (p,n_2)-, $\ldots$, (p,n_{d+1})-gonal, for pairwise distinct $n_1, n_2, \ldots n_{d+1}$ (the geometric dimension of $\mathcal{G}_g^p$) is no larger than the largest number of distinct conjugacy classes of subgroups of order p that can occur within G, where G varies over all possible automorphism groups of surfaces of genus g. It is correct to say 'no larger' rather than 'equal to' since non-conjugate cyclic groups of order p may nonetheless have the same orbit genus. Non-conjugate elements in a p-Sylow subgroup may be conjugate in the full group G, but, for the purpose of determining an upper bound on the number of distinct conjugacy classes in G, it suffices to assume that G is a p-group.

3.1. Kulkarni's result. A group has *exponent* E if it contains elements of order E but no elements of larger order. In the seminal paper [**15**], Kulkarni showed that a p-group P, where p is odd, of order p^k and exponent p^n, $n \leq k$, acts on a surface of genus g only if $g \equiv 1 \pmod{p^{k-n}}$. (A modified result is given in the case $p = 2$.) If the integer $k-n$ (the so-called *cyclic p-deficiency* of P) is small, then P contains a large cyclic subgroup. In the extreme case $k-n = 0$, P is itself cyclic (possibly trivial), containing (if not trivial) a unique cyclic subgroup of order p. So if $g \not\equiv 1 \pmod{p}$, no surface is both (p,n)- and (p,m)-gonal for $n \neq m$, that is, $\mathcal{G}_g^p$ is either empty or has geometric dimension 0. On the other hand, if P is far from cyclic, i.e., if $k-n$ is rather large, there is the potential for a large number of non-conjugate cyclic groups of order p. Kulkarni's result suggests that the geometric dimension of $\mathcal{G}_g^p$ should increase in accordance with the highest power of p dividing $g-1$. To make this precise we need an upper bound on the number of conjugacy classes of cyclic subgroups of order p in a finite p-group.

3.2. Conjugacy classes of subgroups of order p in a p-group. Let P be a finite p-group. Let $K_p(P)$ be the number of conjugacy classes of elements of order p, and $K_{\mathbb{Z}_p}(P)$ the number of conjugacy classes of subgroups of order p in P.

Lemma 3.1. $K_{\mathbb{Z}_p}(P) = K_p(P)/(p-1)$.

PROOF. First, for any element x of order p the conjugacy classes of the powers of x, $C_P(x)$, $C_P(x^2), \dots C_P(x^{p-1})$ are all distinct. For if $\alpha x \alpha^{-1} = x^s$ for some $\alpha \in P$, then the order of α is a divisor of the order of $\mathrm{Aut}(\langle x \rangle)$, which is $p-1$; thus α is the identity and $s = 1$. Next, an element y is conjugate to a power of x if and only if $\langle y \rangle$ and $\langle x \rangle$ are conjugate subgroups. Therefore there is a one-to-one correspondence between distinct sets of conjugacy classes of the form $\{C_P(x), C_P(x^2), \dots C_P(x^{p-1})\}$, of which there are $K_p(P)/(p-1)$, and conjugacy classes of subgroups of order p. □

For the next two lemmas, whose proofs are quite technical (and might be skipped on first reading), the following additional notation is convenient. If G is a finite p-group, H a subgroup, and S an H-invariant subset of G, then $K^H(S)$ denotes the number of H-conjugacy classes of elements in S; and $K_p^H(S)$ the number of H-conjugacy classes of elements of order p in S. For simplicity, we write $K(G)$ for $K^G(G)$.

LEMMA 3.2 (Jaikin [**14**]). *If P contains a central element x of order p^2, then $K_p(P) \leq K_p(P/\langle x^p \rangle)$.*

PROOF. Let $\mathcal{T} = \{T_i/\langle x \rangle \mid T_i \leq P\}$ be a set of representatives of the conjugacy classes of subgroups of order p in the factor group $P/\langle x \rangle$. For each $T_i/\langle x \rangle \in \mathcal{T}$, let $C_i = N_P(T_i)$, the normalizer of T_i in P. Each T_i is an abelian group of order p^3. $T_i \backslash \langle x \rangle$ (note: backslash) denotes the elements of T_i not belonging to $\langle x \rangle$. If T_i is cyclic, then there are no elements of order p in this set. If T_i is not cyclic, fix an element $y_i \in T_i \backslash \langle x \rangle$ of order p. Then

$$K_p^{C_i}(T_i \backslash \langle x \rangle) = K_p^{C_i}(\langle y_i, x^p \rangle \backslash \langle x^p \rangle) = \begin{cases} p^2 - p & \text{if } C_i \text{ centralizes } y_i \\ p - 1 & \text{if } C_i \text{ does not centralize } y_i. \end{cases}$$

Let $\tilde{T}_i = T_i/\langle x^p \rangle$, $\tilde{C}_i = C_i/\langle x^p \rangle$, and let $\bar{x}$ and $\bar{y}_i$ be the images of x and y_i in $\tilde{G} = G/\langle x^p \rangle$. If T_i is cyclic there are no elements of order p in $\tilde{T}_i/\langle \bar{x} \rangle$, and if T_i is not cyclic, then

$$K_p^{\tilde{C}_i}(\tilde{T}_i \backslash \langle \bar{x} \rangle) = p^2 - p,$$

because $\tilde{C}_i$ centralizes $\bar{y}_i$. Thus we obtain

$$K_p(P) = p - 1 + \sum_{T_i/\langle x \rangle \in \mathcal{T}} K_p^{C_i}(T_i \backslash \langle x \rangle) \leq p - 1 + \sum_{T_i/\langle x \rangle \in \mathcal{T}} K_p^{\tilde{C}_i}(\tilde{T}_i \backslash \langle \bar{x} \rangle) = K_p(P/\langle x^p \rangle).$$

□

LEMMA 3.3 (Jaikin [**14**]). *For a p-group P of order p^k and exponent p^n, $K_p(P) \leq p^{k-n+1} - 1$.*

PROOF. The result is clearly true when $n = 1$, so assume $n > 1$. The lemma is proved by induction on n. Let $a \in P$ be an element of order p^n. Put $b = a^{p^{n-2}}$, an element of order p^2. Let $H = C_P(b)$, the centralizer of b in P. Note that if $g \notin H$, then the $\langle a \rangle$-conjugacy class of g has at least p^{n-1} elements. Thus,

$$K^{\langle a \rangle}(P \backslash H) \leq \frac{|P \backslash H|}{p^{n-1}}.$$

Also, by Lemma 3.1, $K_p(H) \leq K_p(H/\langle b^p \rangle)$. Since $a \in H$, $H/\langle b^p \rangle$ has exponent at least p^{n-1}. By induction,

$$K_p(H) \leq K_p(H/\langle b^p \rangle) \leq \frac{|H/\langle b^p \rangle|}{p^{n-2}} - 1 = \frac{|H|}{p^{n-1}} - 1.$$

Thus we obtain

$$K_p(P) \leq K^{\langle a \rangle}(P \backslash H) + K_p(H) \leq \frac{|P \backslash H|}{p^{n-1}} + \frac{|H|}{p^{n-1}} - 1 = p^{k-n+1} - 1.$$

□

Now we need only a short argument to obtain the upper bound on the geometric dimension of $\mathcal{G}_g^p$. Let p^e be the highest power of p dividing $g-1$. For odd p, if $e = 0$, as already already noted, $\mathcal{G}_g^p$ is either empty or has geometric dimension 0. So suppose $e > 0$. Then $g \equiv 1 \pmod{p^e}$. By Kulkarni's result, a p-group P of order p^k acting in genus g has exponent at least p^{k-e}. It follows from Lemma 3.1 combined with Lemma 3.3 that

$$K_{\mathbb{Z}_p}(P) \leq \frac{p^{k-(k-e)+1} - 1}{p-1} = \frac{p^{e+1} - 1}{p-1} = \sigma_{e+1}.$$

Thus the geometrical dimension is at most $\sigma_{e+1} - 1$. Note that this formula holds also in the case $e = 0$.

See [**6**] for related results in the case $p = 2$.

4. Intersections between (p, n_i)- gonal loci

Henceforth, we assume p is an odd prime. By Kulkarni's result, if $g \not\equiv 1 \pmod{p}$, no surface of genus g is both (p,n)-gonal and (p,m)-gonal for $n \neq m$.

Proposition 4.1. *A surface of genus $g \equiv 1 \pmod{p}$ which is both (p,n)-gonal and (p,m)-gonal, $n \neq m$, admits automorphism groups isomorphic to $\mathbb{Z}_p^2$ whose actions induce both the (p,n)-gonal and the (p,m)-gonal automorphism.*

Proof. Let G be the full automorphism group of the surface, and let P be a p-Sylow subgroup of G. P has a non-trivial center which contains a (p,l)-gonal group for some l. If $l = n$ or m, then there are (p,n)-gonal and (p,m)-gonal subgroups of P containing mutually commuting elements (simply take one to be central), generating a subgroup isomorphic to $\mathbb{Z}_p^2$. If $l \neq n$ and $l \neq m$, there is a central (p,l)-gonal group that contains elements which commute with any elements from any (p,n)- and (p,m)-gonal group, generating, with each in turn, a subgroup isomorphic to $\mathbb{Z}_p^2$. □

Surfaces which are (p,n_i)-gonal for more than $p+1$ distinct n_i must admit actions by elementary abelian p-groups of rank greater than 2.

4.1. Existence of $\mathbb{Z}_p^{e+1}$ actions. By Kulkarni's result, if $\mathbb{Z}_p^{e+1}$ acts on a surface of genus g, then $g \equiv 1 \pmod{p^e}$. The following lemma provides a stronger necessary condition for the existence of such an action, and a *canonical signature* for the covering Fuchsian group, having minimal orbit genus and maximal number of periods.

LEMMA 4.2. *Let $g \equiv 1 \pmod{p^e}$, and let $p' = (p-1)/2$. Let $0 \le \kappa < p'$ be the residue of $g \pmod{p'}$. There exists a $\mathbb{Z}_p^{e+1}$ action on a surface of genus g only if*

$$\frac{g-1-p^{e+1}(\kappa-1)}{p'p^e} = R \tag{5}$$

is a nonnegative integer. The covering Fuchsian group of a $\mathbb{Z}_p^{e+1}$ action has signature $(\kappa+tp'; p, \overset{R-tp}{\ldots}, p)$ for some $t = 0, 1, 2, \ldots,$ such that $R-tp \ge 0$ and $R-tp \ne 1$.

PROOF. R is an integer because of the congruences $1 \equiv p \equiv g-1 \equiv \kappa - 1 \pmod{p'}$. If there exists an $H = \mathbb{Z}_p^{e+1}$ action in genus g, the signature of the covering Fuchsian group Λ_H is of the form $(k; p, \overset{r}{\ldots}, p)$ for some k, $r \ge 0$. The Riemann-Hurwitz relation is

$$(g-1)/p^e = p(k-1) + p'r. \tag{6}$$

This together with the fact that $p \equiv 1 \pmod{p'}$ implies $k \equiv g \pmod{p'}$. Hence if $k < p'$, then $k = \kappa$, and otherwise, there exists $t \ge 1$ such that $k = \kappa + tp'$. When $k = \kappa$, (6) coincides with (5) if we put $r = R$. For general $k = \kappa + tp'$, (6) yields $r = R - tp$. Clearly $t \le R/p$, and, of course, if $R < 0$, or if none of the signatures is Fuchsian, there can be no H action. Assuming $R \ge 0$ and the signature is Fuchsian, let $\rho : \Lambda_H \to H$ be a surface-kernel epimorphism corresponding to an H-action. Each of the k commutators $[\alpha_i, \beta_i] \in \Lambda_H$ maps to the trivial element (since H is abelian), hence the product $\gamma_1\gamma_2 \ldots \gamma_r$ of the elliptic generators must be the trivial element. None of the elliptic generators is itself mapped to the trivial element (ρ being surface-kernel), hence $r \ne 1$. □

REMARK 4.3. Necessary and sufficient conditions for the existence of a $\mathbb{Z}_p^{e+1}$ action in terms of R and κ can be given (see [**19**], §7). These conditions are satisfied if g is sufficiently large.

4.2. Induced (p,n)-gonal actions. An $H = \mathbb{Z}_p^{e+1}$ action on a surface X induces (p,n)-gonal actions by its proper non-trivial subgroups. The next lemma gives the possible values of n. Let $\mathbb{Z}_{p,n}$ denote a proper non-trivial subgroup of H whose induced action on X is (p,n)-gonal. The quotient surface $X/\mathbb{Z}_{p,n}$ is a branched covering of the quotient surface X/H, so $n \ge k$, where k is the genus of X/H. To allow for all possible n, we assume k is minimal, that is, we assume the H action has the canonical signature $(\kappa; p, \overset{R}{\ldots}, p)$ with minimal orbit genus κ.

LEMMA 4.4. *Let X be surface of genus $g \equiv 1 \pmod{p^e}$ on which $H = \mathbb{Z}_p^{e+1}$ acts. Let Λ_H be the covering Fuchsian group with signature $(\kappa; p, \overset{R}{\ldots}, p)$ and $\rho\colon \Lambda_H \to H$ the corresponding surface-kernel epimorphism. Let $\mathbb{Z}_{p,n}$ denote a proper, non-trivial subgroup of H whose induced action on X is (p,n)-gonal. Then the possible values of n are*

$$n = n(s) = 1 + p^e(\kappa-1) + p^{e-1}p'(R-s), \tag{7}$$

where

$$s = \begin{cases} 0, 1, 2, \ldots, R-2 & \text{if } \kappa = 0; \text{ or} \\ 0, 1, 2, \ldots, R-2, R & \text{if } \kappa > 0. \end{cases}$$

The parameter s is the number of elliptic generators of Λ_H in the kernel of the map $\chi_s \circ \rho$, where $\chi_s : H \to H/\mathbb{Z}_{p,n(s)} \simeq \mathbb{Z}_p^e$ is the canonical quotient map.

Proof. Let $\Lambda_{p,n}$, with signature $(n; p, \overset{q}{\ldots}, p)$, be the covering Fuchsian group of the induced $\mathbb{Z}_{p,n}$ action. We first show that q is a multiple of p^e. Since the $\mathbb{Z}_{p,n}$ action has q fixed points on X, it follows that the factor group $H/\mathbb{Z}_{p,n} = \mathbb{Z}_p^e$ freely permutes q corresponding points on the quotient surface $X/\mathbb{Z}_{p,n}$, in cycles of length p^e. (Otherwise there would be a point in X fixed by non-cyclic subgroup of $\mathbb{Z}_p^{e+1}$.) Hence $q = sp^e$ for some $s \geq 0$. Now the Riemann-Hurwitz relation for the induced $\mathbb{Z}_{p,n}$ action reduces to

$$n = 1 + (g-1)/p - sp'p^{e-1}. \tag{8}$$

Substituting for $(g-1)/p$ using (6), with $k = \kappa$ and $r = R$, we obtain equation (7). $s \leq R$, since R is the total number of H-orbits on X with a non-trivial isotropy subgroup and s is the number with a particular one, $\mathbb{Z}_{p,n(s)}$. The composition $\chi_s \circ \rho : \Lambda_H \to H/\mathbb{Z}_{p,n(s)}$ maps the elliptic generators of Λ_H fixing those s orbits to the trivial element in $H/\mathbb{Z}_{p,n}$. The relation $\gamma_1 \ldots \gamma_R[\alpha_1, \beta_1] \ldots [\alpha_\kappa, \beta_\kappa] = \mathrm{id}$ in Λ_H induces the relation

$$\chi_s \circ \rho(\gamma_1 \ldots \gamma_r[\alpha_1, \beta_1] \ldots [\alpha_\kappa, \beta_\kappa]) = \chi_s \circ \rho(\gamma_1 \ldots \gamma_R) = \mathrm{id}$$

in $H/\mathbb{Z}_{p,n}$. If $s = R$, the induced relation holds trivially. If $s < R$, we may suppose that the first s canonical elliptic generators are in the kernel of $\chi_s \circ \rho$. Then the induced relation is

$$\chi_s \circ \rho(\gamma_{s+1}\gamma_{s+2} \cdot \cdots \cdot \gamma_R) = \mathrm{id}.$$

This is not possible if $s = R-1$, for then only γ_R has non-trivial image under $\chi_s \circ \rho$, and $\chi_s \circ \rho(\gamma_R) = \mathrm{id}$ is a contradiction. If $\kappa = 0$, $s = R$ is also not possible, since, in this case, $n(s) = n(R) < 0$. □

4.3. Solutions of Accola's relation. Specializing Accola's Theorem 2.2 to $H = \mathbb{Z}_p^{e+1}$, $e \geq 1$, acting with canonical signature $(\kappa; p, \overset{R}{\ldots}, p)$ on a surface of genus $g > 1$, yields the relation

$$g\sigma_e + \kappa p^e = \sum_{i=1}^{\sigma_{e+1}} n(i), \tag{9}$$

where as before $\sigma_N = 1 + p + p^2 + \cdots + p^{N-1}$. The summands $n(i)$ are the orbit-genera of the σ_{e+1} subgroups in the partition of H (Lemma 2.1). By Lemma 4.4, the indices i, with possible repeats, come from the set $\{0, 1, 2, \ldots, R\}$.

Lemma 4.5. *$\{n(i) \mid i \in I\}$, where the index set I has cardinality $|I| = \sigma_{e+1}$, is a solution of Accola's relation* (9) *if and only if $\sum_{i \in I} i = R$.*

Proof. We claim that $\{n(i) \mid i \in I_0\}$, with index set $I_0 = \{0^{(-1+\sigma_{e+1})}, R\}$, is a solution, where the superscript in parentheses denotes the multiplicity of the corresponding index. (This convention is used henceforth.) Note that I_0 has cardinality σ_{e+1}. To see that

$$n(0) \cdot (-1 + \sigma_{e+1}) + n(R)$$

is equal to the left-hand side of (9), use $n(0) = 1 + (g-1)/p$ (from (8)) and the formula for $n(R)$ given in Lemma 4.4. We leave the details to the reader.

If two distinct index sets I and J determine solutions of (9), equality of the sums

$$\sum_{i\in I} n(i) = \left(1+\frac{g-1}{p}\right)|I| - p'p^{e-1}\left(\sum_{i\in I} i\right)$$
$$\sum_{j\in J} n(j) = \left(1+\frac{g-1}{p}\right)|J| - p'p^{e-1}\left(\sum_{j\in J} j\right),$$

implies, since $|I| = |J| = \sigma_{e+1}$, that $\sum_{i\in I} i = \sum_{j\in J} j$. The common sum is R, since that is the case for I_0. □

Lemma 4.5 shows that solutions of (9) are in one-to-one correspondence with additive partitions of R into σ_{e+1} parts chosen from $\{0, 1, \ldots, R\}$. For example, the index sets

$$I_s = \{0^{(-2+\sigma_{e+1})}, s, R-s\}, \quad 0 \le s \le R \tag{10}$$

provide $\lfloor R/2 \rfloor$ distinct solutions in which any possible orbit-genus $n(s)$ appears together with the maximal one $n(0)$. We shall make use of this solution in Section 5. If $j < \sigma_{e+1}$, the index set

$$I_j = \{0^{(-j-1+\sigma_{e+1})}, 1, 2, \ldots, j, R-T_j\}, \tag{11}$$

where T_j is the triangular number $\sum_{i=1}^{j} i = j(j+1)/2$, determines a solution to (9), provided $T_j \le R$. If we define $T_0 = 0$, then for all $R \ge 0$, there is a unique maximal j such that $T_j \le R$ and $0 \le j < -1+\sigma_{e+1}$; with this j, (11) has the maximum possible number of distinct parts, since the smallest indices are used a minimal number of times. Indeed, the $j+2$ indices (including 0) are distinct if $R - T_j > j$; otherwise there is an index j', $0 \le j' \le j$, such that $R - T_j = j'$ and there are only $j+1$ distinct indices.

4.4. Sharpness of the upper bound on the geometric dimension. The existence of solutions (11) to Accola's relation suggests that the upper bound on the geometric dimension of $\mathcal{G}_g^p$ given in Section 3 is attained by surfaces admitting an action of $\mathbb{Z}_p^{e+1}$. We show that this is indeed the case for sufficiently large g.

The rest of this section is devoted to the proof of the following theorem.

THEOREM 4.6. *Let $g \equiv 1 \pmod p$, and let $e \ge 1$ be the largest positive integer such that $g \equiv 1 \pmod{p^e}$. Let R be the integer defined at (5), and let d be the geometric dimension of $\mathcal{G}_g^p$. If g is sufficiently large, then $d \ge j$ when $T_j \le R < T_{j+1}$ and $d = -1+\sigma_{e+1}$ when $R \ge T_{-1+\sigma_e}$.*

We produce an action of $H = \mathbb{Z}_p^{e+1} = \langle x_1, x_2, \ldots, x_{e+1}\rangle$ on a surface of genus g which induces $(p, n(i))$-gonal actions for each i in the index set I_j at (11), where $j \ge 0$ is the largest integer such that $T_j \le R$ and $j < p$. If such an action exists, it has covering Fuchsian group Λ_H with signature $(\kappa; p, \overset{R}{\ldots}, p)$, hyperbolic generators $\alpha_1, \beta_1, \ldots, \alpha_\kappa, \beta_\kappa$, and elliptic generators $\gamma_1, \gamma_2, \ldots, \gamma_R$. To produce the desired action, we define a surface-kernel epimorphism $\rho\colon \Lambda_H \to H$ so that exactly i elliptic generators of Λ_H map into the subgroup $H_i < H$, where $H_i \simeq \mathbb{Z}_p$ is one of the σ_{e+1} subgroups of the partition of H, and $H_i \ne H_k$ for $i \ne k$. By the last sentence of Lemma 4.4, the induced action of H_i is $(p, n(i))$-gonal, as desired.

We first treat the case $e = 1$. Since $\sigma_2 = p+1$, we show that $d = p$ when $R \ge T_p$, and $d = j$ when $T_j \le R < T_{j+1}$, $j < p$. For $R > 3$, Table 1 gives a

schema defining a surface-kernel epimorphism $\rho\colon \Lambda_H \to H = \mathbb{Z}_p^2 = \langle x_1, x_2\rangle$. (We put $x_1 = x$ and $x_2 = y$ to improve legibility.) Because $\mathrm{id} = \prod_{i=1}^{\kappa}[\alpha_i, \beta_i]\prod_{k=1}^{R}\gamma_k$ in Λ_H, the nontrivial powers of x and y must be chosen so that the product of all R elements of H is the identity. We assert that this is always possible under the stated restrictions (see Remark 4.7 below). We do not specify the images of the 2κ hyperbolic generators of Λ_H (if any); they can be assigned arbitrarily, but it is convenient to take them to be κ x's and κ y's. If $R < T_p$, then $j \le p-1$, and in I_j, the index 0 appears together with the indices $1, 2, \ldots, j$. $R - T_j \le j$ by the definition of j, hence there are $j+1$ distinct indices and therefore $d \ge j$. If $R \ge T_p$, we use the index set $I_{p-1} = \{0, 1, 2, \ldots, p-1, R - T_{p-1}\}$, which has $p+1$ distinct indices, since $R - T_{p-1} \ge p$. Hence, again, $d \ge p$.

Index in I_j	Elliptic Generators of Λ_H	Images in H	
		if $T_j < R$:	if $T_j = R$:
1	γ_1	xy	xy
2	γ_2, γ_3	xy^2, xy^2	xy^2, xy^2
3	γ_4, γ_5, γ_6	xy^3, xy^3, xy^3	xy^3, xy^3, xy^3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$j-1$	$\gamma_{1+T_{j-2}}, \ldots, \gamma_{T_{j-1}}$	$xy^{j-1}, \ldots, xy^{j-1}$	powers of x
j	$\gamma_{1+T_{j-1}}, \ldots, \gamma_{T_j}$	powers of x	powers of y
$R - T_j$	$\gamma_{1+T_j}, \ldots, \gamma_R$	powers of y	-

TABLE 1. Schema for defining $\rho : \Lambda_H \to H$, $H = \mathbb{Z}_p^2$, $R > 3$

We next show how the schema in Table 1 generalizes to define actions of $\mathbb{Z}_p^{e+1}$, $e \ge 1$ of the desired type. We make an inductive construction: Let $g_1, g_2, \ldots, g_{\sigma_e}$ be a set of generators for the distinct subgroups of a partition of $\mathbb{Z}_p^e = \langle x_1, \ldots, x_e\rangle$, and assume that

$$g_{\sigma_e} = x_e, \quad g_{\sigma_e - 1} = x_{e-1}, \quad \ldots \quad g_{\sigma_e - e + 1} = x_1.$$

Adjoin a new generator x_{e+1} to form $H = \mathbb{Z}_p^{e+1} = \langle x_1, \ldots, x_{e+1}\rangle$. Form a set of σ_{e+1} generators for the distinct subgroups of a partition of the larger group as follows. Let $h_{ij} = g_i x_{e+1}^j$, $i = 1, \ldots, \sigma_e$, $j = 0, \ldots, p-1$. This amounts to $p\sigma_e$ generators, one less than the necessary $\sigma_{e+1} = p\sigma_e + 1$. Relabel the generators using a single index, as h_t, $t = 1, \ldots, p\sigma_e$. Replace the last e generators by $x_1, \ldots, x_e$, and adjoin one more, defining $h_{\sigma_{e+1}} = x_{e+1}$. We now have a complete set of generators, and a schema for constructing $\rho : \Lambda_H \to H = \mathbb{Z}_p^{e+1}$ is given in Table 2. (Table 1 is the special case $e = 1$.)

REMARK 4.7. In assuming that g is 'sufficiently large' we exclude cases where $R = R(e) < 0$. Since R grows with g, we may further assume $R > T_{e+1}$, which implies $j - e \ge 1$. The latter is the minimal assumption needed for the schema in Table 2 to succeed even in the most restrictive case $\kappa = 0$. For example, taking $e = 2$

Index in I_j	Elliptic Generators of Λ_H	Images in H	
		if $T_j < R$:	if $T_j = R$:
1	γ_1	h_1	h_1
2	γ_2, γ_3	h_2, h_2	h_2, h_2
3	γ_4, γ_5, γ_6	h_3, h_3, h_3	h_3, h_3, h_3
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$j-e$	$\gamma_{1+T_{j-e-1}}, \ldots, \gamma_{T_{j-e}}$	$h_{j-e}, \ldots, h_{j-e}$	powers of x_1
$j-e+1$	$\gamma_{1+T_{j-e}}, \ldots, \gamma_{T_{j-e+1}}$	powers of x_1	powers of x_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$j-1$	$\gamma_{1+T_{j-2}}, \ldots, \gamma_{T_{j-1}}$	powers of x_{e-1}	powers of x_e
j	$\gamma_{1+T_{j-1}}, \ldots, \gamma_{T_j}$	powers of x_e	powers of x_{e+1}
$R-T_j$	$\gamma_{1+T_j}, \ldots, \gamma_R$	powers of x_{e+1}	-

TABLE 2. Schema for defining $\rho : \Lambda_H \to H$, $H = \mathbb{Z}_p^{e+1}$, $e \geq 1$, $R > T_{e+1}$

and $R = 7 = 1 + T_3$, the schema allows the following surface-kernel epimorphism onto $\mathbb{Z}_p^3$: $\gamma_1 \mapsto x_1x_2x_3$, $\gamma_2, \gamma_3 \mapsto x_1, x_1^{-2}$, $\gamma_4, \gamma_5, \gamma_6 \mapsto x_2, x_2, x_2^{-3}$, $\gamma_7 \to x_3^{-1}$. But with $R = 6 = T_3$, no construction is possible.

5. A spanning tree for $\mathcal{G}_g^p$

The results of the previous section show there is a simplex in $\mathcal{G}_g^p$ for almost any additive partition of R into σ_{e+1} parts. Instead of attempting to draw every simplex, we construct a minimal spanning tree. This determines the connected components, and, along with the geometric dimension, provides a rather complete picture.

By Proposition 4.1, any surface which is both $(p, n(i))$- and $(p, n(j))$-gonal, $i \neq j$, admits an action of $\mathbb{Z}_p^2$. Solutions (10) to Accola's relation allow for a $\mathbb{Z}_p^2$ action which induces a $(p, n(0))$-gonal action and a $(p, n(s))$-gonal automorphism, for every non-excluded $s \in \{1, \ldots, R\}$. We use the (maximum) value of R determined by putting $e = 1$ in (5), even if $g \equiv 1 \pmod{p^k}$ for some $k > 1$. The simple schema in Table 3 shows these actions can be realized. (Nontrivial powers of x and y must be chosen so that the total exponent on each is a multiple of p.) By symmetry of I_s in s and $R - s$, we may assume $s \leq \lfloor R/2 \rfloor$. $s = 1$ requires an adjustment, since $\{0^{(p-1)}, 1, R-1\}$ contains the excluded index $R - 1$. One can simply replace the index set by $\{0^{(p-2)}, 1^2, R-2\}$. It is easy to see that an appropriate surface-kernel epimorphism can be constructed in this case.

It follows that any edge from $(p, n(i))$ to $(p, n(j))$ in $\mathcal{G}_g^p$, $i \neq j$, $i, j \neq 0$, can be replaced by a path of length 2 passing through $(p, (n(0))$. So, if $g \equiv 1 \pmod{p}$, there is a path-connected component of $\mathcal{G}_g^p$ with a star-like spanning tree centered at $(p, n(0))$ having an edge from $(p, n(0))$ to each of the vertices $(p, n(s))$. If $\kappa > 0$,

Index in I_s	Elliptic Generators of Λ_H	Images in H
s	$\gamma_1, \ldots, \gamma_s$	powers of x
$R-s$	$\gamma_{s+1}, \ldots, \gamma_R$	powers of y

TABLE 3. Schema for $\rho : \Lambda_H \to H = \mathbb{Z}_p^2, \quad 1 < s \leq \lfloor R/2 \rfloor$

there are other, isolated vertices in $\mathcal{G}_g^p$. This is because (7), with $e = 1$, yields positive values of n for some excluded values of s, corresponding to (p, n)-gonal groups not induced by any $H = \mathbb{Z}_p^2$ action. In particular, $n = 1 + p'$ is obtained by taking $s = R - 1$ (provided $R > 0$); in addition, if $\kappa > 1$, there are $2\kappa - 2 > 0$ smaller positive values of n obtained by taking $s > R$:

$$n(R + t) = \kappa + p'(2\kappa - 2 - t), \quad t = 1, 2, \ldots, 2\kappa - 2.$$

This information is summarized in Figure 1, for $R \geq 3$.

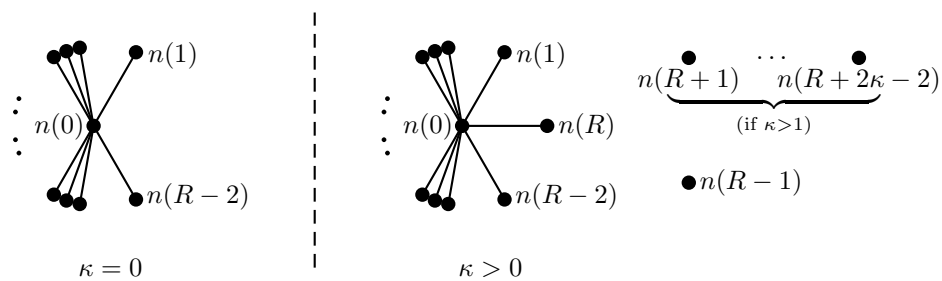


FIGURE 1. Spanning trees for $\mathcal{G}_g^p$, $g \equiv 1 \pmod p$, $R \geq 3$

If $R = 0$, the star-like component reduces to the singleton $n(0)$; if $\kappa > 1$, the additional isolated vertices are $n(2), \ldots, n(2\kappa - 2)$. If $R = 2$, the starlike component is just the edge from $n(0)$ and $n(2)$; if $\kappa > 1$, the additional isolated vertices are $n(1), n(3), \ldots, n(2\kappa - 1)$. If $R = 1$, $\mathcal{G}_g^p$ is empty. Finally, if $g \not\equiv 1 \pmod p$, $\mathcal{G}_g^p$ is either empty or consists of isolated vertices.

In Table 4 we show some spanning trees for $p = 7$.

6. Future directions

There are many ways to "cover" or "stratify" the singular locus in moduli space $\mathcal{M}_g$. Probably the most useful is the *equisymmetricic stratification*: each strata is an irreducible complex algebraic variety consisting of surfaces with isomorphic full automorphism groups whose actions are topologically equivalent (see [**3**], [**4**], [**12**] for further details). The strata are in bijection with conjugacy classes of finite subgroups of the mapping class group in genus g. Obtaining the complete equisymmetric stratification in successive genera presents a series of increasingly challenging problems in finite group theory. For this reason, coarser stratifications are, if nothing else, useful stepping-stones. One might expand our approach in several directions. It is interesting to speculate on how to interpret the cohomology

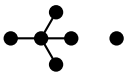
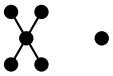
g	κ	R	Spanning Tree	g	κ	R	Spanning Tree
50	2	0		92	2	2	
57	0	5		99	0	7	
64	1	3		106	1	5	
71	2	1	–	113	2	3	
78	0	6		120	0	8	
85	1	4		127	1	6	

TABLE 4. Spanning trees for some $\mathcal{G}_g^7$'s

groups of our finite simplicial complexes. One might also study the intersections between (p_i, n)-gonal and (p_j, m)-gonal loci, allowing the primes as well as the orbit-genera to vary. We plan to pursue some of these ideas in future work.

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