Discrete Groups and Riemann Surfaces

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July 6, 2013

Abstract

These notes summarize four expository lectures delivered at the Advanced School of the ICTS Program Groups, Geometry and Dynamics, December, 2012, Almora, India. The target audience was a group of students at or near the end of a traditional undergraduate math major. My purpose was to expose the types of discrete groups that arise in connection with Riemann surfaces. I have not hesitated to shorten or omit proofs, especially in the later sections, where I thought completeness would interrupt the narrative flow. References and a guide to the literature are provided for the reader who demands all the details.

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1 Prerequisites

To progress beyond the definition of a Riemann surface, one needs to know a little bit about a lot of things. Accordingly, here are the prerequisites necessary to begin these notes. (i) Complex analysis: analytic functions, conformal mappings, Taylor series as in [4]. (ii) Topology: open sets, homeomorphisms, open mappings, the fundamental group, covering spaces as in [28]. (iii) Groups and group actions: permutation groups, normal subgroups, factor groups, isotropy subgroups, group presentations as in [38]. (iv) Hyperbolic geometry: the upper half plane and disk models, the Gauss-Bonnet theorem as in [6]. We start from this broad baseline.

I give a brief guide to further reading in the final section, for those readers whose appetite has been whetted by these brief notes.

2 Riemann surfaces

A Riemann surface is an abstract object that, locally, looks like an open subset of the complex plane $\mathbb{C}$. This means one can do complex analysis in a neighborhood of any point. Globally, a Riemann surface may be very different from $\mathbb{C}$, however. For example, it could be compact, and it need not be simply connected. Here is the technical definition.

Definition 1. A Riemann surface X is a second-countable, connected, Hausdorff space with an atlas of *charts*, $\phi_\alpha : U_\alpha \to V_\alpha$, where U_α, V_α are open subsets of X, $\mathbb{C}$, respectively, and ϕ_α is a homeomorphism. For every pair of charts ϕ_α, ϕ_β with overlapping domains, the *transition map*,

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is bianalytic, that is, analytic with analytic inverse.

Some basic examples follow.

2.1 The Riemann sphere

A two-chart atlas on $S^2 = \{(x, y, w) \in \mathbb{R}^3 \mid x^2 + y^2 + w^2 = 1\}$ is given by *stereographic projection* from the north and south poles:

$$\begin{aligned}
\phi_1 : S^2 \setminus (0,0,1) \to \mathbb{C}, \quad & (x,y,w) \mapsto \frac{x}{1-w} + i\frac{y}{1-w} \\
\phi_2 : S^2 \setminus (0,0,-1) \to \mathbb{C}, \quad & (x,y,w) \mapsto \frac{x}{1+w} - i\frac{y}{1+w}.
\end{aligned}$$

The inverses of these maps are

$$\begin{aligned}
\phi_1^{-1}(z) &= \left(\frac{2\mathrm{Re}(z)}{|z|^2+1}, \frac{2\mathrm{Im}(z)}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right) \\
\phi_2^{-1}(z) &= \left(\frac{2\mathrm{Re}(z)}{|z|^2+1}, \frac{-2\mathrm{Im}(z)}{|z|^2+1}, \frac{1-|z|^2}{|z|^2+1}\right).
\end{aligned}$$

The transition map $\phi_2 \circ \phi_1^{-1}$ is simply $z \mapsto 1/z$.

2.2 The graph of an analytic function

For an analytic function $w = g(z)$ whose domain contains the open set $U \subseteq \mathbb{C}$, the graph $\{(z, g(z)) \mid z \in U\} \subseteq \mathbb{C}^2$, with the single chart $\pi_z : (z, g(z)) \mapsto z$, is a Riemann surface.

2.3 Smooth affine plane curves

Definition 2. An *affine plane curve* X is the zero locus of a polynomial $f(z, w) \in \mathbb{C}[z, w]$. It is *non-singular* or *smooth* if, for all $p = (a, b) \in X$, the partial derivatives $f_z(p)$ and $f_w(p)$ are not simultaneously zero.

By the implicit function theorem, in a neighborhood of every point p on a smooth affine plane curve, at least one of the coordinates z, w is an analytic function of the other, depending on which partial derivative is $\neq 0$. If $f_w(p) \neq 0$, there is an open set U containing p such that, for all $q = (z, w) \in U$, $w = g(z)$, an analytic function of z. Thus $\pi_z : U \to \mathbb{C}$ is a local chart. If, also, $f_z(p) \neq 0$, there is an open set V containing p such that, for all $q = (z, w) \in V$, $z = h(w)$, an analytic function of w. Then $\pi_w : V \to \mathbb{C}$ is also a local chart. The transition functions,

$$\pi_w \circ \pi_z^{-1} : z \mapsto g(z)$$
$$\pi_z \circ \pi_w^{-1} : w \mapsto h(w),$$

defined on $\pi_z(U \cap V)$ and $\pi_w(U \cap V)$, respectively, are, by construction, analytic. Thus a smooth affine plane curve, if connected, is a Riemann surface.

Remark 1. Connectivity can be guaranteed by assuming the polynomial $f(z,w)$ is *irreducible*, that is, not factorable into terms of positive degree. This is a standard result in algebraic geometry which is beyond the scope of this paper. See [36].

2.4 Smooth projective plane curves

The one-dimensional subspaces of the vector space $\mathbb{C}^3$ are the 'points' of the *complex projective plane* $\mathbb{P}^2$. The span of $(x,y,z) \in \mathbb{C}^3$, $(x,y,z) \neq (0,0,0)$, is denoted $[x:y:z]$. For $\lambda \in \mathbb{C}$, $\lambda \neq 0$,

$$[x:y:z] = [\lambda x : \lambda y : \lambda z].$$

x, y and z are *homogeneous coordinates* on $\mathbb{P}^2$: being defined only up to a common scalar multiple, no coordinate takes on any "special" or fixed value. $\mathbb{P}^2$ is a complex manifold of dimension 2, covered by three sets, defined by $x \neq 0$, $y \neq 0$, $z \neq 0$, respectively. In homogenous coordinates, we may assume that $|x|^2 + |y|^2 + |z|^2 = 1$; in particular, that $|x|, |y|, |z| \leq 1$. Thus $\mathbb{P}^2$ is compact.

Definition 3. A polynomial $F(x,y,z) \in \mathbb{C}^3$ is *homogeneous* if, for every $\lambda \in \mathbb{C}^*$, $F(\lambda x, \lambda y, \lambda z) = \lambda^d F(x,y,z)$, where d is the degree of the polynomial.

On $\mathbb{P}^2$, the value of a homogeneous polynomial $F(x,y,z)$ is not well-defined, but the zero locus is.

Definition 4. A *projective plane curve* X is the zero locus in $\mathbb{P}^2$ of a homogeneous polynomial $F(x,y,z) \in \mathbb{C}[x,y,z]$. It is *non-singular* (smooth) if, there is no point $p = [x:y:z] \in X$, at which all three partial derivatives $\partial_x F(p)$, $\partial_y F(p)$, and $\partial_z F(p)$ vanish simultaneously.

An affine plane curve $f(x,y) = 0$ can be "projectivized" (and thereby, compactified) by the following procedure: multiply each term of the defining polynomial f by a suitable power of a new variable z so that all terms have the same (minimal) degree. Then the *affine portion* of the projectivized curve corresponds to $z = 1$, and the *points at infinity* correspond to $z = 0$.

Theorem 1. *A nonsingular projective plane curve is a compact Riemann surface.*

Proof. Let $U_i = \{[x_0 : x_1 : x_2] \subseteq \mathbb{P}^2 \mid x_i \neq 0\}$, $i = 0,1,2$. (Up to a nonzero scalar factor, $x_i \neq 0$ is equivalent to $x_i = 1$.) Let X be a smooth projective plane curve defined as the zero locus of the homogenous polynomial $F(x_0, x_1, x_2)$, and let $X_i = X \cap U_i$. Each X_i is an affine plane curve, e.g.,

$$X_0 = \{(a, b) \in \mathbb{C}^2 \mid F(1, a, b) = 0\}.$$

For a homogeneous polynomial F of degree d,

$$F(x_0, x_1, \dots x_k) = \frac{1}{d} \sum_{i=0}^{k} x_i \partial_i F.$$

This is known as Euler's formula, and it implies (exercise) that X is nonsingular if and only if each X_i is a smooth affine plane curve. Coordinate charts on X_i are ratios of homogeneous coordinates on X, and as such they are well-defined. For example, charts on X_0 are x_1/x_0 or x_2/x_0, and charts on X_2 are x_0/x_2 or x_1/x_2. Transition functions are readily seen to be holomorphic, e.g., near $p \in X_0 \cap X_1$, where $x_0, x_1 \neq 0$, let $z = \phi_1 = x_1/x_0$ and $w = \phi_2 = x_2/x_1$. Then

$$\phi_2 \circ \phi_1^{-1} : z \mapsto [1 : z : h(z)] \mapsto \frac{h(z)}{z} = w,$$

where $h(z)$ is a holomorphic function, and $z \neq 0$, since $p \in X_1$. Connectivity is required to make X_i (and hence X) a Riemann surface. Nonsingular homogeneous polynomials are automatically irreducible [36], so connectivity follows from Remark 1. □

Remark 2. Projective spaces $\mathbb{P}^n$ can be defined for all $n \geq 1$. For example, $\mathbb{P}^1$, the *complex projective line*, is the space of one-dimensional subspaces of $\mathbb{C}^2$,

$$\{[x : y] \mid x, y \in \mathbb{C}, (x, y) \neq (0, 0)\},$$

where $[x : y] = [\lambda x : \lambda y]$, $\lambda \in \mathbb{C}^*$. The two-chart atlas

$$\phi_0 : \mathbb{P}^1 \setminus \{[0 : 1]\} \to \mathbb{C}$$
$$\phi_1 : \mathbb{P}^1 \setminus \{[1 : 0]\} \to \mathbb{C},$$

defined by $[x : y] \mapsto y/x$, resp., $[x : y] \mapsto x/y$, has transition function

$$\phi_1 \circ \phi_0^{-1} : z \mapsto 1/z.$$

This makes $\mathbb{P}^1 \simeq \mathbb{C} \cup \{\infty\}$ a Riemann surface, with ∞ corresponding to the point with coordinates $[1 : 0]$.

3 Holomorphic maps

Definition 5. A map $f : X \to Y$ between Riemann surfaces is *holomorphic* if, for every $p \in X$, there is a chart $\phi : U_p \to \mathbb{C}$ defined on a neighborhood of p, and a chart $\psi : V_{f(p)} \to \mathbb{C}$ defined on a neighborhood of $f(p) \in Y$, such that $\psi \circ f \circ \phi^{-1} : \phi(U_p) \to \psi(V_{f(p)})$ is analytic.

Locally, as we shall see, non-constant holomorphic maps between compact Riemann surfaces look like maps of the form $z \mapsto z^m$. By 'look like,' we mean 'read through suitable local charts,' as in Definition 5. Globally, they look like covering maps, except possibly at a finite set of points.

3.1 Automorphisms

Riemann surfaces X and Y are *isomorphic* or *conformally equivalent* if there exists a holomorphic bijection $f : X \to Y$ with a holomorphic inverse (a *biholomorphism*). For example, it is an easy exercise to show that the complex projective line $\mathbb{P}^1$ and the Riemann sphere are isomorphic (cf. Remark 2 and Section 2.1).

A self-isomorphism $f : X \to X$ of a Riemann surface is called an *automorphism*. The automorphisms form a group $G = \mathrm{Aut}(X)$ under composition. Those fixing a particular point $p \in X$ form a subgroup $G_p \leq G$ called the *isotropy subgroup of* p. The following lemma is, essentially, a consequence of the fact that a finite subgroup of the multiplicative group $\mathbb{C}^*$ of non-zero complex numbers is cyclic (generated by a roots of unity). For a full proof, see, e.g., [34], Proposition 3.1.

Lemma 2. *If G is a finite group of automorphisms of a Riemann surface X, and $G_p \leq G$ is the isotropy subgroup of a point $p \in X$, then G_p is cyclic.*

3.2 Meromorphic functions

A *meromorphic function* on a Riemann surface X is a surjective holomorphic map $f : X \to \mathbb{P}^1$, i.e., it can take the value ∞. We shall see shortly (Lemma 3 below) that when X is compact, 'surjective' is equivalent to 'non-constant.' We collect some examples of meromorphic functions.

- The meromorphic functions on $\mathbb{P}^1$ are the *rational functions* $r(z) = \dfrac{p(z)}{q(z)}$, where $p, q \in \mathbb{C}[z]$, $q \neq 0$.

- The meromorphic functions on the smooth affine plane curve defined by $f(x, y) = 0$ are the rational functions

$$r(x,y) = \frac{p(x,y)}{q(x,y)}, \quad p, q \in \mathbb{C}[x,y],$$

  where $q(x, y)$ does not vanish identically on the curve. Equivalently, $q(x, y)$ is not a divisor of $f(x, y)$.

- The meromorphic functions on a smooth projective plane curve defined by the vanishing of the homogeneous polynomial $F(x, y, z)$, are the rational functions

$$R(x,y,z) = \frac{P(x,y,z)}{Q(x,y,z)}, \quad P, Q \in \mathbb{C}[x,y,z],$$

where P and Q are homogeneous of the same degree, and Q is not a divisor of F.

3.3 The local normal form

Holomorphic maps inherit many properties of analytic maps. Let $f : X \to Y$ be a nonconstant holomorphic map between between Riemann surfaces. Then, as with an analytic map from $\mathbb{C}$ to $\mathbb{C}$,

- f is an *open mapping* (taking open sets to open sets);
- If $g : X \to Y$ is another holomorphic map, and f and g agree on a subset $S \subseteq X$ with a limit point in X, then $f = g$;
- $f^{-1}(y)$, $y \in Y$, is a *discrete* subset of X.

Lemma 3. *If X is a compact Riemann surface and $f : X \to Y$ is a nonconstant holomorphic map, then f is* onto, *Y is* compact, *and $f^{-1}(y) \subseteq X$, $y \in Y$, is a* finite *set.*

Proof. $f(X) \subseteq Y$ is a compact subset of Hausdorff space and hence closed. It is also open since f is an open mapping. Y is connected by definition, hence $f(X)$ is all of Y. Finally, $f^{-1}(y)$ is a discrete subset of a compact space and therefore finite. □

Theorem 4. *If $f : X \to Y$ is a nonconstant holomorphic map, and $p \in X$, there exists a unique positive integer $m = mult_p(f)$ (the* multiplicity of f at *p) and local coordinate charts $\phi : U \subseteq X \to \mathbb{C}$ centered at p (i.e., having $\phi(p) = 0$) and $\psi : V \subseteq Y \to \mathbb{C}$ centered at $f(p)$, such that $\psi \circ f \circ \phi^{-1} : \phi(U) \to \psi(V)$ has the* local normal form *$z \mapsto z^m$.*

Proof. Take arbitrary coordinate charts and center them at p and $f(p)$ by translation coordinate changes. Let $T(w) = \sum_{i=m}^{\infty} c_i w^i$ be the Taylor series of f in the local coordinate w centered at p. Since $T(0) = 0$, $m \geq 1$, and $T(w) = w^m S(w)$, with $S(w)$ analytic at 0 and $S(0) \neq 0$. It follows that $S(w)$ has a local mth root, $R(w)$. Let $z = z(w) = wR(w)$. We have $z(0) = 0$ and $z'(0) = R(0) \neq 0$, so on an open subset containing p, $z(w)$ is a new complex coordinate for a new chart centered at p. Reading through this new chart, f has the form $z \mapsto z^m$. (Uniqueness of m is left to the reader.) □

Definition 6. A point $q \in X$ for which $\text{mult}_q(f) > 1$ is called a *ramification point*; the image of a ramification point (in Y) is called a *branch point*. The set of branch points is called the *branch set*.

In local coordinates $w = h(z)$, ramification points occur at all z_0 for which $h'(z_0) = 0$. These are isolated points, hence the branch set B and its pre-image $f^{-1}(B)$ are *discrete* subsets of Y, X, respectively.

We come to the crucial global property of a holomorphic map between *compact* surfaces:

Theorem 5. *If $f : X \to Y$ is a nonconstant holomorphic map between compact surfaces, there exists a unique positive integer d such that,* for every $y \in Y$,

$$\sum_{p \in f^{-1}(y)} mult_p(f) = d.$$

Proof. The open unit disk $D \subseteq \mathbb{C}$ is a Riemann surface, and for the holomorphic map $f : D \to D$, defined by $z \to z^m$, the theorem is clearly true: 0 is the unique point in $f^{-1}(0)$, and the multiplicity at 0 is m; if $a \in D$, $a \neq 0$, $f^{-1}(a)$ consists of m distinct points (the m mth roots of a), at which the multiplicity of f is 1. Thus the total multiplicity over every point in D is m. A general nonconstant holomorphic map, over each point in its range, is a kind of union of such power maps. That is, for every $y \in Y$ there is a neighborhood V_y containing y such that $f^{-1}(V_y)$ is a union of open sets $U_i \subseteq X$ which can be assumed pairwise disjoint by the the discreteness of $f^{-1}(y) \subset X$. One can replace each U_i by an open disk $D_i \subseteq U_i$ centered at p_i and V_y by an open disk $D_y \subset V_y$ centered at y. Now define $d_y = \sum_i \text{mult}_{p_i}(f)$. There are finitely many summands by discreteness of $f^{-1}(y) \subset X$ and the compactness of X. The map $y \mapsto d_y : Y \to \mathbb{N}$ is locally constant, since it is when restricted to each D_y. Suppose there is $y_1 \in Y$ such that $d_y \neq d_{y_1}$. By the connectedness of Y, there is a path from y to y_0 which can be covered by open sets on which d_y is constant. Hence $d_y = d_{y_1}$, a contradiction. Thus d_y is globally constant, independent of y. □

Remark 3. d is called the *degree* of f. The theorem explains why f is also called a *branched covering map*: the branch locus $B \subset Y$ and its preimage $f^{-1}(B)$ are discrete and hence finite (by compactness of Y). Thus, away from finitely many points, f is a covering map of degree d (every point in $Y \setminus B$ is contained in an open set U whose pre-image is a disjoint union of d open sets, each homeomorphic to U).

Remark 4. An automorphism $f : X \to X$ is a holomorphic map of degree 1.

3.4 The Riemann-Hurwitz relation

Topologically, compact oriented surfaces are completely classified by the *genus* $g \geq 0$. All such surfaces admit triangulations; for any triangulation,

$$\#\{\text{vertices}\} - \#\{\text{edges}\} + \#\{\text{faces}\} = 2 - 2g,$$

a constant, known as the Euler characteristic of the surface. If $f : X_g \to Y_h$ is a covering map of degree d between compact oriented surfaces of genera g, h, resp., then $2g - 2 = d(2h - 2)$. For branched coverings (in particular, for holomorphic maps) we have:

Theorem 6 (Riemann-Hurwitz relation). *If $f : X_g \to Y_h$ is a nonconstant holomorphic map of degree d between compact Riemann surfaces of genera g,h, respectively,* *then*

$$2g-2=d(2h-2)+\sum_{p\in X}(\mathit{mult}_p(f)-1).$$

Proof. Let Y be triangulated so that the branch locus $B\subset Y$ is contained in the vertex set. Let v, e, f be the number of vertices, edges and faces respectively. The triangulation lifts through the covering of degree d to a triangulation of X which has de edges and df faces, but only

$$dv-\sum_{b\in B}(d-|f^{-1}(b)|)$$

vertices. Hence

$$2-2g=dv-de+df-\sum_{b\in B}(d-|f^{-1}(b)|).$$

Since $dv-de+df=d(2-2h)$, it suffices to show that

$$\sum_{b\in B}(d-|f^{-1}(b)|)=\sum_{p\in X}(\mathrm{mult}_p(f)-1).$$

Let $B=\{b_1,b_2,\ldots,b_n\}$. We make use of the trivial fact that $|f^{-1}(b_i)|=\sum_{x\in f^{-1}(b_i)}1$, together with the constancy of the degree $\sum_{x\in f^{-1}(b_i)}\mathrm{mult}_x(f)=d$, to rewrite the sum

$$\begin{aligned}\sum_{b\in B}(d-|f^{-1}(b)|)&=\sum_{i=1}^{n}(d-|f^{-1}(b_i)|)\\&=\sum_{i=1}^{n}\sum_{p\in f^{-1}(b_i)}(\mathrm{mult}_p(f)-1)\\&=\sum_{p\in X}(\mathrm{mult}_p(f)-1).\end{aligned}$$

At the final step, we use the fact that $\mathrm{mult}_p(f)=1$ whenever $p\notin f^{-1}(B)$. □

3.5 Fermat curves

Let X be the smooth projective plane curve which is the zero locus of the polynomial $F(x,y,z)=x^d+y^d+z^d$, $d\geq 2$. Consider the holomorphic map $\pi:X\to\mathbb{P}^1$, given in homogenous coordinates by

$$\pi:[x:y:z]\mapsto[x:y].$$

It has degree d, since $\pi^{-1}([x:y])$ is in bijection with the set of dth roots of $-x^d-y^d$. If $x^d=-y^d$, $|\pi^{-1}([x:y])|=1$ and the multiplicity of π is d. There are d such points, namely, $[1:\omega:0]$, where ω is a dth root of -1. At all other points, the multiplicity of π is 1. The genus of $\mathbb{P}^1$ is 0 so the Riemann-Hurwitz relation $2g_X - 2 = d(-2) + d(d-1)$ yields

$$g_X = \frac{(d-1)(d-2)}{2}.$$

Remark 5. Surprisingly, this *degree-genus formula* holds for *any* smooth projective curve of degree d (see [24], Chapter 4).

3.6 Cyclic covers of the line

Let $h(x)$ be a polynomial of degree k, and consider the affine plane curve $C = \{(x,y) \in \mathbb{C}^2 \mid y^d = h(x)\}$, where $d \geq 2$. If h has distinct roots, the projection $\pi_x : X \to \mathbb{C}$, $(x,y) \mapsto x$ ramifies with multiplicity d over the roots of h, and is a d-fold covering over all other points in $\mathbb{C}$. We compactify C to $\overline{C}$ by projectivization. Then π_x extends to a map $\overline{\pi}_x : \overline{C} \to \mathbb{P}^1$. What happens "at infinity" (i.e., as $x \to \infty$)? Suppose $k = dt$, $t \geq 1$ (a non-trivial assumption). For $x \neq 0$ (i.e., in a neighborhood of ∞), the map $(x,y) \leftrightarrow (1/x, y/x^t)$ is bianalytic and defines new coordinates $z = 1/x$, $w = y/x^t$. The defining equation of C transforms to

$$\begin{aligned} w^d = y^d/x^k = y^d z^k = h(x)z^k = h(1/z)z^k \\ = (1 - za_1)(1 - za_2) \cdot \dots \cdot (1 - za_k) = g(z) \end{aligned}$$

where $a_1, \dots, a_k$ are the roots of $h(x)$. The dth roots of $g(0) \neq 0$ correspond to d points at ∞.

Thus $\overline{\pi}_x : \overline{C} \to \mathbb{P}^1$ is a holomorphic map of degree d between compact Riemann surfaces (in fact, a meromorphic function) which ramifies at k points (over the k distinct zeroes of $h(x)$, but not over ∞) with multiplicity d. The Riemann-Hurwitz relation determines the genus of $\overline{C}$ as follows.

$$\begin{aligned} 2g_{\overline{C}} - 2 &= d(-2) + k(d-1) \\ g_{\overline{C}} &= (d-1)(k-2)/2. \end{aligned}$$

Remark 6. $\overline{C}$ admits a cyclic group of *automorphisms* of order d, which explains the name (cyclic cover). The group is generated by

$$\alpha : (x,y) \mapsto (x, \omega y),$$

where ω is a primitive dth root of unity. It is clear that α preserves the solution set of the defining equation $y^d = h(x)$. α fixes the k ramification points, and permutes all other points in orbits of length d. If $d = 2$, $\overline{C}$ is called *hyperelliptic* and α is the *hyperelliptic involution*, with $k = 2g + 2$ fixed points.

3.7 Resolving singularities

To treat the most general cyclic coverings of the line (and algebraic curves in general), we must deal with singular points, where all partial derivatives of the defining polynomial vanish simultaneously.

Definition 7. A point $p = (x_0, y_0)$ on an affine plane curve $f(x, y) = 0$ is *singular* if $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. A singularity is *monomial* if there are local coordinates (z, w) centered at p in which the defining equation has the form $z^n = w^m$, $n, m > 1$.

Consider again the affine curve defined by $y^d = h(x)$, where $d \geq 2$ and $h(x)$ is a polynomial of degree k. But now, do not assume, as we did in Section 3.6, that h has distinct roots, or that k is a multiple of d. Let

$$h(x) = (x - a_1)^{e_1}(x - a_2)^{e_2} \dots (x - a_r)^{e_r}, \quad a_i \in \mathbb{C},$$

with multiplicities $e_i \geq 1$, and $\sum_{i=1}^{r} e_i = k$; and let

$$k = dt - \epsilon, \quad t \geq 1, \quad 0 \leq \epsilon \leq d - 1.$$

Evidently, C contains singular points whenever $x = a_i$ and $e^i > 1$. In addition, its compactification $\overline{C} \subset \mathbb{P}^2$ may contain monomial singularities *at* ∞. The projection $\pi_x : (x, y) \mapsto x$ is a coordinate chart on the affine portion. For the points at ∞, we change to the coordinates $z = 1/x$, $w = y/x^t$. In the new coordinates, the defining equation

$$y^d = (x - a_1)^{e_1}(x - a_2)^{e_2} \dots (x - a_r)^{e_r} = h(x)$$

transforms to

$$\begin{aligned} w^d &= y^d/x^{k+\epsilon} = y^d z^{k+\epsilon} = h(x) z^{k+\epsilon} = h(1/z) z^{k+\epsilon} \\ &= z^\epsilon (1 - z a_1)^{e_1} (1 - z a_2)^{e_2} \dots (1 - z a_r)^{e_r} = z^\epsilon g(z). \end{aligned}$$

Since $g(0) \neq 0$, in a neighborhood of ∞ (i.e., near $z = 0$), the defining equation is approximately $w^d \approx \text{constant} \cdot z^\epsilon$.

Similarly, near a root a_i of $h(x)$ with multiplicity $e_i > 1$, the equation is

$$\begin{aligned} y^d &= a_1^\epsilon (x - a_1)^{e_1} (a_1 - a_2)^{e_2} \dots (a_1 - a_r)^{e_r} \\ &\approx \text{constant} \cdot X^{e_1}, \end{aligned}$$

where $X = x - a_1$. So there is a monomial singularity of type (d, e_i) here as well.

Theorem 7. *On an affine plane curve, a monomial singularity of type $z^n = w^m$ is resolved by removing the singular point and adjoining gcd(n, m) points.*

Proof. We consider three cases. (i) If $n = m$, $z^n - w^n$ factors as

$$z^n - w^n = \prod_{i=0}^{n-1} (z - \zeta^i w),$$

where ζ is a primitive n th root of unity. Each factor defines a smooth curve. The singularity is resolved by removing the common point $(0, 0)$ and replacing it with n distinct points. (ii) If $\gcd(n, m) = 1$ (relatively prime), there exist $a, b \in \mathbb{Z}$ such that $an+bm = 1$. The map $\phi : (z, w) \mapsto z^b w^a$ defines a "hole chart." This is a chart whose domain is the curve minus the singular point $\{(0, 0)\}$ and whose co-domain is the "punctured" plane $\mathbb{C} \setminus \{0\}$. The inverse chart is $\phi^{-1} : s \mapsto (s^m, s^n)$. By continuity, ϕ extends uniquely to the closure of the domain ("restoring" the singular point). (iii) If $\gcd(n, m) = c$, there exist $a, b \in \mathbb{Z}$ such that $n = ac$ and $m = bc$, and $\gcd(a, b) = 1$. Then

$$z^n - w^m = (z^a)^c - (w^b)^c = \prod_{i=1}^{c} (z^a - \zeta^i w^b),$$

where ζ is a primitive c th root of unity. Case 2 applies to each of the c factors; thus c points are adjoined to fill c holes. □

For the following corollary, we make a simplifying assumption to avoid branching at ∞.

Corollary 8 (Genus of a cyclic cover of the line)**.** *Let $y^d = h(x)$, $d \geq 2$, define the cyclic covering $\pi_x : \overline{C} \to \mathbb{P}^1$. Let the polynomial $h(x)$ have r roots of multiplicities $e_1, \ldots, e_r$. Assume $\sum_i^r e_i \equiv 0 \pmod{d}$ (to avoid branching at ∞). The genus of $\overline{C}$ is*

$$g = 1 + \frac{(r-2)d - \sum_{i=1}^{r} \gcd(d, e_i)}{2}.$$

Proof. $\pi_x : \overline{C} \to \mathbb{P}^1$ is a d-sheeted branched covering; over a zero of multiplicity e, there are $\gcd(d, e)$ points, each of multiplicity $d/\gcd(d, e)$. These are the only branch points, by assumption. The formula for the genus follows from the Riemann-Hurwitz relation. □

Exercise 1. For connectivity of $\overline{C}$, $f(x, y) = y^d - h(x)$ must be irreducible. Prove this is the case iff $\gcd\{d, e_1, \ldots, e_k\} = 1$.

4 Galois groups

4.1 The monodromy group

The monodromy group is a finite permutation group associated with a branched covering $f : X \to Y$ between compact surfaces. It completely determines the covering, up to homeomorphism (or biholomorphism, in the category of Riemann surfaces). It is constructed as follows. Let f have degree d, and let $Y^* = Y - B$, where $B = \{b_1, b_2, \ldots, b_n\} \subset Y$ is the branch set and $X^* = X - f^{-1}(B)$. The restricted map

$$f^* : X^* \to Y^*$$

is an (unramified) d-sheeted *covering map*. Choose a basepoint $y_0 \in Y^*$, and let

$$F = (f^*)^{-1}(y_0) = \{x_1, x_2, \ldots x_d\} \subset X^*,$$

the *fiber* over the basepoint. A loop γ_j, based at y_0 and winding once counterclockwise around the puncture created by the removal of b_j (and not winding around any other puncture), has a *unique lift* to a path $\widetilde{\gamma_{j,i}}$ starting at x_i, $i = 1, 2, \dots d$, with a well-defined endpoint belonging to F. (See [28], Chapter 5.)

Lemma 9. *For each $j \in \{1, 2, \dots, n\}$, the 'endpoint of lift' map*

$$\rho_j : i \mapsto \textit{endpoint of } \widetilde{\gamma}_{j,i} \in F, \quad i \in \{1, 2, \dots, d\}$$

is a bijection (hence, an element of S_d, the symmetric group on d symbols).

Proof. Suppose the endpoint of $\widetilde{\gamma}_{j,i}$, say, x_l, coincides with the endpoint of $\widetilde{\gamma}_{j,k}$. Then there is a path in X from x_i to x_k, namely, $(\widetilde{\gamma}_{j,k})^{-1} \circ \widetilde{\gamma}_{j,i}$, which is a lift of the trivial path $(\gamma_j)^{-1} \circ \gamma_j = \{y_0\} \in Y$. This is only possible if $x_i = x_k$. □

Definition 8. The *monodromy group* of f, denoted $M(f, X, Y)$ or just $M(f)$, is the subgroup of S_d generated by the permutations $\{\rho_1, \dots, \rho_n\}$.

Exercise 2. Show that the monodromy group, up to isomorphism, is independent of the choices made in its construction.

Remarkably, the cycle structures of the ρ_j's encode all the ramification data of the original *branched* covering $f : X \to Y$, as follows. First, if there are n monodromy generators, recover Y from Y^* by adjoining an n-element branch set $B = \{b_1, \dots, b_n\}$. Over b_j, restore the fiber $f^{-1}(b_j)$ by adjoining one point for each cycle of the monodromy generator ρ_j. The multiplicity of f at $p \in f^{-1}(b_j)$ is the length of the corresponding cycle. For example, if $f : X \to Y$ is a 6-sheeted branched covering, branched over $\{b_1, b_2, \dots, b_n\} \subset Y$, and $\rho_2 = (135)(46)(2) \in S_6$, then $f^{-1}(b_2) \subset X$ consists of three points: one of multiplicity 3 (where the sheets 1, 3 and 5 come together); one of multiplicity 2 (where sheets 4 and 6 come together); and one other point (on sheet 2) of multiplicity 1.

The definition of $M(f, X, Y)$ can be given in terms of the fundamental group $\Gamma = \pi_1(Y^*, y_0)$. By standard covering space theory, f induces an imbedding of the fundamental groups $\{\pi_1(X^*, x_i),\ i = 1, \dots, d\}$, (all of them isomorphic), as a conjugacy class of subgroups $\{D_i \leq \Gamma\}$, each of index d. The 'endpoint of lift' map defines an action of Γ on the fiber $F = f^{-1}(y_0) \subset X^*$. The isotropy subgroup of x_i is D_i, and therefore, the kernel of the action is $D^* = \cap_{i=1}^{d} D_i$. It follows that

$$M(f, X, Y) \simeq \Gamma / D^*. \tag{1}$$

It is easy to see that $M(f)$ acts *transitively* on F: Since X^* is connected, there exists a path $l_j \subset X^*$ from x_1 to x_j, for each $j \in \{1, \dots, d\}$, and this path projects to a loop $f(l_j)$ based at y_0, defining an element of Γ which takes x_1 to x_j.

4.2 Two permutation groups

The *Galois group*, also known as the group of *covering transformations* $G(X^*/Y^*)$ for the unbranched covering $f^* : X^* \to Y^*$, is the set of homeomorphisms $h : X^* \to X^*$ such that $f^* = f^* \circ h$. In the category of Riemann surfaces, covering transformations are automorphisms (without fixed points). $G(X^*/Y^*)$, like $M(f, X, Y)$, can be defined in terms of $\Gamma = \pi_1(Y^*, y_0)$. Let $x_i \in F$, and let $D_i \leq \Gamma$ be defined as above. Then

$$G(X^*/Y^*) \simeq N_\Gamma(D_i)/D_i,$$

where $N_\Gamma(D_i) \leq \Gamma$ is the *normalizer* of D_i in Γ, i.e., the largest subgroup of Γ containing D_i as a normal subgroup. This is a special case of a general theorem about homogeneous group actions on sets (see [28], Corollary 7.3 and Appendix B). Conjugate groups have conjugate normalizers, so the definition is independent of the choice of $x_i \in F$.

The action of $G(X^*/Y^*)$ restricts to a group of permutations of F (exercise: why?), hence both $G(X^*/Y^*)$ and $M(f, X, Y)$ can be viewed as subgroups of S_d. What is the relationship between these two groups? There are some clear differences: (i) $M(f)$ can act with fixed points (the isotropy subgroup of x_i is isomorphic to D_i/D^* which is trivial only if $D_i = D^*$, i.e., only if D_i is a normal subgroup of Γ); on the other hand, it can be shown that the only element of $G(X^*/Y^*)$ that fixes a point is the identity. (ii) $M(f)$ acts transitively, as we have seen, while $G(X^*/Y^*)$ need not. The following extended exercise (for the ambitious reader) gives a purely group-theoretic construction which makes the relationship between the two groups precise. (Apply the exercise to the subgroup-group pair $D_i \leq \Gamma$, for any choice of $i \in \{1, 2, \ldots, d\}$.) However, only the last item is really essential for our purposes.

Exercise 3. Let $K \leq H$ be a subgroup-group pair. Let

$$K^* = \cap_{h \in H} h^{-1} K h,$$

the *core of K in H*, and let $N_H(K) = \{h \in H \mid hK = Kh\}$, the normalizer of K in H. Assume that the index $[H : K^*]$ (hence also $[H : K]$, and $[N_H(K) : K]$) is finite. There are two natural finite permutation groups defined on the set $R = \{Kh \mid h \in H\}$ of right cosets of K in H:

- The right (monodromy-type) action $R \times H/K^* \to R$, given by

$$(Kh, K^* h_2) \mapsto Khh_2;$$

- the left (Galois-type) action $N_H(K)/K \times R \to R$, given by

$$(Kh_1, Kh) \mapsto Kh_1 h \quad \text{(where } h_1 \in N_H(K)\text{)}.$$

Show:

1. The actions are *well-defined* and *faithful*, i.e., a group element that fixes every coset in R is the identity.

2. The actions *commute*: $(Kh_1)(Khh_2) = (Kh_1h)(K^*h_2)$.

3. The monodromy-type action is *transitive*.

4. The Galois-type action is *regular*: if $h_1 \in N_H(K)$, and $Kh_1h = Kh$, then $h_1 \in K$ (i.e., all isotropy subgroups are trivial).

5. If K is normal in H (i.e, $K^* = K$, $N_H(K) = H$), the two groups are isomorphic ($\simeq H/K$) and the actions reduce to the left and right *regular* representations of H/K on itself.

4.3 Galois coverings

Item 5 in the previous exercise shows that $M(f, X, Y)$ and $G(X^*, Y^*)$ are isomorphic when the subgroups $D_i \leq \Gamma$ are normal. In this case $f : X \to Y$ is called a *Galois* covering. The covering transformations in $G(X^*, Y^*)$ extend by continuity to automorphisms of the original surface X. The group of extended covering transformations is also called the *Galois group* of f (being isomorphic to $G(X^*, Y^*)$) but the *actions* are distinct. For example, there are nontrivial isotropy subgroups at the restored points $X - X^*$. Recall that the fiber over $b_j \in Y$ is restored by adjoining one point to X^* for each cycle of the monodromy generator ρ_j. Via the isomorphism $M(f) \simeq G(X^*, Y^*)$, ρ_j encodes the local permutation of the sheets in a neighborhood of a restored point in the fiber over b_j. At such a point, the permuted sheets come together, and the length of the corresponding cycle of ρ_j is the order of the local (cyclic!) isotropy subgroup. Moreover, since the points of $f^{-1}(b_j)$ comprise an *orbit* of the Galois group, all the cycles of ρ_j must have the same length (exercise: why?). Let $r_j > 1$ denote the common cycle length of ρ_j. r_j is also the order of ρ_j, and hence it is a nontrivial divisor of the order of the Galois group.

Definition 9. The *branching indices* of the Galois covering $f : X \to Y$ are the integers $r_1, \ldots, r_n$, where n is the cardinality of the branch set $B \subset Y$.

In summary: the index $r_j > 1$ assigned to b_j means that ρ_j is a product of d/r_j cycles of length r_j, where d is the degree of the Galois covering, that is, the order of the Galois group.

Exercise 4. Show that, for a Galois covering, the ramification term in the Riemann-Hurwitz relation (cf. Theorem 6) has the following equivalent form in terms of the branching indices:

$$\sum_{p \in X} \mathrm{mult}_p(f) - 1 = \sum_{i=1}^{n} \frac{|G|}{r_i}(r_i - 1).$$

From this, derive

Theorem 10 (Riemann-Hurwitz relation for a Galois covering). *If $f : X \to Y$ is a Galois covering with Galois group G of order $|G|$ and branching indices $\{r_1, \dots, r_n\}$, then*

$$2g - 2 = |G|\left(2h - 2 + \sum_{i=1}^{n}(1 - 1/r_i)\right), \tag{2}$$

where g is the genus of X and h is the genus of Y.

4.4 A presentation for the Galois group

The fundamental group of a compact surface of genus h has $2h$ generators: there is a loop going 'around' each of h handles, and another going 'through' each handle. If the surface is punctured at n points, there are n additional generators, representing loops winding once around each puncture. For example, the fundamental group Γ of $Y^* = Y - B$ has generators

$$a_1, b_1, \dots, a_h, b_h, \gamma_1, \dots, \gamma_n \tag{3}$$

and the single relation

$$\prod_{i=1}^{h}[a_i, b_i]\prod_{j=1}^{n}\gamma_j = \mathrm{id}, \tag{4}$$

where $[a, b]$ denotes the commutator $a^{-1}b^{-1}ab$. The relation comes from the standard topological construction of a compact surface of genus $h > 0$ as the quotient space of a $4h$-gon. The oriented edges are labelled, in order, by the elements a_i, b_i, a_i^{-1}, b_i^{-1}, $i = 1, \dots, h$. The 'bouquet' $\prod_j^n \gamma_j$ is homotopic to single loop winding once around *all* of the punctures, which in turn is homotopic to the polygonal boundary (see [28], Chapter 1). In the case $h = 0$, there is a more intuitive explanation of the relation: a loop winding once around all the punctures can be shrunk to a point "around the back" of the sphere.

Since the Galois group G is isomorphic to Γ/D_i (recall (1)) there is a surjective homomorphism $\theta : \Gamma \to G$ which carries $\gamma_j \to \rho_j$. This yields a partial presentation of G, in terms of the generators

$$\rho_1, \dots, \rho_n, \quad g_1, k_1, \dots, g_h, k_h \tag{5}$$

(the g_i, h_i being images under θ of the $a_i, b_i \in \Gamma$) and the relations

$$\rho_i^{r_i} = \mathrm{id}, \quad j = 1, \dots, n, \tag{6}$$

(given by the branching indices), and

$$\prod_{i=1}^{h}[g_i, k_i]\prod_{j=1}^{n}\rho_j = \mathrm{id}, \tag{7}$$

corresponding to (4). There are no further relations, but we postpone the proof (see Corollary 20). For our immediate purpose it doesn't matter.

Rather than starting with a Galois covering $f : X \to Y$, we can instead start with a finite group G which has an actual (not partial) *presentation* of the form (5), (6), (7), and *recover* the corresponding Galois covering. The next section is an extended example.

4.5 The dihedral group as a Galois group

A *dihedron* is a polyhedron with two faces. It collapses to a flat polygon in Euclidean space, but it can be realized on the Riemann sphere as follows. Let $n \geq 2$ be an integer. Divide the equator of the sphere into n segments of equal length by marking n equally-spaced points (vertices). These equatorial segments comprise the edges, and the upper and lower hemispheres the two n-sided faces. The *dihedral group* is the group of rotations of the sphere which transform the dihedron into itself. Take for example $n = 3$. The 3-dihedron is preserved by a counterclockwise 3-fold *rotation* about the polar axis (oriented, say, from south pole to north pole) and by any of three *half-turns* about a line joining one of the three vertices to the midpoint of the opposite edge. This is a total of 6 distinct rotations, including the identity. Any two distinct half-turns, performed consecutively, result in a 3-fold rotation. A rotation conjugated by a half-turn is a rotation through the same angle but in the opposite sense (i.e., clockwise as opposed to counterclockwise about the oriented polar axis). It follows that the 3-dihedral group has order 6 and presentation

$$\langle H_1, H_2, R \mid H_1^2 = H_2^2 = R^3 = H_1H_2R = \mathrm{id}\rangle,$$

where R stands for a 3-fold rotation H_i $i = 1, 2$ for distinct half-turns. One of the generators is redundant due to the final relation, but we keep all of them because they give a presentation of the form (5), (6), (7) (with $h = 0$ and $n = 3$) required for a Galois group. Verify that the branch indices $\{2, 2, 3\}$, together with $g = h = 0$ and $|G| = 6$ form a set of data which satisfies the Riemann-Hurwitz relation (2).

Exercise 5. Generalize the discussion above to the dihedral group of order $2n$, $n \geq 2$, acting on $\mathbb{P}^1$, with branch indices $\{2, 2, n\}$. Hint: the cases n odd and n even are different: in the even case opposite vertices and opposite edge midpoints determine two conjugacy classes of half-turns.

Exercise 6. Verify that, besides $\{2, 2, n\}$, the only other *triples* of branching indices which satisfy (2) with $g = h = 0$ are: $\{2, 3, 3\}$, $\{2, 3, 4\}$, and $\{2, 3, 5\}$. Determine $|G|$ in each case. Recover corresponding Galois coverings $\mathbb{P}^1 \to \mathbb{P}^1$ by inscribing, respectively, a regular tetrahedron, octahedron, and icosahedron on the sphere, and determining the rotations of the sphere that transform the polyhedra to themselves. Hint: the Galois groups are, respectively, A_4 (alternating group), S_4, and A_5.

4.6 Galois coverings of the line

The examples in the previous section were all Galois coverings of the complex line $\mathbb{P}^1$ *by itself*. It is also of interest to study coverings of the line by surfaces of higher genus. We have analyzed one case already: those for which the Galois group is cyclic (Sections 3.6, 3.7). For a d-fold cyclic covering of the line ($G \simeq \mathbb{Z}_d$), the branching indices could be any nontrivial divisors of d, provided elements of those orders (a) generate $\mathbb{Z}_d$ and (b) have product equal to the identity. These are simply relations (6) and (7), with $h = 0$. The following theorem of W. Harvey is quite useful.

Theorem 11 (Harvey [16]). *Let $A = \{a_1, a_2, \dots, a_n\}$, $n \geq 2$, be a multi-set of integers with $a_i > 1$. Then A is the set of branching indices of a d-fold cyclic covering of the line if and only if $d = lcm(A) = lcm(A - \{a_i\})$, $i = 1, \dots, n$.*

Proof. A set of elements of a cyclic group of order d generates the whole group (not a subgroup) if and only if $\text{lcm}(A) = d$. If the product of the elements of such a set is the identity, one of them is redundant. Hence the removal of any one of the generators cannot not reduce the lcm of the orders. $\square$

To construct a Galois covering of the line with arbitrary finite Galois group G, take a finite generating set of non-trivial elements. G itself (minus the identity) will always do. Whatever generating set is used, suppose the elements have orders $\{r_1, \dots, r_n\}$. If their product is not the identity, adjoin one more element, which is the inverse of their product (if needed, let its order be r_{n+1}). Construct the Galois covering Riemann surface whose genus g is determined by (2) using $h = 0$ and branching indices $\{r_1, \dots, r_n, (r_{n+1})\}$. This gives a proof of the following theorem.

Theorem 12. *Every finite group is a group of automorphisms of a compact Riemann surface.*

Remark 7. There is another proof of this fact due to Hurwitz which does not use coverings of the line (see [2], Theorem 4.8). Given any finite group G with any finite generating set $S = \{s_1, \dots s_h\} \subseteq G - \{\text{id}\}$, let Γ be the fundamental group of a compact surface Y of genus $h = |S|$. Γ is generated by $2h$ elements $a_1, b_1, \dots, a_h, b_h$ with $\prod_{i=1}^{h} [a_i, b_i] = \text{id}$ (see (3) and (4), with $n = 0$). Let $\theta : \Gamma \to G$ map $a_i \mapsto s_i$ and $b_i \mapsto \text{id}$. θ is clearly a surjective homomorphism, with kernel ker(θ) a normal subgroup of Γ. The (unramified) regular covering of Y corresponding to ker(θ) has Galois group G, hence Y is a compact surface admitting G as a (fixed point free) group of automorphisms.

4.7 The Galois extension problem

Having constructed a Riemann surface with a given group of automorphisms, can we tell if it is the *full* group of automorphisms? A less general, but related question is: given finite-sheeted Galois coverings $f : X \to Y$, and $g : Y \to Z$, with Galois groups G_1 and G_2, with orders d_1 and d_2, respectively, under what conditions is $g \circ f : X \to Z$ a Galois covering of degree $d_1 d_2$? A necessary condition is the existence a group $G_0 \leq \mathrm{Aut}(X)$, containing G_1 as a normal subgroup of index d_1 , such that

$$\frac{G_0}{G_1} \simeq G_2.$$

This is the Galois or Riemann surface version of the problem of group extensions. To address it, one also needs conditions under which an automorphism of Y can be 'lifted' through the covering $f : X \to Y$ to an automorphism of X. Such conditions can be formulated (see, e.g., [2], Theorem 4.11) but it turns out to be much simpler to use the uniformization approach described in the next section.

5 Uniformization

There are just three simply connected Riemann surfaces. This classical result, due to Klein, Poincarè and Koebe, is known as the uniformization theorem [8]. The three surfaces are, up to conformal equivalence:

1. the complex plane $\mathbb{C}$;
2. the Riemann sphere $\mathbb{P}^1$;
3. the upper half plane $\mathbb{U} = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$.

Each of these surfaces has a complete metric of constant curvature. On $\mathbb{U}$, the metric is $|dz|/\mathrm{Im}(z)$, with curvature $\equiv -1$. The real line $z = 0$ is the *ideal boundary*, denoted $\partial\mathbb{U}$. There is a conformal bijection taking $\mathbb{U}$ to the interior of the unit disk, and ∂U to the unit circle; occasionally this alternate model of $\mathbb{U}$ is more convenient.

The uniformization theorem implies that every Riemann surface is conformally equivalent to a quotient $\tilde{X}/\Gamma$, where $\tilde{X}$ is one of the simply connected surfaces, and Γ is a discrete subgroup of $\mathrm{Isom}^+(\tilde{X})$ (orientation-preserving isometries), acting properly discontinuously. Here *discrete* means that any infinite sequence $\{\gamma_n \in \Gamma\}$ which converges (in the subspace topology) to the identity, is eventually constant, i.e., there exists $N < \infty$ such that $\gamma_n = \mathrm{id}$ for all $n \geq N$. Γ acting *properly discontinuously* on $\tilde{X}$ means for every compact $K \subseteq \tilde{X}$, the set $\{\gamma \in \Gamma \mid \gamma K \cup K \neq \emptyset\}$ is finite.

By proper discontinuity, the set $D \subset \tilde{X}$ of points having non-trivial isotropy subgroup is *discrete* (possibly empty). Deleting D makes the quotient map into an unramified (usually, infinite-sheeted) covering

$$\tilde{X} - D \to (\tilde{X} - D)/\Gamma.$$

which can be used to transfer the conformal structure on $\tilde{X}$ to the quotient. Hence $(\tilde{X} - D)/\Gamma$ is, uniquely, a Riemann surface, punctured at a discrete set of points. The conformal structure is easily extended to the compactification, by 'filling in' the punctures.

When $\tilde{X} = \mathbb{U}$, $\mathrm{Isom}^+(\tilde{X})$ is the real Möbius group

$$\left\{ z \mapsto \frac{az+b}{cz+d} \;\middle|\; a,b,c,d \in \mathbb{R}, \quad ad-bc=1 \right\} \simeq \mathrm{PSL}(2,\mathbb{R}),$$

and discrete subgroups are called *Fuchsian groups*. There are three types of elements in $\mathrm{PSL}(2,\mathbb{R})$: *elliptic* elements, with trace $= 2$ and a single fixed point in $\mathbb{U}$; *parabolic* elements, with trace < 2 and a single fixed point in $\partial\mathbb{U}$; and *hyperbolic* elements, with trace > 2 and two fixed points in $\partial\mathbb{U}$. Hyperbolic and parabolic elements have infinite order; an elliptic element may have infinite order, however

Lemma 13. *An elliptic element* in a Fuchsian group *must have finite order.*

Proof. Otherwise, the group is not discrete. □

In general, when a group acts on a set, commuting elements preserve each other's fixed point set. A much stronger statement is true for $\mathrm{PSL}(2,\mathbb{R})$ acting on $\mathbb{U}$.

Lemma 14. *Non-trivial elements of PSL$(2,\mathbb{R})$ commute if and only if they have the same fixed point set.*

For a proof, see, e.g., [18], Theorem 5.7.4.

Corollary 15. *An abelian Fuchsian group is cyclic.*

Proof. (Sketch) By the classification of elements of $\mathrm{PSL}(2,\mathbb{R})$, commuting elements are either both elliptic, or both parabolic, or both hyperbolic. By the lemma, they share, respectively, a fixed point in $\mathbb{U}$, or one fixed point in $\partial\mathbb{U}$, or two fixed points in $\partial\mathbb{U}$. Thus each is a power of a single element. □

Co-compact Fuchsian groups are those having compact quotient space, and they cannot contain parabolic elements: the single fixed point on $\partial\mathbb{U}$ would correspond to a cusp or puncture on the quotient surface. *Co-finite area* groups are those for which the hyperbolic area of the quotient surface (in the induced metric) is finite. In the next section we construct a *fundamental domain* (the Dirichlet region) for a co-compact, co-finite area Fuchsian group Γ acting on $\mathbb{U}$. The geometry of this region (a convex geodesic polygon with finitely many sides, none of which touch $\partial\mathbb{U}$) will yield:

- a finite presentation of Γ;
- a formula for the area of the quotient surface $\mathbb{U}/\Gamma$;
- another form of the Riemann-Hurwitz relation;
- a proof that the automorphism group of a compact Riemann surface is finite;
- a convenient approach to the extension question for automorphism groups.

5.1 The Dirichlet region

Let Γ be a co-compact, co-finite area Fuchsian group (henceforth, we will just say "Fuchsian group"). Recall: a fundamental domain for Γ acting on $\mathbb{U}$ is a closed subset $D \subset \mathbb{U}$ such that (i) $\cup_{\gamma \in \Gamma}(\gamma D) = \mathbb{U}$; and (ii) $\mathrm{Int}(D) \cap \mathrm{Int}(\gamma D) = \emptyset$ unless $\gamma = \mathrm{id}$.

Choose $p \in \mathbb{U}$ which is not fixed by any nontrivial element of Γ. The *Dirichlet region* for Γ, based at p, is the set

$$D_p = \{z \in \mathbb{U} \mid d(z,p) \le d(\gamma z, p), \forall \gamma \in \Gamma\},$$

where d denotes hyperbolic distance. It is straightforward to verify that D_p is a fundamental domain for Γ, and that it is a finite intersection of half-planes bounded by geodesics. Recall that the geodesics in $\mathbb{U}$ are either vertical half lines or semicircles intersecting $\partial\mathbb{U}$ orthogonally. A bounding geodesic segment of the Dirichlet region is called a *side*. A point where two distinct sides intersect is called a *vertex*. The collection $\{\gamma D_p \mid \gamma \in \Gamma\}$ is called a *Dirichlet tesselation* of $\mathbb{U}$. A particular γD_p is called a *face* of the tesselation. Faces sharing a common side are called *neighboring* faces.

Let $q \in \mathbb{U}$ be the fixed point of a nontrivial elliptic element $\gamma \in \Gamma$. Then the orbit Γq must intersect the Dirichlet region D at a point u on its boundary. Let k be the order of γ ($k < \infty$ by Lemma 13). If $k \ge 3$, u must be a vertex of D, at which three or more sides meet at angles $\le 2\pi/k < \pi$. If $k = 2$, u might be the midpoint of a side; in this case, it is convenient to adjoin u to the vertex set, creating, from the "half-sides," a pair of new sides meeting at an angle π.

The set of vertices of D is partitioned into subsets (*vertex cycles*) whose elements belong to the same Γ orbit. Vertices are in the same cycle have conjugate isotropy subgroups. Hence there is a *period* associated with each vertex cycle; it is the common order of the elliptic generator of the isotropy subgroup.

Exercise 7. Show that the vertex cycles with period > 1 are in bijection with conjugacy classes of nontrivial elliptic elements of maximal order in Γ.

Lemma 16. *The internal angles at the vertices of a vertex cycle of period k in a Dirichlet region sum to $2\pi/k$.*

Proof. Let $v_1, \ldots, v_t$ be the vertices in a cycle, and let θ_i be the internal angle at v_i, $i = 1, \ldots, t$. Let $H \le \Gamma$ be the (finite, cyclic) isotropy subgroup of v_1. Then there are $|H| = k$ faces containing vertex v_1 and having internal angle θ_1 at v_1; similarly, there are k faces containing v_j and having internal angle θ_j at v_j. There exists $\gamma_j \in \Gamma$ such that $\gamma_j v_j = v_1$. Thus γ_j adds k more faces to the total set of faces surrounding v_1. Of course, the total angle around v_1 is 2π. Summing over all j, we have

$$k(\theta_1 + \theta_2 + \cdots + \theta_t) \le 2\pi.$$

The proof is completed by showing that every face containing v_1 has been counted in this procedure, hence the inequality is actually equality. This is left to the reader. □

Sides s_1, s_2 of a Dirichlet region D for Γ are *congruent* if there is a *side-pairing* $\gamma \in \Gamma$ such that $s_2 = \gamma s_1$. In this case, D and γD are neighboring faces. A side may be congruent to itself (if its midpoint is fixed by an elliptic element of order 2). No more than two sides can be congruent. For if a side s were congruent with $s_1 = \gamma_1 s$ and $s_2 = \gamma_2 s$, then s would belong to three faces, namely, D, $\gamma_1^{-1}D$, and $\gamma_2^{-1}D$, an impossibility (unless $\gamma_1 = \gamma_2$). Hence, counting a side whose midpoint is fixed by an elliptic element of order 2 as a pair of (congruent) sides, the number of sides of D is even.

Lemma 17. *The k side-pairing elements of a $2k$-sided Dirichlet region for Γ are a finite* generating set *for Γ.*

Proof. Let $\Lambda \le \Gamma$ be the subgroup generated by the side-pairing elements of a Dirichlet region D for Γ. The strategy of the proof (see [18], Theorem 5.8.7) is to show that the connected set $\mathbb{U}$ is the *disjoint* union of two *closed* sets,

$$X = \cup_{\lambda \in \Lambda} \lambda D \quad \text{and} \quad Y = \cup_{\gamma \in \Gamma - \Lambda} \gamma D.$$

(Exercise: a union of faces is closed.) Clearly $X \neq \emptyset$. Thus if we show that $X \cap Y = \emptyset$, it will follow that $Y = \emptyset$, i.e., $\Lambda = \Gamma$. Let $\lambda \in \Lambda$ be arbitrary, and suppose γD, $\gamma \in \Gamma$, is a neighboring face of λD. Then D is a neighboring face of $\gamma^{-1}\lambda D$. Hence $\gamma^{-1}\lambda \in \Lambda$, which forces $\gamma \in \Lambda$. This is true for each of the finitely many neighbors of λD. There are possibly finitely many other faces which share only a vertex with λD. Let $\gamma_1 D$ be one of them. Since $\gamma_1 D$ is a "a neighbor of a neighbor of $\ldots$ a neighbor of" λD (finitely many!), the previous argument, applied finitely many times, shows that $\gamma_1 \in \Lambda$. Thus all the faces surrounding any vertex of λD are Λ-translates of D, and none is a $(\Gamma - \Lambda)$-translate. This shows that $X \cap Y = \emptyset$. $\square$

Let Γ have a Dirichlet region D with $2k \ge 4$ sides, $r \ge 0$ vertex cycles with periods $m_i > 1$, $i = 1, 2, \ldots r$, and $s \ge 0$ other vertex cycles (with period 1). The Gauss-Bonnet theorem, which gives the hyperbolic area of a geodesic polygon in terms of its internal angles, shows that the hyperbolic area of D is

$$\begin{aligned}
\mu(D) &= \pi(2k-2) - \sum \text{internal angles} \\
&= \pi(2k-2) - \left(\sum_{i=1}^{r} \frac{2\pi}{m_i}\right) - 2\pi s \\
&= 2\pi\left[k - 1 - s - \sum_{i=1}^{r} \frac{1}{m_i}\right] \\
&= 2\pi\left[\mathbf{k} - \mathbf{1} - \mathbf{s} - \mathbf{r} + \sum_{i=1}^{r} 1 - \frac{1}{m_i}\right].
\end{aligned}$$

Lemma 18. *The integer $k - 1 - s - r$ appearing in brackets above is equal to the Euler characteristic of the compact quotient surface $\mathbb{U}/\Gamma$. Hence the genus of $\mathbb{U}/\Gamma$ is $h = (k + 1 - s - r)/2$*

Proof. Consider the space of orbits of Γ on its Dirichlet region, known as the *orbifold* D/Γ. This space is homeomorphic to a compact surface of some genus $h \geq 0$, with r *cone points*, where the total angle surrounding a point is $< 2\pi$, corresponding to the vertex cycles with period $n > 1$. There are s other distinguished points, corresponding to the vertex cycles of period 1. These $s + r$ 'vertices' are joined by k 'edges', corresponding to k pairs of identified sides. There is 1 simply connected 'face.' The Euler characteristic $(2h - 2)$ of the orbifold, # vertices - # edges + # faces, is therefore equal to $s + r - k + 1$, from which the formula for h follows. It remains to show that D/Γ is homeomorphic to the quotient surface $\mathbb{U}/\Gamma$. This is done by defining an open, continuous, bijective mapping between the two spaces. That this is possible is due to the *local finiteness* of D: every point has an open neighborhood which meets only finitely many of its Γ-translates. $\square$

Evidently a Dirichlet region encodes a great deal of information about Γ: (i) the genus (h) of the compact quotient surface $\mathbb{U}/\Gamma$; (ii) the number of conjugacy classes of elliptic elements of maximal order (r); and (iii) the orders of those maximal elliptic elements ($m_1, \ldots, m_r$). In fact, this information turns out to be sufficient to determine Γ uniquely up to isomorphism. It is clear that the data,

$$(h; m_1, \ldots, m_r) \qquad h, r \geq 0; \quad m_i > 1, \tag{8}$$

known as the *signature* of Γ, must be the same for isomorphic groups. Moreover, by the Gauss-Bonnet theorem and Lemma 18, the hyperbolic area of a Dirichlet region is given by the formula

$$\mu(D) = 2\pi \left[2h - 2 + \sum_{i=1}^{r} 1 - \frac{1}{m_i} \right], \tag{9}$$

which depends on the signature alone. Since there are many possible Dirichlet regions (depending on the initial choice of a point $p \in \mathbb{U}$), and, indeed, many other types of fundamental domains, it had better be true that the area of any 'sufficiently nice' fundamental domain is a numerical invariant of Γ. In fact, it is (see , e.g., [18], Theorem 5.10.1). Remarkably, any set of data of the form (8) for which the expression (9) is positive, determines a unique Fuchsian group. This was known to Poincaré, but it was not until 1971 that B. Maskit gave the first complete and correct proof [33].

Theorem 19. *There exists a Fuchsian group with signature* $(h; m_1, \ldots, m_r)$ *if and only if*

$$\left[2h - 2 + \sum_{i=1}^{r} 1 - \frac{1}{m_i} \right] > 0.$$

Proof. (Sketch of the 'if' part.) Construct a $4h + r$-sided regular hyperbolic polygon (it is convenient to work in the unit disk model of the hyperbolic plane). In counterclockwise order, label the first $4h$ sides $\alpha_1, \beta_1, \alpha_1^{-1}, \beta_1^{-1}, \ldots,$ $\alpha_h, \beta_h, \alpha_h^{-1}, \beta_h^{-1}$. On the last r sides, erect external isosceles triangles with apex angles $2\pi/m_i$. Delete the bases and label the equal sides of the isosceles triangles ξ_i, ξ_i^{-1}. Expand or contract the resulting polygonal region (which has $4h+2r$ sides) until it has the required area. Let $a_i, b_i \in \mathrm{PSL}(2,\mathbb{R})$ pair α_i with α_i^{-1} and β_i with β_i^{-1}, respectively. Let $e_i \in \mathrm{PSL}(2,\mathbb{R})$ pair ξ_i with ξ_i^{-1}. Let $\Gamma \leq \mathrm{PSL}(2,\mathbb{R})$ be the group generated by these elements. Claim: Γ is Fuchsian, and the polygonal region is a fundamental polygon for Γ, with r singleton vertex cycles of periods m_i (the apices of the isosceles triangles) and one other vertex cycle (the $4h+r$ vertices of the original regular polygon) with period 1. □

Corollary 20. *The Fuchsian group Γ with signature $(h; m_1, \ldots, m_r)$ has presentation*

$$\Gamma = \langle a_1, b_1, \ldots, a_h, b_h, e_1, \ldots, e_r \mid$$
$$e_1^{m_1} = e_2^{m_2} = \cdots = e_r^{m_r} = \prod_{i=1}^{h}[a_i, b_i] \prod_{j=1}^{r} e_j = id\rangle. \quad (10)$$

Proof. We follow the proof given by Greenberg in [14], Theorem 1.5.1. Γ is *generated* by the given (side-pairing) elements, by Lemma 17. It is clear from our previous discussions the given relations hold; we must verify that no further relations are needed to define Γ. If $r > 0$, remove from $\mathbb{U}$ all the fixed points of elliptic elements of Γ, and remove from $S = \mathbb{U}/\Gamma$ the images of those points, obtaining S_0. Let $\phi' : \mathbb{U}_0 \to S_0$ be the restriction of the the quotient map $\phi : \mathbb{U} \to \mathbb{U}/\Gamma$. ϕ' is an unbranched Galois covering (infinite sheeted), with Galois group Γ. From the theory of covering spaces,

$$\Gamma \simeq \pi_1(S_0)/\phi'_*(\pi_1(\mathbb{U}_0)),$$

where ϕ'_* is the imbedding of fundamental groups induced by ϕ' (basepoints suppressed). Since S_0 is a surface of genus g punctured at $r > 0$ points,

$$\pi_1(S_0) = \langle a_1, b_1, \ldots, a_h, b_h, e_1, \ldots, e_r \mid \prod_{i=1}^{h}[a_i, b_i] \prod_{j=1}^{r} e_j = \mathrm{id}\rangle.$$

We claim that $\phi'_*(\pi_1(\mathbb{U}_0))$ is the *smallest* normal subgroup of $\pi_1(S_0)$ containing $e_1^{m_1}, \ldots, e_r^{m_r}$, that is, no relations other than $e_j^{m_j} = \mathrm{id}$, $j = 1, \ldots, r$, are needed to define $\pi_1(S_0)/\phi'^*(\pi_1(\mathbb{U}_0)) = \Gamma$. $\pi_1(\mathbb{U}_0)$ is freely generated by infinitely many loops $\lambda_1, \lambda_2, \ldots$ winding once around each of infinitely many punctures. If λ_i winds once around a puncture lying over the jth puncture in S_0, then, up to conjugacy, $\phi'_*(\lambda_i) = (e_j^{m_j})$. Now let $u = \phi'_*(\tilde{u}) \in \phi'_*(\pi_1(\mathbb{U}_0))$ be arbitrary. Then $\tilde{u} = (\lambda_1)^{k_1}(\lambda_2)^{k_2}\ldots$, for integers $k_1, k_2, \ldots$. Hence u is a product of powers of conjugates of $e_1^{m_1}, \ldots, e_r^{m_r}$. This completes the proof in the case $r > 0$. If $r = 0$, $\mathbb{U}_0 = \mathbb{U}$ and $\pi_1(\mathbb{U})$ is the trivial group, so that $\Gamma = \pi_1(S_0)/\langle \mathrm{id} \rangle = \pi_1(S)$, the fundamental group of a compact surface of genus h, which has the standard presentation. □

5.2 Surface groups

A *torsion-free* Fuchsian group has signature $(g; -)$, $g > 1$, and presentation

$$\Lambda_g = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = \mathrm{id} \rangle.$$

It is called a surface group, since it is isomorphic to the fundamental group of a compact surface of genus g.

We state two well-known results involving surface groups. The first is sometimes called the uniformization theorem, even though it is not the most general statement. The second translates the classification of compact surfaces of genus g up to conformal equivalence into a problem in pure group theory.

Theorem 21. *Any compact Riemann surface X_g of genus $g > 1$ is conformally equivalent to the orbit space $\mathbb{U}/\Lambda_g$, where Λ_g is a surface group of genus g.*

Proof. The orbifold D/Λ_g is a manifold (since there are no "cone" points). It inherits a conformal structure from $\mathbb{U}$. □

Theorem 22. *Let $\Lambda, \Lambda' \leq PSL(2, \mathbb{R})$ be two surface groups of fixed genus $g > 1$. The compact surfaces $\mathbb{U}/\Lambda$ and $\mathbb{U}/\Lambda'$ are conformally equivalent if and only if Λ and Λ' are conjugate subgroups of $PSL(2, \mathbb{R})$.*

Proof. Let $\rho : \mathbb{U}/\Lambda \to \mathbb{U}/\Lambda'$ be a conformal homeomorphism between the two compact surfaces. Any homeomorphism, in particular, ρ, lifts to the universal cover, i.e., there exists $T \in \mathrm{PSL}(2, \mathbb{R})$ such that

$$\rho[z]_\Lambda = [T(z)]_{\Lambda'},$$

where $[z]_\Lambda \in \mathbb{U}/\Lambda$ denotes the Λ-orbit of z and and $[T(z)]_{\Lambda'} \in \mathbb{U}/\Lambda'$ denotes the Λ'-orbit of $T(z)$. For any $S \in \Lambda$, $\rho[S(z)]_\Lambda = \rho[z]_\Lambda = [TS(z)]_{\Lambda'} = [T(z)]_{\Lambda'}$. Hence $TS(z) = VT(z)$ for some $V \in \Lambda'$. This is true for all $z \in \mathbb{U}$, hence, $TST^{-1} = V$. Thus $T\Lambda T^{-1} \leq \Lambda'$. In fact, equality must hold, since Λ and Λ' are isomorphic. Thus Λ and Λ' are conjugate in $\mathrm{PSL}(2, \mathbb{R})$. Conversely, if $T\Lambda T^{-1} = \Lambda'$, the map $[z]_\Lambda \mapsto [T(z)]_{\Lambda'}$ is a conformal homeomorphism. □

5.3 Triangle groups

A Fuchsian group with orbit-genus 0 and only three periods is called a *triangle group*. Triangle groups are constructed as follows. Let $\Delta \in \mathbb{U}$ be a geodesic triangle with vertices $a, b, c \in \mathbb{U}$, at which the interior angles are π/n, π/m, π/r respectively. Reflections in the sides of Δ generate a discrete group of isometries of $\mathbb{U}$ having Δ as fundamental domain. The orientation-preserving subgroup (of index 2) is a Fuchsian group with signature $(0; n, m, r)$. To see why, let e_1 be the product of the two reflections in the sides incident with vertex a; geometrically, this is a rotation (orientation-preserving) about vertex a through an angle $2\pi/n$. Define e_2 and e_3 similarly as rotations about b and c through angles

$2\pi/m$, $2\pi/r$, respectively. The product $e_1e_2e_3$ is easily seen to be trivial (write it as the product of six side reflections). Let D be the four-sided region formed by the union of Δ with its reflection across the side ab. e_1 and e_3 pair the sides of D, so D is a Dirichlet region for the group $\Gamma_\Delta = \langle e_1, e_3\rangle$ with presentation $\langle e_1, e_2, e_3 \mid e_1^n = e_2^m = e_3^r = e_1e_2e_3 = \mathrm{id}\rangle$, and signature $(0; n, m, r)$.

By Theorem 19, there is a Fuchsian triangle group $(0; n, m, r)$ if and only if

$$1 - \left(\frac{1}{n} + \frac{1}{m} + \frac{1}{r}\right) > 0.$$

Remark 8. The geometric construction of Γ_Δ works as just well if the initial geodesic triangle is in $\mathbb{C}$ or $\mathbb{P}^1$. In these cases, the quantity above is ≤ 0, and there are just finitely many possible triples, yielding *euclidean* and *spherical* triangle groups. We have already encountered the spherical triangle groups (Exercise 6). The euclidean triangle groups are

$$(2,4,4),\ (3,3,3),\ (2,3,6),$$

corresponding to tesselations of the Euclidean plane by squares, equilateral triangles, and regular hexagons.

Exercise 8. Prove that the Fuchsian group whose Dirichlet region has smallest hyperbolic area is the triangle group with signature $(0; 2, 3, 7)$. Hint: minimize $\mu(D) > 0$ by starting from the general signature $(h; m_1, \ldots, m_r)$ and showing, successively, that the following must be true: $h = 0$; $3 \leq r \leq 4$; $r = 3$ and $m_1 = 2$; $m_2 = 3$, etc.

5.4 Automorphisms via uniformization

Let Γ be Fuchsian, and $\Gamma_1 \leq \Gamma$ a subgroup of finite index d. If D_1 and D are (respective) Dirichlet regions, a simple geometric argument shows that the hyperbolic area of D_1 must be d times the hyperbolic area of D, that is,

$$\mu(D_1) = d\mu(D).$$

The reader might be pleasantly surprised to discover that this is none other than familiar Riemann-Hurwitz relation governing the holomorphic map

$$\rho : \mathbb{U}/\Gamma_1 \to \mathbb{U}/\Gamma, \qquad \rho : [z]_{\Gamma_1} \mapsto [z]_\Gamma. \tag{11}$$

If one puts $\Lambda_g \leq N(\Lambda_g)$ in place of $\Gamma_1 \leq \Gamma$, where Λ_g is a surface group of genus $g > 1$ and $N(\Lambda_g)$ denotes the normalizer of Λ_g in $\mathrm{PSL}(2,\mathbb{R})$, then (11) is a Galois covering with Galois group

$$N(\Lambda_g)/\Lambda_g \simeq \mathrm{Aut}(\mathbb{U}/\Lambda_g). \tag{12}$$

To prove that this is the *full* automorphism group of the compact surface $\mathbb{U}/\Lambda_g$, and that it is *finite*, we need the following lemma.

Lemma 23. *Let Γ be a Fuchsian group. Then $N(\Gamma)$ is also Fuchsian and the index $[N(\Gamma) : \Gamma]$ is finite.*

Proof. If $N(\Gamma)$ is not Fuchsian, there is an infinite sequence of distinct elements $n_i \in N(\Gamma)$ tending to id. For $\gamma \in \Gamma$, $\gamma \neq$ id, $n_i^{-1}\gamma n_i$ is an infinite sequence in Γ tending to γ, which must be eventually constant, since Γ is Fuchsian. Thus for all sufficiently large i, n_i and γ commute. Γ is not cyclic (recall our standing assumption that Γ is co-compact), hence, by Corollary 15, Γ is nonabelian, i.e., there is an element $\gamma' \in \Gamma$ which does not commute with γ. On the other hand, imitating the first part of the proof, for sufficiently large i, n_i commutes with γ' as well. Hence both γ and γ' have the same fixed point set, which implies that they commute (cf. Lemma 14), a contradiction. Thus $N(\Gamma)$ is Fuchsian. A very similar argument shows that $N(\Gamma)$ contains no parabolic elements. Hence $N(\Gamma)$ has a compact fundamental domain of finite area. The index $[N(\Gamma) : \Gamma]$, being equal to the ratio of two finite areas, is finite. □

Corollary 24 (Hurwitz)**.** *The automorphism group of a compact Riemann surface of genus $g > 1$ is finite, with order $\leq 84(g-1)$.*

Proof. The normalizer $N(\Lambda_g)$ of a surface group is Fuchsian with a Dirichlet region of finite area A. By exercise 8, $A \geq \pi/21$. The area of a Dirichlet region for Λ_g is $2\pi(2g-2)$. It follows by the Riemann-Hurwitz relation that

$$|\mathrm{Aut}(\mathbb{U}/\Lambda_g))| = [N(\Lambda_g) : \Lambda_g] \leq \frac{2\pi(2g-2)}{A} \leq 84(g-1).$$

□

Remark 9. A group of $84(g-1)$ automorphisms of a compact surface of genus $g > 1$ is called a *Hurwitz group*. The smallest Hurwitz group is PSL$(2,7)$ (order 168) acting in genus $g = 3$. There are infinitely many genera g having surfaces with $84(g-1)$ automorphisms, and also infinitely many genera in which no such surfaces exist [29]. M. Conder has determined all the Hurwitz genera $<$ 301, and many infinite families of Hurwitz groups [11]. It has been shown that Hurwitz genera occur (asymptotically) as often as perfect cubes in the sequence of natural numbers [27].

5.5 Surface-kernel epimorphisms

An action $G \times X_g \to X_g$ by a group G of automorphisms of a compact Riemann surface X_g of genus g is called a *Riemann surface transformation group*. We have just seen that any Riemann surface transformation group can be uniformized. If $g > 1$, this means it can be represented entirely in terms of Fuchsian groups acting on the universal covering space $\mathbb{U}$:

$$\frac{\Gamma}{\Lambda_g} \times \frac{\mathbb{U}}{\Lambda_g} \to \frac{\mathbb{U}}{\Lambda_g}, \qquad \overline{\gamma} : [z] \mapsto [\gamma z].$$

Here $\Gamma \geq \Lambda_g$ is a subgroup of $N(\Lambda_g)$, where Λ_g is a surface group, and $G \simeq \Gamma/\Lambda_g$. $\overline{\gamma}$ denotes the element $\gamma\Lambda_g$ of the factor group; $[z]$, $[\gamma z]$ denote the Λ_g-orbits of $z, \gamma z \in \mathbb{U}$. Since Λ_g could imbed as a normal subgroup of Γ in more than one way, it is more precise to associate a Riemann surface transformation group with a *short exact sequence*

$$\{\mathrm{id}\} \to \Lambda_g \hookrightarrow \Gamma \xrightarrow{\rho} G \to \{\mathrm{id}\}.$$

The epimorphism ρ, which imbeds Λ_g in Γ as $\ker(\rho)$, is called a *smooth* or *surface-kernel* epimorphism, and determines the transformation group up to conformal conjugacy.

5.6 Topological conjugacy

Suppose two surface kernel epimorphisms, $\rho, \rho' : \Gamma \to G$ differ by pre- and post composition by automorphisms α, β of Γ, G, respectively. That is, suppose the diagram of short exact sequences

$$\begin{array}{ccccccccc}
\{\mathrm{id}\} & \to & \Lambda_g & \overset{i}{\hookrightarrow} & \mathbf{\Gamma} & \overset{\rho}{\to} & \mathbf{G} & \to & \{\mathrm{id}\} \\
 & & \| & & \alpha\downarrow & & \beta\downarrow & & \\
\{\mathrm{id}\} & \to & \Lambda_g & \overset{j}{\hookrightarrow} & \mathbf{\Gamma} & \overset{\rho'}{\to} & \mathbf{G} & \to & \{\mathrm{id}\}
\end{array}$$

commutes. By a deep result going back to Nielsen [35] (see also [43]), there exists an orientation-preserving homeomorphism $h : \mathbb{U}/i(\Lambda_g) \to \mathbb{U}/j(\Lambda_g)$ (not necessarily conformal!) such that

$$\begin{array}{ccccc}
G & \times_\rho & \mathbb{U}/i(\Lambda_g) & \to & \mathbb{U}/i(\Lambda_g) \\
\beta\downarrow & & h\downarrow & & h\downarrow \\
G & \times_{\rho'} & \mathbb{U}/j(\Lambda_g) & \to & \mathbb{U}/j(\Lambda_g)
\end{array}$$

commutes.

Transformation groups related in this way are called *topologically conjugate*. This is a weaker equivalence relation than conformal conjugacy. In the latter case, h is conformal and the two G-actions are conjugate within the full automorphism group of a single (conformal equivalence class of) surface. In contrast, topologically conjugate G-actions may occur on conformally distinct surfaces. This is the case whenever $i(\Lambda_g)$ and $j(\Lambda_g)$ are not conjugate within $\mathrm{PSL}(2, \mathbb{R})$ (cf. Theorem 22).

The classification of group actions up to topological conjugacy is analogous to (indeed, a special case of) the classification of surfaces up to *quasi-conformal* equivalence. We touch on this large and important subject in the next section.

5.7 Teichmüller spaces

Let Γ be a Fuchsian group, $\mathcal{L} = \mathrm{PSL}(2, \mathbb{R})$, and let $R(\Gamma)$ be the *representation space* of all injective homomorphisms $r : \Gamma \to \mathcal{L}$ such that the image $r(\Gamma)$ is again Fuchsian. If the signature of Γ is $(h; m_1, m_2, \dots m_r)$, then $R(\Gamma)$ can be topologized as a subspace of the product of $2h + r$ copies of $\mathcal{L}$, by assigning to $r \in R(\Gamma)$ the point

$$(r(a_1), r(b_1), \dots, r(a_h), r(b_h), r(e_1), \dots r(e_r)) \in \mathcal{L}^{2h+r}.$$

$r_1, r_2 \in R(\Gamma)$ are *equivalent* if their images are conjugate in $\mathcal{L}$.

Definition 10. The *Teichmüller space of* Γ, denoted $T(\Gamma)$, is the set of equivalence classes $[r : \Gamma \to \mathcal{L}]$, endowed with the quotient topology from $R(\Gamma)$.

Let $\mathrm{Aut}^+(\Gamma)$ be the group of automorphisms of Γ which are both *type-* and *orientation-preserving*. Type-preserving automorphisms preserve elliptic, parabolic, hyperbolic types. Orientation-preserving automorphisms carry the final 'long' relator in (10) to a conjugate of itself but not of its inverse. $\alpha \in \mathrm{Aut}^+(\Gamma)$ induces a homeomorphism of $T(\Gamma)$ defined by

$$[\alpha] : [r] \mapsto [r \circ \alpha].$$

The subgroup $\mathrm{Inn}(\Gamma) \leq \mathrm{Aut}^+(\Gamma)$ of inner automorphisms acts trivially by the definition of $T(\Gamma)$. We define the *Teichmüller modular group* for Γ as

$$\mathrm{Mod}(\Gamma) = \frac{\mathrm{Aut}^+(\Gamma)}{\mathrm{Inn}(\Gamma)} = \mathrm{Out}^+(\Gamma).$$

Theorem 25. *Mod*(Γ) *acts properly discontinuously on* $T(\Gamma)$. *The stabilizer of a point* $[r] \in T(\Gamma)$ *is isomorphic to the finite subgroup* $N_{\mathcal{L}}(r(\Gamma))/r(\Gamma)$.

Proof. See [32]. We prove only the second statement here. If $[\alpha] \in \mathrm{Mod}(\Gamma)$ fixes $[r]$, then $[r \circ \alpha] = [r]$ and there exists $t \in \mathcal{L}$ such that, for all $\gamma \in \Gamma$, $r \circ \alpha(\gamma) = tr(\gamma)t^{-1}$. It follows that $t \in N_{\mathcal{L}}(r(\Gamma))$. If $t \in r(\Gamma)$, $\alpha \in \mathrm{Inn}(\Gamma)$ and hence $[\alpha]$ is the identity in $\mathrm{Mod}(\Gamma)$. Thus the stabilizer of $[r]$ is isomorphic to a subgroup of $N_{\mathcal{L}}(r(\Gamma))/r(\Gamma)$. On the other hand, if $t \in N_{\mathcal{L}}(r(\Gamma))$, the map $\beta_t : r(\gamma) \mapsto tr(\gamma)t^{-1}$ is a type- and orientation-preserving automorphism of $r(\Gamma)$, whence $\alpha_t = r^{-1} \circ \beta_t \circ r$ is a type- and orientation-preserving automorphism of Γ. α_t is inner if and only if $t \in r(\Gamma)$. This establishes the isomorphism. □

The motivating example occurs when $\Gamma = \Lambda_g$, a surface group of genus $g > 1$. $T(\Lambda_g)$ is homeomorphic to $\mathcal{T}_g$, the (Teichmüller) space of *marked* Riemann surfaces of genus g [3]. A 'marking' is an explicit choice of generators (up to orientation-preserving homeomorphisms) of the fundamental group of the surface. $\mathrm{Mod}(\Lambda_g)$ is known as the *mapping class group*.

The action of $\mathrm{Mod}(\Gamma)$ on $T(\Gamma)$ is almost always *faithful*, that is, only the trivial element fixes every point in $T(\Gamma)$. This is the case for $\Gamma = \Lambda_g$, $g > 2$. ($g = 2$ is an important exception – see Example 1 below.) The orbit or *moduli* spaces

$$\mathcal{M}_g = T(\Lambda_g)/\mathrm{Mod}(\Lambda_g),$$

are higher dimensional orbifolds which parametrize Riemann surfaces of genus g up to conformal equivalence. The *singular set* of $\mathcal{M}_g$, where the manifold structure breaks down, is the analogue of the set of cone points of an orbifold. Away from the singular set, $\mathcal{M}_g$ looks like a manifold of complex dimension $3g-3$. This 'parameter count' goes back to Riemann; see [34], Chapter VII, §2 for a modern treatment.

The attentive reader may have noticed that the isotropy subgroup of $[r] \in T(\Lambda_g)$, namely $N_{\mathcal{L}}(r(\Lambda_g))/r(\Lambda_g)$, is isomorphic to $\mathrm{Aut}(\mathbb{U}/r(\Lambda_g))$, the automorphism group of the (conformal equivalence class of) surface determined by $[r]$. This follows from the deep and satisfying theorem below, which shows that automorphism group actions in a given genus $g > 1$, up to topological conjugacy, are in bijection with conjugacy classes of finite subgroups of the corresponding mapping class group. The theorem in its full generality remained a conjecture (of Nielsen) until 1983, when it was proved by S. Kerckhoff.

Theorem 26 ([23]). *A subgroup $H \leq Mod(\Lambda_g)$ has a non-empty fixed point set in $T(\Lambda_g)$ if and only if H is finite.*

We state, without proof, two further results which will be needed in the next section.

Theorem 27 ([7], [30]). *The Teichmüller space of a Fuchsian group Γ with signature $(h; m_1, \ldots, m_r)$ is homeomorphic to an open ball in the Euclidean space $\mathbb{C}^{3h-3+r}$.*

Definition 11. The complex number $3h-3+r$ is the *Teichmüller dimension* of Γ.

Theorem 28 ([13]). *An inclusion $i : \Gamma \to \Gamma_1$ of Fuchsian groups induces a imbedding of Teichmüller spaces,*

$$\bar{i} : T(\Gamma_1) \to T(\Gamma), \quad \bar{i} : [r] \mapsto [r \circ i],$$

with closed image.

It follows that the *branch locus* in $\mathcal{T}_g$ (pre-image of the singular set in $\mathcal{M}_g$) is (non-disjoint) union of imbedded Teichmüller spaces $T(\Gamma) \subseteq \mathcal{T}_g$, one for each conjugacy class of Fuchsian group Γ containing a surface group of genus g as a normal subgroup of finite index. Describing this locus in each genus is a problem of long-standing and current interest (see, e.g., [17, 9, 5, 41]).

6 Greenberg-Singerman extensions

We return to the problem of determining whether a group of automorphisms of a Riemann surface extends to a larger group, and whether that larger group is the full group of automorphisms. These questions were left dangling in Section 4.7.

The relevance of Theorem 28 to the extension problem is as follows: Let $\Lambda_g \leq \Gamma \leq \Gamma_1$ be a chain of inclusions of Fuchsian groups, with Λ_g normal in both Γ_1 and Γ. If the Teichmüller dimensions of $T(\Gamma)$ and $T(\Gamma_1)$ are *equal*, the imbedding $\bar{i} : T(\Gamma_1) \to T(\Gamma) \subseteq \mathcal{T}_g$ induced by the inclusion $i : \Gamma \hookrightarrow \Gamma_1$, is a *surjection* even if $i(\Gamma)$ is a proper subgroup of Γ_1. In this case, the group action uniformized by Γ on the Riemann surfaces in $T(\Gamma)$, might extend *on all the surfaces* to larger group action uniformed by Γ_1. In other words, the G action is not the full automorphism group of any surface. All triangle groups have Teichmüller dimension 0, so any inclusion of one triangle group in another is a potential instance of this situation. Before specializing to triangle group inclusions, we give an example, of independent interest, where the Teichmüller dimensions are nonzero.

Example 1. $\Gamma(2;-)$ is a subgroup of index 2 in $\Gamma_1(0;2,2,2,2,2,2)$. One can check that the Teichmüller dimensions are both $= 3$. Now $\Gamma(2;-) = \Lambda_2$ 'covers' the trivial action on every surface of genus 2. But all surfaces of genus 2 are hyperelliptic (2-fold cyclic coverings of $\mathbb{P}^1$); hence the trivial action extends, on every genus 2 surface, to a $\mathbb{Z}_2$-action with $2g+2=6$ branch points.

The list of subgroup pairs $\Gamma < \Gamma_1$ for which the Teichmüller dimensions are equal is quite small, although it contains some infinite families. It was partially determined L. Greenberg [13] in 1963 and completed by D. Singerman [40] in 1972. In Table 1 we give a sublist involving only certain triangle groups. σ is the signature of a triangle group $\Gamma(\sigma)$, and σ_1 the signature of an over group $\Gamma(\sigma_1)$. The index of the smaller group in the larger is also given. In cases N6 and N8, $\Gamma(\sigma)$ is a *normal* subgroup of $\Gamma(\sigma_1)$; in the remaining cases, the inclusions are non-normal. 'Cyclic admissible' indicates that the sub-signatures (σ) are possible signatures for a cyclic group action (cf. Theorem 11).

Case	σ	σ_1	$[\Gamma(\sigma_1):\Gamma(\sigma)]$	Conditions
N6	$(0;k,k,k)$	$(0;3,3,k)$	3	$k\geq 4$
N8	$(0;k,k,u)$	$(0;2,k,2u)$	2	$u\vert k$, $k\geq 3$
T1	$(0;7,7,7)$	$(0;2,3,7)$	24	-
T4	$(0;8,8,4)$	$(0;2,3,8)$	12	-
T8	$(0;4k,4k,k)$	$(0;2,3,4k)$	6	$k\geq 2$
T9	$(0;2k,2k,k)$	$(0;2,4,2k)$	4	$k\geq 3$
T10	$(0;3k,k,3)$	$(0;2,3,3k)$	4	$k\geq 3$

Table 1: Cyclic-admissible signatures (σ) and extensions (σ_1)

It is not obvious, given two signatures, whether one is the signature of a subgroup of the other, or what the index is. Some geometric intuition can be gained from examining fundamental domains. We do this for the T9 inclusion from Table 1. For simplicity, we write (a,b,c) for the signature $(0;a,b,c)$. The symbol $\triangleleft$ denotes a normal inclusion.

Example 2. Observe that T9 is equivalent to two successive extensions of the N8 type:

1. $(2k,2k,k)\triangleleft(2,2k,2k)$; followed by

2. $(2, 2k, 2k) \triangleleft (2, 4, 2k)$.

There exists a hyperbolic isosceles triangle (in $\mathbb{U}$) with apex angle $2\pi/k$ and base angles π/k ($k \geq 3$). This is half of a Dirichlet region for the triangle group $(2k, 2k, k)$ (cf. Section 5.3). We subdivide this into four congruent triangles as follows.

1. Drop a perpendicular from the apex to the midpoint m of the base, creating two congruent right triangles (with angles π/k at the apex and $\pi/2$ at m). Each of these is half a Dirichlet region for $(2, 2k, 2k)$

2. Draw a perpendicular from m to each of the two opposite sides.

We now have four congruent triangles with angles $\pi/2, \pi/4, \pi/k$, each of which is half of a Dirichlet region for $(2, 4, 2k)$. Hence we have the index 4 inclusion $(2k, 2k, 2) \leq (2, 4, 2k)$.

Recall from Section 5.5 that an action of a finite group G on a Riemann surface $X = \mathbb{U}/\Lambda_g$, uniformized by a Fuchsian group Γ of signature $\sigma(\Gamma)$, corresponds to a short exact sequence

$$\{\mathrm{id}\} \;\rightarrow\; \Lambda_g \;\hookrightarrow\; \Gamma \;\overset{\rho}{\rightarrow}\; G \;\rightarrow\; \{\mathrm{id}\}.$$

where ρ is a surface-kernel epimorphism. Suppose $\sigma(\Gamma)$ appears as the first member of a Greenberg-Singerman pair $\{\sigma, \sigma_1\}$. Then the surface-kernel epimorphism ρ might extend to ρ_1, having the *same kernel*, onto a larger group G_1, uniformized by Γ_1 with signature σ_1. In that case, we have a commuting diagram of short exact sequences,

$$\begin{array}{ccccccccc}
\{\mathrm{id}\} & \rightarrow & \Lambda_g & \hookrightarrow & \mathbf{\Gamma} & \overset{\rho}{\rightarrow} & \mathbf{G} & \rightarrow & \{\mathrm{id}\} \\
 & & \| & & \mu\downarrow & & \nu\downarrow & & \\
\{\mathrm{id}\} & \rightarrow & \Lambda_g & \hookrightarrow & \mathbf{\Gamma_1} & \overset{\rho'}{\rightarrow} & \mathbf{G_1} & \rightarrow & \{\mathrm{id}\}
\end{array}$$

where μ, ν are inclusion maps. The inclusion μ can be given explicitly, since the signatures and hence presentations of Γ, Γ_1 are given. The problem then is to determine conditions on G which permit an extension to G_1 so that the diagram commutes. This has been done recently for all of the Greenberg-Singerman pairs [10]. There is no general algorithm; the problem must be handled on a case-by-case method.

In the last two sections, we consider three variations of an extended example in which the action of a cyclic group of automorphisms extends to the action of a larger group. The actions take place on cyclic covers of the line, and the covering Fuchsian groups are triangle groups. These and many other examples are treated comprehensively in [21], which is also an excellent general reference for several of the topics treated in this paper.

6.1 Generalized Lefschetz curves

The generalized Lefschetz curves are n-fold cyclic covers of the line with equations

$$y^n = x(x-1)^b(x+1)^c,$$

where $1+b+c \equiv 0 \pmod{n}$, and $1 \le b, c \le n-1$. By Theorem 11, we must have $\mathrm{lcm}(\gcd(n,b), \gcd(n,c)) = n$. By Corollary 8, the genus of the curve is

$$g = [(n+1-\gcd(n,b)-\gcd(n,c)]/2. \tag{13}$$

The quotient map modulo the cyclic automorphism group $\mathbb{Z}_n \simeq \langle (x,y) \mapsto (x, \zeta y)\rangle$, where ζ is a primitive n-th root of unity, is an n-fold branched covering with branching indices

$$(n, n/\gcd(n,b), n/\gcd(n,c)). \tag{14}$$

This is also the signature of the Fuchsian triangle group Δ covering the $\mathbb{Z}_n$ action. We have the short exact sequence

$$\{\mathrm{id}\} \quad \to \quad \Lambda_g \quad \hookrightarrow \quad \Delta \quad \stackrel{\rho}{\to} \quad \mathbb{Z}_n \quad \to \quad \{\mathrm{id}\},$$

where $\rho : \Delta \to \mathbb{Z}_n$ is a surface-kernel epimorphism. Let x_1, x_2, x_3 be the three elliptic generators of Δ, and let

$$\mathbb{Z}_n = \langle a \mid a^n = \mathrm{id}\rangle.$$

ρ determines a *generating vector*

$$\langle \rho(x_1), \rho(x_2), \rho(x_3)\rangle \in \mathbb{Z}_n \times \mathbb{Z}_n \times \mathbb{Z}_n.$$

We may assume, up to an automorphism of $\mathbb{Z}_n$, that $\rho(x_1) = a$. If $\rho(x_2) = a^i$ and $\rho(x_3) = a^j$, then, since ρ is a surface-kernel epimorphism, $\rho(x_1)\rho(x_2)\rho(x_3) = a^{1+i+j} = \mathrm{id}$. Equivalently, $1+i+j \equiv 0 \pmod{n}$.

We want to study cases where the signature (14) is the first member of a Greenberg-Singerman pair, so that there is a potential extension of the $\mathbb{Z}_n$ action.

Suppose, for a concrete example, that $n = 2k \ge 6$, $b = 1$, $c = n-2$. Then Δ has signature $(2k, 2k, k)$ and there is a potential extension of type T9 of the $\mathbb{Z}_{2k}$-action to a G_{8k}-action with covering group Δ_1, of signature $(2, 4, 2k)$. Let y_1, y_2, y_3 be the elliptic generators of Δ_1. An explicit imbedding of $\mu : \Delta \to \Delta_1$ is given by

$$\mu : x_1 \to y_2^2 y_3 y_2^2, \quad x_2 \mapsto y_3, \quad x_3 \mapsto y_2 y_3^2 y_2^{-1}.$$

We seek a group G_{8k}, and an inclusion $\nu : \mathbb{Z}_{2k} \to G_{8k}$, such that

$$\begin{array}{ccccccccc}
\{\mathrm{id}\} & \to & \Lambda_g & \hookrightarrow & \Delta & \stackrel{\rho}{\to} & \mathbb{Z}_{2k} & \to & \{\mathrm{id}\} \\
 & & \| & & \mu\downarrow & & \nu\downarrow & & \\
\{\mathrm{id}\} & \to & \Lambda_g & \hookrightarrow & \Delta_1 & \stackrel{\rho'}{\to} & G_{8k} & \to & \{\mathrm{id}\}
\end{array}$$

commutes.

From Example 2, the T9 extension is equivalent to two successive normal (index 2) extensions of type N8. The first of these must cover an extension of $\mathbb{Z}_{2k}$ to a group $G_{4k} \triangleright \mathbb{Z}_{2k}$ which can be constructed as follows: let $\alpha \in \mathrm{Aut}(\mathbb{Z}_{2k})$ have order ≤ 2. Let t be a new generator of order 2 such that conjugation by t acts on $\mathbb{Z}_{2k} = \langle a \rangle$ as α does. Then

$$G_{4k} = \langle a, t \mid a^{2k} = t^2 = 1, tat^{-1} = \alpha(a) \rangle.$$

If $\alpha(a) = a^{-1}$, then $G_{4k} \simeq D_{4k}$, the dihedral group of order $4k$; if $\alpha(a) = a$, then $G_{4k} \simeq \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. If $k \neq p^s$ (p an odd prime) there exists an involutory automorphism α, $\alpha(a) \neq a, a^{-1}$. In this case G_{4k} is a (non-dihedral, non-abelian) *semi-direct product* $\mathbb{Z}_2 \ltimes_\alpha \mathbb{Z}_{2k}$.

Let Δ_0 be the intermediate triangle group with signature $(2, 2k, 2k)$ and elliptic generators z_1, z_2, z_3 An imbedding $\mu_0 : \Delta \to \Delta_0$ is given by

$$\mu_0 : x_1 \to z_3^{-1} z_2 z_3, \quad x_2 \mapsto z_2, \quad x_3 \mapsto z_3^2.$$

We seek a surface kernel epimorphism $\rho_0 : \Delta_0 \to \langle a, t \rangle = G_{4k}$ such that

$$\begin{array}{ccccccccc}
\{\mathrm{id}\} & \to & \Lambda_g & \hookrightarrow & \Delta & \overset{\rho}{\to} & \langle a \rangle & \to & \{\mathrm{id}\} \\
 & & \| & & \mu_0 \downarrow & & \downarrow & & \\
\{\mathrm{id}\} & \to & \Lambda_g & \hookrightarrow & \Delta_0 & \overset{\rho_0}{\to} & \langle a, t \rangle & \to & \{\mathrm{id}\}
\end{array}$$

commutes. It is not difficult to verify that

$$\rho_0 : z_1 \mapsto t, \quad z_2 \mapsto ta, \quad z_3 \mapsto a^{-1}$$

will do. That is, $\langle t, ta, a^{-1} \rangle$ is a Δ_0-generating vector for the G_{4k}-action.

For a second N8 extension (of the G_{4k} action), we need $\beta \in \mathrm{Aut}(G_{4k})$, of order 2, which interchanges ta and a^{-1} (the last two elements of the G_{4k} generating vector). Hence let s be a new generator such that conjugation by s acts as β does, i.e.,

$$sas^{-1} = a^{-1}t.$$

Equivalently, $(sa)^2 = ts^2$. Since $s^2 \in \langle a, t \rangle$ (for an index 2 extension), and $s^2 \notin \langle a \rangle$, either $s^2 = t$, or $s^2 = \mathrm{id}$. Let $s^2 = t$. Then $(sa)^2 = \mathrm{id}$, and hence we have an extended group

$$G_{8k} = \langle s, a \mid s^4 = a^{2k} = (sa)^2 = \mathrm{id},\ s^2 a s^2 = \alpha(a) \rangle,$$

containing $G_{4k} = \langle s^2, a \rangle$, acting with Δ_1- generating vector

$$\langle sa, s, a \rangle \quad (2, 4, 2k).$$

Note that the Riemann-Hurwitz relation (equivalently, (13)), shows that $k = g + 1$, so in this section we have extended a $\mathbb{Z}_{2g+2}$ action to a G_{8g+8}-action on the Lefschetz curve

$$y^{2g+2} = x(x-1)(x+1)^{2g}$$

of genus $g \geq 2$.

6.2 Accola-Maclachlan and Kulkarni curves

These well-known curves arise from certain definite choices of $\alpha \in \mathrm{Aut}(\mathbb{Z}_{2k})$ as considered in the previous section.

Case 1. $\alpha(a) = a$, i.e., α is trivial and $G_{4k} = \mathbb{Z}_2 \times \mathbb{Z}_{2k}$. With $k = g+1$, we have a curve with equation $y^{2g+2} = x(x-1)(x+1)^{2g}$, and full automorphism group

$$G_{8g+8} = \langle s, a \mid s^4 = a^{2g+2} = (sa)^2 = [s^2, a] = \mathrm{id}\rangle.$$

The curve is hyperelliptic with hyperelliptic involution s^2. It was identified by Accola and Maclachlan (independently) in 1968 [1, 31]. Note that $G_{8g+8}/\langle s^2\rangle \simeq D_{4g+4}$, the dihedral group of order $4g+4$. The latter group acts on the quotient sphere, as in Section 4.5.

Case 2. If $g \equiv -1 \pmod 4$, $\alpha(a) = a^{g+2}$ defines an automorphism of $\mathbb{Z}_{2g+2}$ (exercise). In this case we have a nonhyperellitpic curve with full automorphism group

$$G_{8g+8} = \langle s, a \mid s^4 = a^{2g+2} = (sa)^2 = \mathrm{id},\ s^2 a s^2 = a^{g+2}\rangle.$$

This curve was identified by R.S. Kulkarni in 1991. An equation of the curve is

$$y^{2g+2} = x(x-1)^{g+2}(x+1)^{g-1}.$$

The existence of the Accola-Maclachlan curve in each genus $g > 1$ provides a *lower bound* for the order of a group of automorphisms of a surface of genus g.

Theorem 29. *Let $m(g)$ be the order of the largest group of automorphisms of a compact Riemann surface of genus $g > 1$. Then $8g+8 \leq m(g) \leq 84(g-1)$.*

Remark 10. Accola and Maclachlan showed that the lower bound is *sharp*, that is, there exist genera g for which $8g+8$ is the *largest* order of an automorphism group.

7 Further reading

The books [19], [24] and [34] and are excellent self-contained introductory texts with minimal prerequisites. The latter two have an algebraic-geometric slant. Other basic, but somewhat more dense texts on Riemann surfaces are [12], and [2]. Leon Greenberg's paper [14] is a very useful short treatment of Fuchsian and Kleinian groups, and their relation to automorphism groups. The recent paper [21] by Kallel and Sjerve fills in several gaps in my own presentation.

For Teichmüller theory, a vast area, the papers by Ahlfors and Bers [3], [7] are foundational; see also [8], and the more recent book [15].

Lack of space forced me to forgo a treatment of *dessin d'enfants*, Belyĭ curves, and graph embeddings, which comprise a closely related area of much current interest. The recent book [26] is an excellent introduction. A shorter but still comprehensive treatment is given in [20]. [18] is a foundational paper, along with the papers in [37]. My own recent paper [42] makes a connection between Greenberg-Singerman extensions and dessins.

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