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ORIGINAL PAPER

On gonality of Riemann surfaces

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Abstract A compact Riemann surface X is called a (p, n)-gonal surface if there exists a group of automorphisms C of X (called a (p, n)-gonal group) of prime order p such that the orbit space X/C has genus n. We derive some basic properties of (p, n)-gonal surfaces considered as generalizations of hyperelliptic surfaces and also examine certain properties which do not generalize. In particular, we find a condition which guarantees all (p, n)-gonal groups are conjugate in the full automorphism group of a (p, n)-gonal surface, and we find an upper bound for the size of the corresponding conjugacy class. Furthermore we give an upper bound for the number of conjugacy classes of (p, n)-gonal groups of a (p, n)-gonal surface in the general case. We finish by analyzing certain properties of quasiplatonic (p, n)-gonal surfaces. An open problem and two conjectures are formulated in the paper.

Keywords Automorphism groups of Riemann surfaces \cdot Hyperelliptic Riemann surfaces \cdot *p*-Hyperelliptic Riemann surfaces \cdot *p*-Gonal Riemann surfaces \cdot Fuchsian groups \cdot Uniformization theorem

Mathematics Subject Classification (2000) 14J10 · 14J50 · 30F10 · 30F20

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1 Introduction

A compact Riemann surface X is called (p, n)-gonal if there exists a cyclic group of automorphisms C of X, called a (p, n)-gonal group, of prime order p, such that the orbit space X/C has genus n. (p, n)-gonal surfaces and families of (p, n)-gonal surfaces have been the focus of a number of different studies over the last half a century. (p, n)-gonal surfaces were first considered in Kuribayashi [19], where it was shown that for a fixed genus g, the space of (p, n)-gonal surfaces is a complex analytic submanifold of complex dimension 3n - 3 + r of the Teichmüller space T_g of surfaces of genus g where r is the number of branch points of the quotient map $X \rightarrow X/C$. In the same paper, Kuribayashi also considered the corresponding algebraic curves for (p, 0)-gonal surfaces, or what are currently referred to as p-gonal surfaces (by extension of "trigonal"), and their defining equations. Later, Cornalba in [9] used a restricted version of (p, n)-gonality, where the branching indices of the quotient map are specified, to completely describe the components of the singular loci in the moduli spaces of Riemann surfaces. Other results focus on specific families of (p, n)-gonal surfaces such as p-hyperelliptic surfaces which were introduced in [11] and coincide with the notion of (2, p)-gonality, and elliptic-hyperelliptic surfaces, or (2, 1)-gonal surfaces.

If X is a compact Riemann surface and X admits any non-trivial automorphism, then X will be (p, n)-gonal for some n and p. So describing the spaces of (p, n)-gonal surfaces is equivalent to describing the singular loci in the moduli spaces of Riemann surfaces. This observation provides the principal motivation for determining properties of (p, n)-gonal surfaces.

The (2, 0)-gonal, or hyperelliptic surfaces have been analyzed in great detail over the last century, with the most recent results classifying the different automorphism groups which can act on such surfaces, see for example [4,6]. The (3, 0)-gonal, or *cyclic trigonal* surfaces, were first considered in [1], with the classification of their automorphism groups and defining equations provided in [2,7,10,35]. More generally, (p, 0)-gonal, or *cyclic p-gonal* surfaces, were considered in [13], with complete classification results providing defining equations and full groups of automorphisms appearing in [33,34]. (2, 1)-gonal, or *elliptic-hyperelliptic* surfaces were considered in [31] and more recently in [5].

These prior studies of families of (p, n)-gonal surfaces have revealed many interesting, and sometimes surprising properties of such surfaces or the groups acting conformally on them. For example, any two (p, 0)-gonal groups of a surface X must be conjugate in the full automorphism group of X, see [14], or [15], and in fact all p-gonal groups coincide if the genus of X is large enough [8,26]. In [16] an upper bound is given for the number of such groups. As another example, it was shown in [32] that a fixed surface X can be (p, 0)-gonal for at most two distinct values of p, and all surfaces which satisfy this property were classified. One of our primary goals is to take previous results which hold for specific families, and provide generalizations to all (p, n)-gonal surfaces for arbitrary p and n, or show that no such generalization exists.

The so called *quasiplatonic surfaces* (see Sect. 2 for the formal definition) are of great interest due to their close relation with maps and hypermaps on surfaces [18], the inverse Galois problem [24], and the Grothendieck-Teichmüller theory of dessins d'enfants [25]. A number of observations regarding quasiplatonic surfaces considered as (p, n)-gonal surfaces appear in the literature. For example, it is shown in [31] that for all positive integers N, there exists a genus g such that the number of quasiplatonic (2, 1)-gonal surfaces of genus g is strictly larger than N. This is in stark contrast to the (2, 0)-gonal case where for generic g, there are just three quasiplatonic (2, 0)-gonal surfaces of genus g, and also in contrast to

the (p, n)-gonal case, n > 1, where for sufficiently large genus, there are no quasiplatonic (p, n)-gonal surfaces (see Sect. 6).

We note that the concept of (p, n)-gonality holds for arbitrary positive p and n, but we shall assume throughout that p is a prime. Unless the contrary is explicitly stated, we shall also only consider surfaces of genus $g \ge 2$. This is due to the fact that results regarding gonality of surfaces of genus 0 and 1 are well known. The unique surface of genus 0 is quasiplatonic and (p, 0)-gonal for all primes p. All surfaces of genus 1 are (2, 0)-gonal and (p, 1)-gonal for all p. In addition, there is a unique (3, 0)-gonal surface of genus 1 and one additional genus 1 surface which is (4, 0)-gonal (both of which are quasiplatonic).

2 Preliminaries

A co-compact *Fuchsian group* is a discrete group of orientation preserving isometries of the upper half-plane \mathcal{H} with compact orbit space. The orbit space inherits the complex structure from \mathcal{H} and hence is a compact Riemann surface. Let Λ be a Fuchsian group. We define the signature of Λ to be $(h; m_1, \ldots, m_r)$ where the orbit space \mathcal{H}/Λ has genus h and the quotient map $\pi : \mathcal{H} \to \mathcal{H}/\Lambda$ branches over r points with ramification indices m_1, \ldots, m_r . h is known as the *orbit genus* of Λ . When r = 0, we write (h; -) and Λ is called a Fuchsian group Λ with signature $(h; m_1, \ldots, m_r)$ has presentation

$$\langle \alpha_1, \beta_1, \ldots, \alpha_h, \beta_h, \gamma_1, \ldots, \gamma_r : \gamma_1^{m_1}, \ldots, \gamma_r^{m_r}, \gamma_1 \ldots \gamma_r[\alpha_1, \beta_1] \ldots [\alpha_h, \beta_h] \rangle.$$

The elements $\alpha_1, \beta_1, \ldots, \alpha_h, \beta_h$ are *hyperbolic* generators and the elements $\gamma_1, \ldots, \gamma_r$ are *elliptic* generators for Λ . Any element of finite order in Λ is conjugate to a power of one of the elliptic generators.

By the uniformization theorem, a compact Riemann surface X of genus $g \ge 2$ is conformally equivalent to the orbit space \mathcal{H}/Γ of the hyperbolic plane with respect to the action of a Fuchsian surface group Γ with signature (g; -) called a *uniformizing group* for X. Under such a realization, a finite group G is a group of automorphisms of X if and only if $G \cong \Lambda/\Gamma$ for some Fuchsian group Λ containing Γ as a normal subgroup and so if and only if there exists an epimorphism $\theta : \Lambda \to G$ with Γ as the (torsion-free) kernel. In such a situation, the well known *Riemann–Hurwitz formula* holds:

$$|G| = \mu(\Gamma)/\mu(\Lambda),$$

where $\mu(\Lambda)$ is the hyperbolic area of a fundamental region which equals

$$2\pi \left(2h - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{m_i}\right)\right) \tag{1}$$

for Λ with signature $(h; m_1, \ldots, m_r)$.

In the moduli space of genus g surfaces, the dimension of the locus of surfaces admitting the action of the group $G = \Lambda / \Gamma$ is equal to the Teichmüller dimension of Λ which is determined by its signature and is equal to 3h - 3 + r. Observe that when h = 0 and r = 3, so Λ is a triangle group, the dimension is 0 and the corresponding locus is a finite set of points, corresponding to *quasiplatonic* surfaces.

Definition 2.1 A compact Riemann surface *X* is quasiplatonic if a uniformizing surface group Γ for *X* is a normal subgroup of a triangle group [30].

We treat these surfaces in Sect. 6.

3 Elementary properties of (p, n)-gonal surfaces

We start by gathering some basic facts about (p, n)-gonal surfaces.

Lemma 3.1 A compact Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g \ge 2$ where Γ is a torsion free uniformizing group for X is a (p, n)-gonal surface for a prime p if and only if there exists a Fuchsian group with signature (n; p, .r., p), where r = 2(g + p - 1 - np)/(p - 1), containing Γ as a normal subgroup of index p.

Proof Suppose X is (p, n)-gonal and let it be represented as the orbit space \mathcal{H}/Γ . Let $C = \langle \varphi \rangle$ denote a (p, n)-gonal group of X. Then, $C = \Lambda/\Gamma$ for some Fuchsian group Λ . By assumption, the orbit space $\mathcal{H}/\Lambda = X/C$ has genus n, so Λ must have signature $(n; m_1, \ldots, m_r)$ for some m_1, \ldots, m_r . Since Γ is torsion free, the corresponding generators $\gamma_1, \ldots, \gamma_r$ must preserve their orders in the quotient group Λ/Γ and therefore, since C has prime order p, it follows that all elements $\gamma_1, \ldots, \gamma_r$ must have order p. Therefore, Λ has signature (n; p, .r., p), where r can be found using the Riemann–Hurwitz formula.

Conversely, suppose there exists a Fuchsian group Λ with signature (n; p, .r., p), where r = 2(g + p - 1 - np)/(p - 1), containing Γ as a normal subgroup of index p. Then the quotient group Λ/Γ acts conformally on the compact Riemann surface $X = \mathcal{H}/\Gamma$ which has genus g by the Riemann–Hurwitz formula. By construction, X will be a (p, n)-gonal surface with (p, n)-gonal group $C = \Lambda/\Gamma$.

Theorem 3.2 An integer $g \ge 2$ is a genus of a (p, n)-gonal Riemann surface X if and only if

$$g = pn - p + \frac{r(p-1)}{2} + 1,$$

where $r \neq 1$ and r is even for p = 2.

Proof Suppose that X is a (p, n)-gonal surface of genus $g \ge 2$ and let Γ denote a Fuchsian group such that the orbit space \mathcal{H}/Γ is conformally equivalent to X. Then by the previous lemma, there exists a Fuchsian group with signature (n; p, .r., p), where r =2(g + p - 1 - np)/(p - 1), containing a Fuchsian group Γ with signature (g; -) as a normal subgroup of index p. Calculation of g is a straightforward application of the Riemann– Hurwitz formula. Now suppose g = pn - p + [r(p-1)/2] + 1 and let Λ be a Fuchsian group with signature (n; p, .r., p) for some r. If $p \neq 2$ and $r \neq 1$, we can define an epimorphism $\theta: \Lambda \to C = \langle a \rangle$ in the following way: we map all the hyperbolic generators, α_i and β_i to a, and the elliptic generators $\gamma_1, \ldots, \gamma_r$ to $a, \ldots, a, a^2, a^{p-r}$, respectively, if $r \equiv 1 (p)$ and to $a, \ldots, a, a, a^{p-r+1}$, respectively, otherwise. If p = 2 and r is even then we can define an epimorphism $\theta: \Lambda \to C = \langle a \rangle$ mapping all the hyperbolic generators and elliptic generators to a. Finally if r is odd and p = 2, no epimorphisms from a Fuchsian group with signature (n; p, ..., p) onto C with torsion-free kernel exist since all elliptic elements have to be sent onto a, while the product of all commutators of hyperbolic elements is mapped to the identity of G. Similarly such epimorphisms do not exist for r = 1.

If r = 0 in Theorem 3.2, the (p, n)-gonal group acts freely, i.e., without fixed points, and the quotient map is a regular covering. It is often convenient to exclude this case, so we make the following definition.

Definition 3.3 A (p, n)-gonal group acting on a surface X is called *properly* (p, n)-gonal if it acts with fixed points, and *improperly* (p, n)-gonal otherwise. If there is no need to make

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the surface X explicit, we simply say that the group *acts properly* (or improperly). If there is no need to make the group explicit, we call the surface itself *properly* (or improperly) (p, n)-gonal.

If a (p, n)-gonal group *C* acts properly, then the number of fixed points of an element of *C* can be determined using a formula due to Macbeath [21], a result which we will use extensively in our calculations, see Lemma 5.1 for a complete statement of this result.

Let $\mu(p, n)$, $\mu_{\text{prop}}(p, n)$ denote the minimum genera ≥ 2 of a (p, n)-gonal, and a properly (p, n)-gonal surface, respectively. As an immediate corollary of Theorem 3.2 we obtain the following result providing the minimal genus for which there exists a (p, n)-gonal surface and a properly (p, n)-gonal surface.

Corollary 3.4 With the above notations we have

$$\mu(p,n) = \begin{cases} (p-1)/2 & \text{if } n = 0, p \ge 5, \\ 2 & \text{if } n = 0, p = 2, 3 \\ p & \text{if } n = 1, \\ p(n-1)+1 & \text{if } n > 1. \end{cases}$$
$$\mu_{\text{prop}}(p,n) = \begin{cases} (p-1)/2 & \text{if } n = 0, p \ge 5, \\ 2 & \text{if } n = 0, p = 2, 3 \\ p & \text{if } n = 1, \\ pn & \text{if } n > 1. \end{cases}$$

Proof The Riemann–Hurwitz relation for a (p, n)-gonal action on a surface of genus g is

$$g - 1 = p(n - 1) + \frac{r(p - 1)}{2}$$

We require g - 1 > 1. If n > 1, $g - 1 \ge p(n - 1) > 1$, with equality holding for improper actions. For proper actions, r > 0 and hence, by Theorem 3.2, $r \ge 2$. Taking r = 2 yields the minimum (proper) genus. If n = 1, we must have $r \ge 2$ so that g - 1 > 0, and taking r = 2 yields the minimum (proper) genus p. If n = 0, and $p \ge 5$, we must take $r \ge 3$ to make g - 1 > 0. If n = 0 and p = 2 (resp. 3), we must take $r \ge 6$ (resp. 4) to make g - 1 > 0. Taking the minimum r in these cases yields the minimum (proper) genus.

4 Conjugacy classes of (p, n)-gonal Groups

It was shown in [14] that for arbitrary genus g, there is a unique conjugacy class of (p, 0)gonal groups in the full automorphism group of a (p, 0)-gonal surface X. In this section we shall derive a partial generalization of this result for arbitrary n. To do this, we shall need a technical result due to Sah [23] and Maclachlan [22], which provides a relationship between the signatures of a Fuchsian group Λ and a normal subgroup Γ in terms of the index, the orbit genera and the orders of the images of the elliptic generators of Λ in the quotient group Λ/Γ .

Lemma 4.1 Let Λ be a Fuchsian group with signature $(g; m_1, ..., m_r)$ and suppose Γ is a normal subgroup of Λ of finite index N, and let $\gamma_1, ..., \gamma_r$ denote the elliptic generators of Λ . If the image of γ_i in the quotient group Λ / Γ has order t_i , then the orbit genus g' of Γ is given by

$$g' - 1 = N(g - 1) + \frac{N}{2} \sum_{i=1}^{r} \left(1 - \frac{1}{t_i}\right),$$
(2)

and the periods of Γ are $f_{ij} = m_i/t_i$, $1 \le j \le N/t_i$, $1 \le i \le r$, where $f_{ij} = 1$ are deleted.

The following useful corollaries are immediate.

Corollary 4.2 Let Λ be a Fuchsian group of orbit genus g containing a Fuchsian group Λ' of orbit genus g' as a normal subgroup. Then $g' \ge g$ and the inequality is strict for $g \ge 2$.

Corollary 4.3 Let Λ be a Fuchsian group of orbit genus 1 containing a Fuchsian group Λ' of orbit genus 1 as a normal subgroup. Then all elliptic generators of Λ belong to Λ' .

Proof Observe that, as Fuchsian groups, both groups must contain elliptic generators. Then formula (2) gives $\sum_{i=1}^{r} (1 - 1/t_i) = 0$, hence the result.

The following well-known lemma is a consequence of Corollary 3.4 (see also [17]).

Lemma 4.4 Suppose C is a cyclic group of a prime order p acting on a surface of genus $g \ge 2$. Then $g \ge (p-1)/2$.

Theorem 4.5 Let X be a (p, n)-gonal surface of genus $g \ge 2$ with p > 2n + 1. Then all (p, n)-gonal groups are conjugate in the full automorphism group G of X.

Proof The case n = 0 was proved in [14]. Henceforth assume that $n \ge 1$.

Suppose that the full automorphism group *G* of *X* contains at least two conjugacy classes of (p, n)-gonal groups. Applying the Sylow theorems, it follows that p^2 divides the order of *G*. Let *P* denote a Sylow subgroup of *G* and let *C* denote a (p, n)-gonal subgroup of *P*. Note that since *G* contains more than one subgroup of order *p*, *P* cannot be cyclic and in particular, it must contain more than one subgroup of order *p*. If *C* is central, let *K* denote the elementary abelian group of order p^2 generated by *C* and any element of order *p* not belonging to *C*. If *C* is not central, let *K* denote the elementary abelian group of order p^2 generated by *C* and any element of order *p* in the center of *P*. To finish the proof, we consider separately two cases n > 1 and n = 1.

First suppose that n > 1. The group K/C acts on the surface X/C which by assumption has genus *n* bigger than 1. In particular, there is an action of a cyclic group of prime order *p* on a surface of genus *n*. By Lemma 4.4, $n \ge (p-1)/2$ which in turn gives $2n + 1 \ge p$, contrary to our assumption.

Now suppose n = 1 and let Γ be a fixed uniformizing surface Fuchsian surface group for X, so $X = \mathcal{H}/\Gamma$. Let Λ_P and Λ_C denote the Fuchsian groups with $P = \Lambda_P/\Gamma$ and $C = \Lambda_C/\Gamma$. The group Λ_C has signature (1; p, .', ., p) for some r. Since no proper subgroup of P is its own normalizer (see, e.g. [27], 6.3.9), there exists a finite subnormal chain of subgroups $P = P_0 \triangleright P_1 \triangleright \cdots \triangleright P_{m-1} \triangleright P_m = C$. Let $\Lambda_P = \Lambda_0 \triangleright \Lambda_1 \triangleright \cdots \triangleright \Lambda_{m-1} \triangleright \Lambda_m = \Lambda_C$ be the chain of corresponding Fuchsian groups, say with the orbit genera g_i . By Corollary 4.2, $g_i \leq g_{i+1}$ and so the genus of Λ_P is 0 or 1.

In the first case there exists a pair K, K' of subgroups of P such that $K' \triangleleft K$, [K : K'] = p, and the corresponding Fuchsian groups Λ and Λ' have orbit genera 0 and 1, respectively. Now, since all non-trivial elements of $K/K' \cong \Lambda/\Lambda'$ have order p, we have by Lemma 4.1

$$0 = -p + \frac{p}{2} \sum_{i=1}^{t} \left(1 - \frac{1}{p} \right)$$

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for some t. But the above is equivalent to

$$p = \frac{t}{t-2}$$

which has integral solutions only if p = 2, 3, neither of which satisfy the bound p > 2n + 1 for n = 1, so this case is impossible.

In the second case all Λ_i have genera 1 by Corollary 4.2 and so, by Corollary 4.3, all the elliptic generators of Λ_i are contained in Λ_{i+1} for i = 0, ..., m - 1. We conclude that all the elliptic generators of Λ_P are contained in Λ_C .

Now let C' be another subgroup of P of order p, not conjugate to C, and let $\Lambda_{C'}$ be the corresponding Fuchsian group. Since $\Lambda_C \cap \Lambda_{C'} = \Gamma$, $\Lambda_{C'}$ contains none of the elliptic generators of Λ_P . Thus $\Lambda_{C'}$ is a torsion free Fuchsian group, with orbit genus at least 2 since the value (1) from Sect. 2 must be positive. It follows that C is the unique subgroup of P with orbit genus 1. Since all Sylow p-subgroups are conjugate, there is just one conjugacy class of (p, 1)-gonal groups in the full automorphism group of X, which completes the proof. \Box

Though our initial result holds for all n, if we add the condition n > 1, there are a number of additional results we can derive.

Corollary 4.6 Suppose X is a (p, n)-gonal surface with n > 1 and full automorphism group G. If p > 2n + 1, then p^2 does not divide the order of G.

Proof If p^2 divides the order of *G*, then there is a Sylow subgroup *P* of *G* of order p^k for some $k \ge 2$. As in the proof of Theorem 4.5, it is easy to see that there exists some subgroup *K* of *P* of order p^2 which contains a (p, n)-gonal subgroup *C*. It follows that the group K/C is a group of order *p* acting on a surface of genus *n* bigger than 1 and so we must have $p \le 2n + 1$.

Remark 4.7 If we allow n = 0, 1, Corollary 4.6 is false. Indeed, one can easily construct surfaces which are both (p^2, n) - and (p, n)-gonal for n = 0, 1. For example if K denotes an elementary abelian group of order p^2 , there is a K action with signature $(0; p, p^2, p^2)$ which restricts to a C action with signature $(0; p, \frac{p+2}{2}, p)$. Similarly, there is a K action with signature (1; p, p) which restricts to a C action with signature $(1; p, \frac{2p}{2}, p)$.

The classical theorem of Castelnuovo-Severi [8,26] implies that a (p, n)-gonal group acting on a surface of genus $g > 2pn + (p - 1)^2$ is unique and hence normal in the full automorphism group. (Due to its complexity, we do not state the classical version of the Castelnuovo-Severi here, and instead refer the reader to [3] for a modern exposition of the Castelnuovo-Severi theorem. For a related fact about arbitrary *p*-subgroups of the full automorphism group of a surface, see [20]). This motivates the following definition.

Definition 4.8 A (p, n)-gonal surface of genus $g \ge 2$ is called *strongly* (p, n)-gonal if $g > 2pn + (p-1)^2$.

Our results combined with the Castelnuovo-Severi theorem allow (an inductive) classification of automorphism groups of strongly (p, n)-gonal surfaces of sufficiently high genus.

Corollary 4.9 Suppose X is a strongly (p, n)-gonal surface with full automorphism group G and n > 1. If p > 2n + 1, then G is isomorphic to the semi-direct product $C \rtimes A$, where C is the normal (p, n)-gonal group, A is a group of automorphisms acting on a surface of genus n, and p does not divide the order of A.

Proof By the Castelnuovo-Severi inequality, the (p, n)-gonal group C is normal in G. Now G/C acts on a surface of genus n and by Corollary 4.6, p^2 does not divide the order of G. It follows that G satisfies the short exact sequence

$$1 \to C \to G \to A \to 1$$

where *A* acts on a surface of genus *n*. By the Schur–Zassenhaus theorem (see e.g. [27], Theorem 9.3.6) the sequence splits over *C* and hence $G \cong C \rtimes A$.

Theorem 4.5 shows there is a unique conjugacy class of (p, n)-gonal groups in the full automorphism group when p > 2n + 1. In contrast, the next theorem shows it is possible to construct a (p, n)-gonal surface with n > 1 for which the number of non-conjugate (p, n)-gonal groups is equal to k for any preassigned positive integer $k \ge 2$.

Theorem 4.10 For any fixed prime p and any $k \ge 2$, there exists a (p, n)-gonal surface, for some n > 1, whose full automorphism group contains exactly k non-conjugate properly (p, n)-gonal groups.

Proof Let Λ be a Fuchsian group with signature $(0; p, \stackrel{4k}{\dots}, p), k \ge 2$, and let $\gamma_1, \dots, \gamma_{4k}$ denote a set of elliptic generators for Λ . Let P denote an elementary abelian group of order p^k with generators a_1, \dots, a_k . We now show that there exists a Fuchsian surface group Γ with genus g > 1 such that $P = \Lambda / \Gamma$. We do this by defining an epimorphism $\theta \colon \Lambda \to P$ with torsion free kernel. For $0 \le i < k, 0 \le j < 4$, we define θ as follows:

$$\theta: \gamma_{1+4i+j} \mapsto \begin{cases} a_{i+1} & j = 0, 1\\ a_{i+1}^{-1} & j = 2, 3 \end{cases}$$

Using the Riemann–Hurwitz formula, it follows that P acts on a surface X of genus

$$g = 2kp^{k-1}(p-1) - p^k + 1.$$

Moreover, since $k \ge 2$, Λ can always be chosen as a finitely maximal group (see [29]), from which it follows that *P* acts as the full automorphism group of some surface *X*. We shall now analyze this action.

Using Lemma 4.1, since Λ has signature $(0; p, \stackrel{4k}{\dots}, p)$, all elliptic generators of Λ have order p, so if Γ is any normal subgroup of index N in Λ , then for its orbit genus g' we have:

$$g' = Ns \frac{p-1}{p} - N + 1$$
(3)

where *s* denotes the number of elliptic generators of Λ with non-trivial image under the quotient map $\pi : \Lambda \to \Lambda/\Gamma$. We can use Eq. (3) to determine the genus *n* of each surface X/C for each subgroup *C* of *P* of order *p*. For every *C*, we have $N = p^{k-1}$. For $C = \langle a_i \rangle$, i = 1, 2, ..., r, we have s = 4k - 4 and so X/C has genus

$$n = 4p^{k-2}(p-1)(k-1) - p^{k-1} + 1 > 1.$$

The Riemann–Hurwitz relation for *C* acting on *X* with the quotient of genus *n*, shows that *C* acts with $4p^{k-1}(k-1) \ge 2p$ fixed points, so these *C* actions are proper. Moreover, the groups are non-conjugate, since *P* is abelian. For any other *C* we have s = 4k and thus the genus of *X*/*C* is different from *n*. It follows that *P* has exactly *k* nonconjugate proper (p, n)-gonal subgroups.

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Theorem 4.10 shows that if we are prepared to accept a large genus, we can specify as many non-conjugate (p, n)-gonal groups as desired. However, not surprisingly, if the genus is fixed, there is a bound on the number of nonconjugate properly (p, n)-gonal groups in the full automorphism group of a (p, n)-gonal surface of genus g.

Theorem 4.11 The number of conjugacy classes of properly (p, n)-gonal groups of a Riemann surface of genus $g \ge 2$ is bounded above by

$$\frac{2p - 1 + \sqrt{(2p - 1)^2 + 8gp}}{2(p - 1)}$$

Proof Let *s* be the number of conjugacy classes of proper (p, n)-gonal groups of a Riemann surface $X = \mathcal{H}/\Gamma$ of genus *g*, with full automorphism group $G = \Lambda/\Gamma$. Since *G* has *s* nonconjugate subgroups of order *p* acting with fixed points, Λ has at least *s* periods which are multiples of *p*, and so $\mu(\Lambda) \ge 2\pi(-2 + s(p-1)/p)$. Since $g \ge 2$, Λ is a Fuchsian group with $\mu(\Lambda) > 0$. Now, as distinct subgroups of order *p* intersect trivially, we have $|G| \ge s(p-1) + 1$. So by the Riemann–Hurwitz formula we have

$$s(p-1) + 1 \le |G| = \frac{\mu(\Gamma)}{\mu(\Lambda)} \le \frac{4\pi(g-1)}{2\pi(-2+s(p-1)/p)}.$$

This leads to a quadratic inequality in s. The upper bound on s is the positive root of the corresponding quadratic equation.

Problem 4.12 The bound in Theorem 4.11 is rather generous for n = 0 and the absence of n suggests it may be generous in general. It would be interesting to determine a bound involving n which is attained for infinitely many triples (g, p, n).

5 On the number of conjugate (p, n)-gonal groups

Under the conditions of Theorem 4.5 there is a unique conjugacy class of (p, n)-gonal groups. Here we shall give a bound on the number of such conjugate groups under these conditions. Bounds for n = 0 have been found in [16] and can be also derived from the classification given in [34]. So we we shall assume $n \ge 1$ and p > 2n + 1. We shall use the approach from [16], where the principal tool was the following lemma due to Macbeath [21] on the number of fixed points of an automorphism φ of X in terms of a group G of automorphisms of X with $\varphi \in G$ and the ramification data of the G-action.

Lemma 5.1 Let $X = \mathcal{H}/\Gamma$ be a Riemann surface with automorphism group $G = \Lambda/\Gamma$ and let $\gamma_1, \ldots, \gamma_r$ be a set of elliptic canonical generators of Λ whose periods are m_1, \ldots, m_r , respectively. Let $\theta : \Lambda \to G$ be the canonical projection. Then the number F(f) of points of X fixed by $f \in G$ is given by the formula

$$\mathbf{F}(f) = |\mathbf{N}_G(\langle f \rangle)| \sum \frac{1}{m_i},$$

where N denotes the normalizer and the sum is taken over those i for which f is conjugate to a power of $\theta(\gamma_i)$.

Theorem 5.2 The maximum number of conjugate properly (p, n)-gonal groups, p > 2n+1, $n \ge 1$, of the full automorphism group of a properly (p, n)-gonal surface of genus $g \ge 2$ is

$$\frac{28(g-1)}{(g+2-3n)} \quad \text{if } p = 3, \\ \frac{16(g-1)}{(g+4-5n)} \quad \text{if } p = 5, \\ \frac{6(g-1)(p-1)}{(g+p-1-np)(p-6)} \quad \text{if } p \ge 7.$$
(4)

Proof Let X be a proper (p, n)-gonal Riemann surface of genus $g \ge 2$, with p > 2n + 1, n > 1. Let $\langle \varphi \rangle$ be a (p, n)-gonal group of X. Let $X = \mathcal{H}/\Gamma$ and $A = \Lambda/\Gamma$ denote the full automorphism group of X for Fuchsian groups Γ and Λ with signatures (g; -) and $(h; m_1, \ldots, m_r)$, respectively. Let N be the order of A and let M be the number of (p, n)-gonal subgroups of A. By Theorem 4.5, all of them are conjugate, so $M = [A : N_A(\langle \varphi \rangle)]$ and thus by Lemma 5.1, every period of Λ produces at most N/pM fixed points of φ . Thus, using the Riemann–Hurwitz relation for the (p, n)-gonal group $\langle \varphi \rangle$ acting on a surface of genus g we get

$$\mathcal{F}(\varphi) = \frac{2(g+p-1-np)}{p-1} \le \frac{sN}{pM}$$
(5)

fixed points, where s is the number of periods which are multiples of p.

The Riemann-Hurwitz relation for A acting on a surface of genus g is $4\pi(g-1) = N\mu(\Lambda) = sN\mu(\Lambda)/s$. By (5), $sN \ge pM \cdot F(\varphi)$. It follows that

$$M \le \frac{(g-1)(p-1)}{p(g+p-1-np)} \cdot \frac{2\pi s}{\mu(\Lambda)}.$$
(6)

The maximum value of *M* is obtained by minimizing $\mu(\Lambda)$.

For $h \neq 0$, $\mu(\Lambda) \geq 2\pi s(p-1)/p$, and we obtain

$$M \le \frac{g-1}{g+p-1-np}.\tag{7}$$

If $h = 0, r \ge 3$. First, let $r \ge 4$. Clearly $s \ge 1$ and since $(0; 2, \stackrel{r}{\ldots}; 2, p, \stackrel{s}{\ldots}, p)$ is the signature of a Fuchsian group with a minimal area under these conditions, we have

$$\mu(\Lambda) \ge 2\pi(-2 + (r - s)/2 + s(p - 1)/p)$$

$$\ge 2\pi(-2 + (4 - s)/2 + s(p - 1)/p)$$

$$= \pi s(p - 2)/p,$$

which gives

$$M \le \frac{2(g-1)(p-1)}{(g+p-1-np)(p-2)}.$$
(8)

Now let r = 3 and consider first the case $p \ge 5$. If s = 3, then $\mu(\Lambda) \ge 2\pi(p-3)/p$ since under these conditions, (0; p, p, p) is the signature of a Fuchsian group with the minimal area. Thus

$$M \le \frac{3(g-1)(p-1)}{(g+p-1-np)(p-3)}.$$
(9)

If s = 2 then (0; 2, p, p) is the signature of a Fuchsian group with the minimal area, and so $\mu(\Lambda) \ge \pi(p-4)/p$ and hence

$$M \le \frac{4(g-1)(p-1)}{(g+p-1-np)(p-4)}.$$
(10)

If s = 1 and $p \ge 7$, then a Fuchsian group with signature (0; 2, 3, p) has the minimal possible area and so $\mu(\Lambda) \ge \pi(p-6)/3p$ and so

$$M \le \frac{6(g-1)(p-1)}{(g+p-1-np)(p-6)}.$$
(11)

If s = 1 and p = 5, then a Fuchsian group with signature (0; 2, 4, 5) has the minimal possible area. So $\mu(\Lambda) \ge \pi/10$ and thus

$$M \le \frac{16(g-1)}{g+4-5n} \tag{12}$$

If r = p = 3, and s = 3, then a Fuchsian group with signature (0; 3, 3, 6) has the minimal possible area. So $\mu(\Lambda) \ge \pi/3$ and hence

$$M \le \frac{12(g-1)}{g+2-3n}.$$
(13)

If s = 2, then a Fuchsian group with signature (0; 2, 3, 9) has the minimal possible area. So $\mu(\Lambda) \ge \pi/9$ and therefore,

$$M \le \frac{24(g-1)}{g+2-3n}.$$
(14)

Finally if s = 1, then a Fuchsian group with signature (0; 2, 3, 7) has the minimal possible area. So $\mu(\Lambda) \ge \pi/2$ and thus

$$M \le \frac{28(g-1)}{g+2-3n}.$$
(15)

Comparing inequalities (7)–(15) we obtain the result.

6 Quasiplatonic surfaces

Recall that a compact Riemann surface is quasiplatonic if it is uniformized by a surface group which is a normal subgroup of a triangle group.

Theorem 6.1 Let X be a quasiplatonic, properly (p, n)-gonal surface of genus $g \ge 2$, with full automorphism group G. Then

$$|G| \le \begin{cases} 84(g-1) & \text{if } p = 2, 3, 7, \\ 40(g-1) & \text{if } p = 5, \\ 12p(g-1)/(p-6) & \text{if } p \ge 7. \end{cases}$$

Proof Let $X = \mathcal{H}/\Gamma$, $G = \Lambda/\Gamma$ where Λ is a triangle group and Γ a surface group with signature (g; -). Let C be a (p, n)-gonal group and let $C = \Lambda_C/\Gamma$. Since X is properly (p, n)-gonal, C acts with fixed points. Thus Λ_C has signature (n; p, ..., p), with r > 0. Then $\Lambda_C \leq \Lambda$ has an elliptic element of order p, and so Λ has an elliptic element of order a multiple of p. The signature for a Fuchsian group Λ with minimal area

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with the additional condition that p divides one of the periods is (0; 2, 3, p) for $p \ge 7$, hence $\mu(\Lambda) \ge 2\pi(1 - (1/2) - (1/3) - (1/p)) = 2\pi(p - 6)/6p$ and so the result follows from the Riemann-Hurwitz formula. If p = 2 or 3 (or 7), the signature for a Fuchsian group Λ with minimal area with the additional condition that p divides one of the periods is (0; 2, 3, 7), so the result follows from the Riemann-Hurwitz formula and since $\mu(\Lambda) \ge 2\pi(1 - (1/2) - (1/3) - (1/7))$. Finally, if p = 5, the signature for a Fuchsian group Λ with minimal area with the additional condition that 5 divides one of the periods is (0; 2, 4, 5), so again the result follows from the Riemann-Hurwitz formula and since $\mu(\Lambda) \ge 2\pi(1 - (1/2) - (1/4) - (1/5))$.

Theorem 6.2 For every prime p and n > 1 there are just finitely many quasiplatonic strongly (p, n)-gonal surfaces. More precisely, no surface of genus g > 42p(n - 1) is quasiplatonic and strongly (p, n)-gonal.

Proof Let $X = \mathcal{H}/\Gamma$ be a quasiplatonic, strongly (p, n)-gonal surface with full automorphism group $G = \Lambda/\Gamma$, where Λ is a triangle group and Γ a surface group with signature (g; -). Let *C* be the unique (p, n)-gonal group. The quotient group G/C acts on a surface of genus n > 1 and hence $|G/C| \le 84(n - 1)$. It follows that $|G| \le 84p(n - 1)$. Since *G* acts with a triangular signature (k, l, m), by the Riemann–Hurwitz relation,

$$2g - 2 = |G|(1 - (1/k) - (1/l) - (1/m)) < 84p(n - 1).$$

Hence, g < 1+42p(n-1). If there were infinitely many quasiplatonic, strongly (p, n)-gonal surfaces, there would be infinitely many of a given fixed genus $g_0 < 1+42p(n-1)$ which is clearly absurd.

Remark 6.3 If we allow n = 0, 1 the theorem is false: For n = 0 and p = 2, the strong conditon is merely g > 1, and there are at least 3 quasiplatonic hyperelliptic surfaces for every $g \ge 2$ (the Bolza, Wiman, and Accola-Maclachlan surfaces). For n = 1 and p = 2, and any positive integer N, there are infinite sequences of genera g in which there are more than N distinct strongly (2, 1)-gonal quasiplatonic surfaces [31].

We conjecture that the strong hypothesis is essential.

Conjecture 6.4 For every prime p and $n \ge 0$ there are infinitely many genera $g \ge 2$ for which there are quasiplatonic (p, n)-gonal surfaces.

7 Examples

We finish by considering a couple of examples to illustrate our results. In each of the examples, the signatures of subgroups of Fuchsian groups were calculated using Theorem 1 of [28]. Other calculations were performed using the computer algebra system GAP [12].

Example 7.1 Let X denote Bring's genus 4 surface. Then X is uniformized by a surface group Γ which is a normal subgroup of index 120 of a Fuchsian group Λ with signature (0; 2, 4, 5), and the group $G = \Lambda / \Gamma$ is isomorphic to the symmetric group S₅. We tabulate the conjugacy classes of each (p, n)-gonal group in Table 1.

Our calculations show that *X* is (2, 1)-gonal, (2, 2)-gonal, (3, 2)-gonal and (5, 0)-gonal. Note that *X* is not strongly (p, n)-gonal for any *p*, and for each (p, n), we do not have p > 2n + 1. Also, the action is not properly (3, 2)-gonal since there are no fixed points of the action of a (3, 2)-gonal group on *X*.

Signature

(2:2.2)

(1;2,2,2,2,2,2)

n

1

2

р

2

2

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Table 1 (p, n)-gonal classes for

Bring's curve

	3	2	(2;-)	10
	5	0	(0;5,5,5,5)	6
Table 2 (p, n)-gonal classes for a Hurwitz curve of genus 14	р	п	Signature	Number of groups in class
	2	6	(6;2,2,2,2,2,2,2,2)	91
	3	4	(4;3,3,3,3)	91
	7	2	(2;7,7)	78
	13	2	(2;-)	14
	13	2	(2;)	14
<i>Example 7.2</i> Let X be a Hu	rwtiz cr	urve of	genus 14 (there are t	hree such surfaces, up to
conformal equivalence, but it gonal properties). Then X is	is straig uniform	ghtforw nized by	ard to show that all the a surface group Γ with	ree share the same (p, n) - hich is a normal subgroup
of index 1092 of a Fuchsian	group /	A with s	signature $(0, 2, 3, 7)$	and the group $G = \Lambda / \Gamma$

conformal equivalence, but it is straightforward to show that all three share the same (p, n)gonal properties). Then X is uniformized by a surface group Γ which is a normal subgroup
of index 1092 of a Fuchsian group Λ with signature (0; 2, 3, 7), and the group $G = \Lambda / \Gamma$ is isomorphic to the group $L_2(13)$. We tabulate the conjugacy classes of each (p, n)gonal group in Table 2. Our calculations show that X is (2, 6)-gonal, (3, 4)-gonal, (7, 2)-gonal and
improperly (13, 2)-gonal.

For the proper action of the (7, 2)-gonal group, we have $7 > 2 \cdot 2 + 1 = 5$, so that Theorem 5.2 applies. A simple calculation shows that the upper bound on the number of conjugate properly (7, 2)-gonal groups is attained in this case.

We finish the paper with a conjecture motivated by the last example.

Conjecture 7.3 The bound from Theorem 5.2 is attained for infinitely many triples (g, p, n).

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Number of groups in class

10

15

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