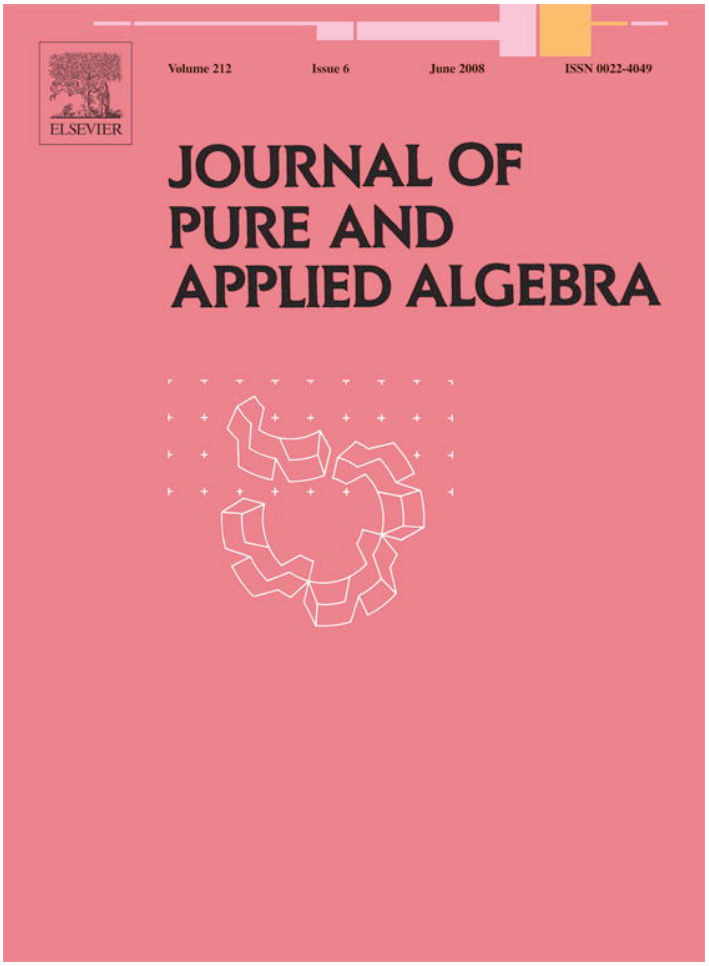

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Journal of Pure and Applied Algebra 212 (2008) 1415–1426
JOURNAL OF PURE AND APPLIED ALGEBRA
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Exceptional points in the elliptic–hyperelliptic locus

Ewa Tyszkowska[a], Anthony Weaver[b,*]

[a] *Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, 80-952 Gdańsk, Poland*
[b] *Department of Mathematics and Computer Science, Bronx Community College, City University of New York, University Avenue and West 181st Street, Bronx, NY, 10453, USA*

Received 26 February 2007; received in revised form 24 July 2007
Available online 3 December 2007

Communicated by R. Parimala

Abstract

An exceptional point in the moduli space of compact Riemann surfaces is a unique surface class whose full automorphism group acts with a triangular signature. A surface admitting a conformal involution with quotient an elliptic curve is called elliptic–hyperelliptic; one admitting an anticonformal involution is called symmetric. In this paper, we determine, up to topological conjugacy, the full group of conformal and anticonformal automorphisms of a symmetric exceptional point in the elliptic–hyperelliptic locus. We determine the number of ovals of any symmetry of such a surface. We show that while the elliptic–hyperelliptic locus can contain an arbitrarily large number of exceptional points, no more than four are symmetric.

MSC: 30F10; 32G15

1. Introduction

Exceptional points in moduli space are unique surface classes whose full group of conformal automorphisms acts with a triangular signature. Such points are of interest for many reasons, one being that their defining equations (as algebraic curves) have coefficients in a number field [1]. Determining the exceptional points in $\mathcal{M}_g$ (the moduli space in genus g) is not a simple matter. Although there are just finitely many possible triangular signatures satisfying the Riemann–Hurwitz relation in genus g, not all of them correspond to a group action. Furthermore, several distinct groups may act with the same signature, or one group may have topologically distinct actions with the same signature. The problem can be attacked piecemeal by restricting attention to certain subloci in $\mathcal{M}_g$.

The *n-hyperelliptic locus* $\mathcal{M}_g^n \subseteq \mathcal{M}_g$ consists of surfaces admitting a conformal involution (the *n-hyperelliptic involution*), with quotient a surface of genus n. When $n = 0$, these are the classical hyperelliptic surfaces. When $n = 1$, these are the *elliptic–hyperelliptic surfaces*. If $g > 4n + 1$, the n-hyperelliptic involution is unique and central in the full group of conformal automorphisms of the surface, and this puts a strong structural restriction on any larger

* Corresponding author.
E-mail addresses: ewa.tyszkowska@math.univ.gda.pl (E. Tyszkowska), anthony.weaver@bcc.cuny.edu (A. Weaver).

0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2007.09.013

group of automorphisms of the surface. The n-hyperelliptic loci are important in building up a stratification of $\mathcal{M}_g$, since if $g > 4n + 3$, the intersection $\mathcal{M}_g^n \cap \mathcal{M}_g^{n+1}$ is empty ([8], Cor V.1.9.2; see also [13,19]).

For $n > 1$, there is a constant bound $168(n - 1)$ on the order of the full automorphism group of an n-hyperelliptic surface of genus g. It follows that for sufficiently large g, and $n > 1$, $\mathcal{M}_g^n$ contains no exceptional points. The number of exceptional points in $\mathcal{M}_g^0$ is always between three and five (inclusive) and is precisely three for all $g > 30$ [18]. By contrast, we shall show (Theorem 6.2) that for infinitely many g, the number of exceptional points in $\mathcal{M}_g^1$ is larger than any preassigned positive integer (but, also, for infinitely many g, the number of exceptional points in $\mathcal{M}_g^1$ is 0).

A *symmetry* of a Riemann surface is an antiholomorphic involution; a surface is *symmetric* if it admits a symmetry. Under the correspondence between curves and surfaces, the fact that a surface X is symmetric means that the corresponding curve is definable over $\mathbb{R}$. In the group of conformal and anticonformal automorphisms of X, nonconjugate symmetries correspond bijectively to real curves which are nonisomorphic (over $\mathbb{R}$), and whose complexifications are birationally equivalent to X. It is natural to ask which exceptional points are also symmetric. In the 0-hyperelliptic locus with $g > 30$, the answer is: all of them. In Section 6 we show that if the elliptic–hyperelliptic locus contains exceptional points, at most four of them are also symmetric.

If X has genus g, and ϱ is a symmetry of X, the set of fixed points $\mathrm{Fix}(\varrho)$ of ϱ consists of k disjoint Jordan curves called *ovals*, where $0 \le k \le g + 1$, by a theorem of Harnack [10]. In Section 7, using the topological classification of conformal actions on elliptic–hyperelliptic Riemann surfces given in [17] (with a corrigendum supplied in Section 5), together with a result of Singerman characterizing symmetric exceptional points [16], and a formula of Gromadzki [9], we give presentations of the full group of conformal and anticonformal automorphisms of symmetric exceptional points in $\mathcal{M}_g^1$, $g > 5$, and count the number of ovals of each conjugacy class of symmetry in such a group.

The outline of the paper is as follows. In Section 2 we give necessary preliminaries on NEC groups. In Section 3, we study actions, reduced by the 1-hyperelliptic involution, on the quotient elliptic curves; the algebra and number theory of the Gaussian and Eisenstein integers yield natural presentations of the reduced groups. In Section 4 we give the topological classification of conformal actions with triangular signature on symmetric elliptic–hyperelliptic surfaces. In Section 5 we state a corrigendum to a theorem in [17], which contained errors with respect to maximality of actions. We thus obtain a complete and correct classification (Theorem 5.1) of full conformal actions on elliptic–hyperelliptic surfaces. In the final two sections, we determine, up to topological conjugacy, the full group of conformal and anticonformal automorphisms of the symmetric exceptional points in $\mathcal{M}_g^1$, and count the ovals corresponding to the symmetries in such groups (Theorems 6.3 and 7.2).

We call attention to the related papers [2–5].

2. NEC groups

Every compact Riemann surface X of genus $g \ge 2$ can be represented as the orbit space of the hyperbolic plane $\mathcal{H}$ under the action of a discrete, torsion-free group Γ, called a *surface group of genus* g, consisting of orientation-preserving isometries of $\mathcal{H}$, and isomorphic to the fundamental group of X. Any group of conformal and anticonformal automorphisms of $X = \mathcal{H}/\Gamma$ can be represented as Λ/Γ, where Λ is a *noneuclidean crystallographic (NEC) group* containing Γ as a normal subgroup. An NEC group is a cocompact discrete subgroup of the full group $\mathcal{G}$ of isometries (including those which reverse orientation) of $\mathcal{H}$. Let $\mathcal{G}^+$ denote the subgroup of $\mathcal{G}$ consisting of orientation-preserving isometries. An NEC group is called a *Fuchsian group* if it is contained in $\mathcal{G}^+$, and a *proper NEC group* otherwise.

Wilkie [20] and Macbeath [12] associated to every NEC group a *signature* which determines its algebraic structure and the geometric nature of its action. It has the form

$$(g; \pm; [m_1, \ldots, m_r]; \{(n_{11}, \ldots, n_{1s_1}), \ldots, (n_{k1}, \ldots, n_{ks_k})\}), \tag{1}$$

where the numbers $m_i \ge 2$ are called the *proper periods*, the brackets () (which may be empty) are called the *period cycles*, the numbers $n_{ij} \ge 2$ are called the *link periods*, and $g \ge 0$ is the *orbit genus*. An NEC group with signature of the form $(g; [-]; \{(-), \ldots, (-)\})$ is called a *surface NEC group of genus* g. A Fuchsian group is an NEC group with signature of the form

$$(g; +; [m_1, \ldots, m_r]; \{-\}). \tag{2}$$

In the particular case $g = 0$ we shall write briefly $[m_1, \ldots, m_r]$. A group with signature $[m_1, m_2, m_3]$ is called a *triangle group*, and the signature is called *triangular*. If Λ is a proper NEC group with the signature (1), its *canonical*

Fuchsian subgroup $\Lambda^+ = \Lambda \cap \mathcal{G}^+$ has the signature

$$(\gamma; +; [m_1, m_1, \ldots, m_r, m_r, n_{11}, \ldots n_{1s_1}, \ldots, n_{k1}, \ldots, n_{ks_k}]; \{-\}), \tag{3}$$

where $\gamma = \alpha g + k - 1$ and $\alpha = 2$ if the sign is $+$ and $\alpha = 1$ otherwise. The group with the signature (1) has a presentation given by generators:

(i)	$x_i, i = 1, \ldots, r,$	(elliptic generators)
(ii)	$c_{ij}, i = 1, \ldots, k;\ j = 0, \ldots s_i,$	(reflection generators)
(iii)	$e_i, i = 1, \ldots, k,$	(boundary generators)
(iv)	$a_i, b_i, i = 1, \ldots, g$ if the sign is $+$,	(hyperbolic generators)
	$d_i, i = 1, \ldots, g$ if the sign is $-$,	(glide reflection generators)

and relations

(1) $x_i^{m_i} = 1, i = 1, \ldots, r,$
(2) $c_{is_i} = e_i^{-1} c_{i0} e_i, i = 1, \ldots, k,$
(3) $c_{ij-1}^2 = c_{ij}^2 = (c_{ij-1}c_{ij})^{n_{ij}} = 1, i = 1, \ldots, k;\ j = 1, \ldots, s_i,$
(4) $x_1 \ldots x_r e_1 \ldots e_k a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1,$
$x_1 \ldots x_r e_1 \ldots e_k d_i^2 \ldots d_g^2 = 1.$

Any system of generators of an NEC group satisfying the above relations will be called *a canonical system* of generators.

Every NEC group has a fundamental region, whose hyperbolic area is given by

$$\mu(\Lambda) = 2\pi \left(\alpha g + k - 2 + \sum_{i=1}^{r} (1 - 1/m_i) + 1/2 \sum_{i=1}^{k} \sum_{i=1}^{s_i} (1 - 1/n_{ij}) \right), \tag{4}$$

where α is defined as in (3). It is known that an abstract group with the presentation given by the generators (i)–(iv) and the relations (1)–(4) can be realized as an NEC group with the signature (1) if and only if the right-hand side of (4) is positive. If Γ is a subgroup of finite index in an NEC group Λ then it is an NEC group itself and the *Riemann–Hurwitz relation* is

$$[\Lambda : \Gamma] = \mu(\Gamma)/\mu(\Lambda). \tag{5}$$

3. Conformal actions on elliptic curves

Every elliptic curve is a quotient of the additive group $\mathbb{C}$ by a *lattice* $\mathcal{L} = \mathcal{L}(\tau_1, \tau_2) \subset \mathbb{C}$ of cofinite area with basis $\{\tau_1, \tau_2\} \subset \mathbb{C}$, and *modulus* $\mu = \tau_1/\tau_2 \notin \mathbb{R}$ (see, e.g., [11], Section 5.8). $\mathbb{C}/\mathcal{L}$ and $\mathbb{C}/\mathcal{L}'$ are conformally equivalent if and only if the moduli μ and μ' of $\mathcal{L}$ and $\mathcal{L}'$ are equivalent under a Möbius transformation defined by an integer matrix of determinant 1. In such a case, the lattices $\mathcal{L}, \mathcal{L}'$ are *similar*, which means $\mathcal{L}' = \lambda\mathcal{L}$ for some $\lambda \in \mathbb{C} \setminus \{0\}$. If $\mathcal{L}$ is *self-similar* ($\lambda\mathcal{L} = \mathcal{L}, \lambda \neq 1$), then $\mathbb{C}/\mathcal{L}$ admits a conformal automorphism with a fixed point. Every lattice $\mathcal{L} = \mathcal{L}(\tau_1, \tau_2)$ is self-similar by $\mathcal{L} = -1\mathcal{L}$, thus every elliptic curve admits the conformal involution $h_{-1} : [z] \mapsto [-z]$, where $[z]$ denotes the coset $z + \mathcal{L}$. If $\mathcal{L}$ has modulus i, $\mathcal{L} = i\mathcal{L}$ and h_{-1} is the *square* of the automorphism $h_i : \quad [z] \mapsto [iz]$ of order 4. Similarly, if $\mathcal{L}$ has modulus $\omega = e^{i\pi/3}$, $\mathcal{L} = \omega\mathcal{L}$ (since $\omega^2 = \omega - 1$) and h_{-1} is the *cube* of the automorphism $h_\omega : \quad [z] \mapsto [\omega z]$ of order 6. Conjugates of h_i, h_ω and their powers are the only automorphisms with fixed points on elliptic curves. (This fact is also known as the *crystallographic restriction*, see [7], Section 4.32.)

Lattices with moduli equivalent to i or ω are closed under multiplication and hence admit ring structures. The rings $\mathcal{G} \simeq \{a+bi \mid a, b, \in \mathbb{Z}\}$ and $\mathcal{E} \simeq \{a+b\omega \mid a, b, \in \mathbb{Z}\}$, are known as the Gaussian and Eisenstein integers, respectively (see, e.g. [6], Sections 174, 175). Each ring has a multiplicative norm δ on the non-zero elements (see Table 1) taking values in the positive integers; the elements of norm 1 are called *units*. Every nonzero, nonunit factorizes uniquely (up to reordering and unit multiples) into primes; primes which differ by a unit multiple are called *associates*. Every ideal is *principal*; a *prime ideal* is one which, when it contains a product, also contains one of the factors. An element is a *prime* if and only if the principal ideal it generates is a prime ideal. Associate primes generate the same prime ideal.

Table 1
The rings $\mathcal{G}$ and $\mathcal{E}$

Ring	Units	Norm	Splitting primes
$\mathcal{G}$	$\pm 1, \pm i$	$\delta_{\mathcal{G}}(a+bi) = a^2+b^2$	$2, \quad p \equiv 1 \bmod 4$
$\mathcal{E}$	$\pm 1, \pm\omega, \pm\omega^2$	$\delta_{\mathcal{E}}(a+b\omega) = a^2+ab+b^2$	$3, \quad p \equiv 1 \bmod 6$

Finally, every proper ideal factorizes uniquely as a product of proper prime ideals. A rational prime p *splits* in $\mathcal{G}$ or $\mathcal{E}$ if it factorizes nontrivially. p splits in $\mathcal{G}$ if and only if $p = 2$ or $p \equiv 1 \bmod 4$, for then p is expressible in the form $p = a^2 + b^2$, $a, b \in \mathbb{Z}$, and thus $p = (a+bi)(a-bi)$. Similarly, a prime p splits in $\mathcal{E}$ if and only if $p = 3$ or $p \equiv 1 \bmod 6$, for then p is expressible in the form $p = a^2 + ab + b^2$, and thus $p = (a+b\omega)(a-b\omega^2)$. 2, 3 are the unique splitting primes in $\mathcal{G}, \mathcal{E}$, respectively, whose prime factors are associates. See Table 1.

If $\mathcal{L}' \subset \mathbb{C}$ is a lattice and $\mathcal{L} \subset \mathcal{L}'$ is a sublattice of finite index, the group $\mathcal{L}'/\mathcal{L} = T$, abelian of rank ≤ 2, acts by *translations* on $\mathbb{C}/\mathcal{L}$ (that is, without fixed points or short orbits). Conjugates of translations are translations, hence a finite group of automorphisms of an elliptic curve has semidirect product structure $T \rtimes H$, where H is a cyclic group generated by a power of h_i or h_ω. If $|H| > 2$, the moduli of $\mathcal{L}$ and $\mathcal{L}'$ are both i or both ω and $\mathcal{L}$ must be an *ideal* in $\mathcal{L}'$: otherwise the action of H on $\mathbb{C}/\mathcal{L}'$ cannot lift through the covering $\mathbb{C}/\mathcal{L} \to \mathbb{C}/\mathcal{L}'$ (with Galois group T) to an action of $T \rtimes H$ on $\mathbb{C}/\mathcal{L}$.

Let $\mathcal{R} = \mathcal{G}$ or $\mathcal{R} = \mathcal{E}$ and let $z\mathcal{R}$ denote the principal ideal generated by $z \in \mathcal{R}$. Let $\tilde{w}$ denote the element $w + z\mathcal{R}$ in the additive group $\mathcal{R}/z\mathcal{R} = T$. Since T decomposes as a direct sum of abelian p-groups, the following theorem (communicated to us by Ravi S. Kulkarni) leads to a complete classification of finite groups of automorphisms of elliptic curves.

Theorem 3.1. (a) *If p is a rational prime,*

$$\frac{\mathcal{G}}{p^k\mathcal{G}} = \langle \tilde{1}, \tilde{i} \rangle \simeq \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}; \qquad \frac{\mathcal{E}}{p^k\mathcal{E}} = \langle \tilde{1}, \widetilde{-\omega} \rangle \simeq \mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}.$$

(b) *If $\pi \in \mathcal{R} \setminus \mathbb{Z}$ is a factor of a splitting prime p,*

1. $p > 3 : \frac{\mathcal{R}}{\pi^k\mathcal{R}} = \langle \tilde{1} \rangle \simeq \mathbb{Z}_{p^k}$
2. $p = 2 : \frac{\mathcal{G}}{\pi^{2k}\mathcal{G}} = \langle \tilde{1}, \tilde{i} \rangle \simeq \mathbb{Z}_{2^k} \oplus \mathbb{Z}_{2^k}; \quad \frac{\mathcal{G}}{\pi^{2k-1}\mathcal{G}} = \langle \widetilde{1-i}, \tilde{i} \rangle \simeq \mathbb{Z}_{2^{k-1}} \oplus \mathbb{Z}_{2^k}$
3. $p = 3 : \frac{\mathcal{E}}{\pi^{2k}\mathcal{E}} = \langle \tilde{1}, \widetilde{-\omega} \rangle \simeq \mathbb{Z}_{3^k} \oplus \mathbb{Z}_{3^k}; \quad \frac{\mathcal{E}}{\pi^{2k-1}\mathcal{E}} = \langle \widetilde{1+\omega}, \tilde{\omega} \rangle \simeq \mathbb{Z}_{3^{k-1}} \oplus \mathbb{Z}_{3^k}.$

The most general finite automorphism group $\tilde{G}$ of an elliptic curve has the structure

$$\mathbb{Z}_n \oplus \mathbb{Z}_m \rtimes \mathbb{Z}_t, \quad m|n, t|4 \text{ or } t|6. \tag{6}$$

If $n/m \leq 3$, a presentation of $\tilde{G}$ is obtained by arguments of the following type. Let $\tilde{G}_A = \langle \tilde{1}, \tilde{i} \rangle \rtimes \langle h_i \rangle$, $\tilde{G}_B = \langle \widetilde{1-i}, \tilde{i} \rangle \rtimes \langle h_i \rangle$, $\tilde{G}_C = \langle \tilde{1}, \tilde{\omega} \rangle \rtimes \langle h_\omega \rangle$, $\tilde{G}_D = \langle \widetilde{1+\omega}, \tilde{\omega} \rangle \rtimes \langle h_\omega \rangle$, $\tilde{G}_E = \langle \tilde{1}, \tilde{\omega} \rangle \rtimes \langle h_{\omega^2} \rangle$ and $\tilde{G}_F = \langle \widetilde{1+\omega}, \tilde{\omega} \rangle \rtimes \langle h_{\omega^2} \rangle$. Express h_i as conjugation by an element c, and denote the elements $\tilde{i}, \tilde{1}$ by x, y. Then the relations $h_i(\tilde{1}) = \tilde{i}$, $h_i(\tilde{i}) = \tilde{i}^2 = -\tilde{1}$ become $cyc^{-1} = x$, $cxc^{-1} = y^{-1}$. In a similar way we obtain all the presentations given in Table 2.

Groups of the more general type (6) ($t > 2$) are obtained by adjoining a relation of the form $x^m = y^{mk}$ to one of the presentations in Table 2, where k is a positive integer such that $(k, n/m) = 1$. A finite group is thereby defined if and only if k satisfies certain other conditions: if $t = 4$, we must have $k^2 + 1 \equiv 0 \bmod n/m$; if $t = 3$ or 6, we must have $k^2 - k + 1 \equiv 0 \bmod n/m$. The existence of such k's is equivalent to the number-theoretical condition that all prime factors of n/m split in the appropriate ring ($\mathcal{G}$ if $t = 4$, $\mathcal{E}$ if $t = 3, 6$).

We note that 2 splits in $\mathcal{G}$ and $k^2 + 1 \equiv 0 \bmod 2$ has the solution $k = 1$, so we obtain a group isomorphic to $\tilde{G}_B$ by adjoining the relation $x^{n/2} = y^{n/2}$ to the presentation of $\tilde{G}_A$. Similarly, 3 splits in $\mathcal{E}$ and $k^2 - k + 1 \equiv 0 \bmod 3$ has the solution $k = 2$, so we obtain a group isomorphic $\tilde{G}_D$ or $\tilde{G}_F$ by adjoining the relation $x^{n/3} = y^{2n/3}$ to the presentation of $\tilde{G}_C$ or $\tilde{G}_E$, respectively.

Table 2
Some finite automorphism groups of elliptic curves

Group	Relators
$\tilde{G}_A = (\mathbb{Z}_n \oplus \mathbb{Z}_n) \rtimes \mathbb{Z}_4$	$x^n, c^4, [x, y], cyc^{-1}x^{-1}, cxc^{-1}y$
$\tilde{G}_B = (\mathbb{Z}_n \oplus \mathbb{Z}_{n/2}) \rtimes \mathbb{Z}_4$	$c^4, v^n, w^{n/2}, cvc^{-1}vw, cwc^{-1}v^{-2}w^{-1}, [v, w]$
$\tilde{G}_C = (\mathbb{Z}_n \oplus \mathbb{Z}_n) \rtimes \mathbb{Z}_6$	$y^n, c^6, cxc^{-1}y^{-1}, cyc^{-1}y^{-1}x, [x, y]$
$\tilde{G}_D = (\mathbb{Z}_n \oplus \mathbb{Z}_{n/3}) \rtimes \mathbb{Z}_6$	$c^6, v^n, w^{n/3}, cvc^{-1}wv^{-2}, cwc^{-1}wv^{-3}, [v, w]$
$\tilde{G}_E = (\mathbb{Z}_n \oplus \mathbb{Z}_n) \rtimes \mathbb{Z}_3$	$c^3, x^n, cxc^{-1}y^{-1}x, cyc^{-1}x, [x, y]$
$\tilde{G}_F = (\mathbb{Z}_n \oplus \mathbb{Z}_{n/3}) \rtimes \mathbb{Z}_3$	$c^3, v^n, w^{n/3}, cvc^{-1}wv^{-1}, cwc^{-1}w^2v^{-3}, [v, w]$

Table 3
(2, 4, 4)-triangular symmetric actions

Case in [17]	α	γ	μ	$n \equiv 0 \bmod 4$	$n \equiv 2 \bmod 4$	$n \equiv 1 \bmod 2$
4.2	0	0	1	$A_0^n, A_1^n, B_{00}^n, B_{01}^n$	$A_0^n, A_1^n, B_{00}^n, B_{01}^n$	A_0^n, A_1^n
4.3	1	0	0	$A_0^n, A_1^n, B_{00}^n, B_{01}^n$	$A_0^n, A_1^n, B_{10}^n, B_{11}^n$	
4.4	0	1	0	$A_0^n, A_1^n, B_{00}^n, B_{01}^n$		
4.5	1	1	0	$A_0^n, A_1^n, B_{00}^n, B_{01}^n$		
4.6	1	0	1	$A_0^n, A_1^n, B_{00}^n, B_{01}^n$	$A_0^n, A_1^n, B_{10}^n, B_{11}^n$	

4. Triangular symmetric actions

If a group G of conformal automorphisms of an elliptic–hyperelliptic surface $X = \mathcal{H}/\Gamma$ of genus $g > 5$ acts with a triangular signature, then G is isomorphic to Λ/Γ, where Λ has one of the fifteen distinct triangular signatures

$$[k, l, m] = [2\varepsilon_1, 4\varepsilon_2, 4\varepsilon_3], [3\varepsilon_1, 3\varepsilon_2, 3\varepsilon_3] \text{ or } [2\varepsilon_1, 3\varepsilon_2, 6\varepsilon_3], \quad \varepsilon_i \in \{1, 2\}, \tag{7}$$

where at least one ε_i is equal to 2. If the signature of Λ arises from [2, 4, 4] (resp. [2, 3, 6], [3, 3, 3]) in this way, we shall say that G determines a (2, 4, 4)-*action* (resp., a (2, 3, 6)-, (3, 3, 3)-*action*). Let $\theta : \Lambda \to G$ be an epimorphism with kernel Γ. Then θ preserves the (finite) orders of the generators of Λ and so G is generated by two elements g_1 and g_2 of orders k and l respectively whose product has order m. Singerman [16] showed that X is symmetric if and only if either of the maps

$$g_1 \mapsto g_1^{-1}, g_2 \mapsto g_2^{-1} \quad \text{or} \quad g_1 \mapsto g_2^{-1}, g_2 \mapsto g_1^{-1} \tag{8}$$

induces an automorphism of G. Here we list all actions with triangular signature on elliptic–hyperelliptic Riemann surfaces for which the group $G = \langle g_1, g_2 \rangle$ satisfies condition (8). We call them *triangular symmetric actions*.

Theorem 4.1. *The topological type of a triangular symmetric* (2, 4, 4)*-action on an elliptic–hyperelliptic Riemann surface is determined by a finite group $G = A_\varepsilon^n$ or $G = B_{\varepsilon\delta}^n$ with presentation*

$$A_\varepsilon^n = \langle x, y, c, \rho : \rho^2, x^n\rho^\varepsilon, c^4\rho^\mu, [x, y]\rho^\gamma, cyc^{-1}x^{-1}, cxc^{-1}y\rho^\alpha, R\rangle, \tag{9}$$

or

$$B_{\varepsilon\delta}^n = \langle w, v, c, \rho : \rho^2, c^4\rho^\mu, v^n\rho^\varepsilon, w^{n/2}\rho^\delta, cvc^{-1}wv\rho^\alpha, cwc^{-1}v^{-2}w^{-1}\rho^{\alpha+\gamma}, [v, w]\rho^\gamma, R\rangle, \tag{10}$$

a Fuchsian group $\Lambda = \Lambda_{\alpha,\gamma,\mu}$ with signature $[2(|\alpha-\mu|+1), 4(\mu+1), 4(|\gamma-\mu|+1)]$ generated by x_1, x_2, x_3, and an epimorphism $\theta : \Lambda \to G$ defined by $\theta(x_1) = c^{-2}x$, $\theta(x_2) = c$, $\theta(x_3) = y^{-1}c$ or $\theta(x_1) = c^{-2}v$, $\theta(x_2) = c$, $\theta(x_3) = v^{-1}w^{-1}c$, respectively, where R is the set of relations making ρ central and $\varepsilon, \delta \in \{0, 1\}$. All nonequivalent actions are listed in Table 3.

Theorem 4.2. *The topological type of the triangular symmetric* (2, 3, 6)*-action on an elliptic–hyperelliptic Riemann surface is determined by a finite group of automorphisms $G = C_\varepsilon^n$ or $G = D_{\varepsilon\varepsilon'}^n$ with the presentation*

$$C_\varepsilon^n = \langle x, y, c, \rho : \rho^2, y^n\delta^\varepsilon, c^6\rho^\mu, cxc^{-1}y^{-1}\rho^\alpha, cyc^{-1}y^{-1}x\rho^\beta, [x, y]\rho^{\alpha+\mu}, R\rangle \tag{11}$$

Table 4
(2, 3, 6)-triangular symmetric actions

Case in [17]	α	β	μ	n mod 12						
				0	±1	±2	±3	±4	±5	6
6.2	0	1	0	C_0^n, D_{00}^n	C_1^n	C_0^n	C_1^n, D_{10}^n	C_0^n	C_1^n	C_0^n, D_{00}^n
6.3	0	0	1	C_0^n, D_{00}^n		C_1^n		C_0^n		C_1^n, D_{11}^n
6.4	0	1	1	C_0^n, D_{00}^n		C_1^n		C_0^n		C_1^n, D_{11}^n
6.5	1	0	0	C_0^n, D_{00}^n		C_1^n		C_0^n		C_1^n, D_{11}^n
6.6	1	1	0	C_0^n, D_{00}^n		C_1^n		C_0^n		C_1^n, D_{11}^n
6.7	1	0	1	C_0^n, D_{00}^n	C_1^n	C_0^n	C_1^n, D_{11}^n	C_0^n	C_1^n	C_0^n, D_{00}^n
6.8	1	1	1	C_0^n, D_{00}^n	C_0^n	C_0^n	C_0^n, D_{01}^n	C_0^n	C_0^n	C_0^n, D_{00}^n

Table 5
(3, 3, 3)-triangular symmetric actions

Case in [17]	β	γ	μ	n mod 12						
				0	±1	±2	±3	±4	±5	6
3.2	0	1	0	E_0^n, F_{00}^n		E_1^n		E_0^n		E_1^n, F_{11}^n
3.3	1	0	1	E_0^n, F_{00}^n	E_0^n	E_0^n	E_0^n, F_{01}^n	E_0^n	E_0^n	E_0^n, F_{00}^n
3.4	1	1	1	E_0^n, F_{00}^n		E_1^n		E_0^n		E_1^n, F_{11}^n

or

$$D_{\varepsilon\varepsilon'}^n = \langle w, v, c, \rho : \rho^2, c^6\rho^\mu, v^n\rho^\varepsilon, w^{n/3}\rho^{\varepsilon'}, cvc^{-1}wv^{-2}\rho^{\alpha+\mu+\beta}, cwc^{-1}wv^{-3}\rho^{\beta+\mu}, [v, w]\rho^{\alpha+\mu}, R\rangle, \quad (12)$$

a Fuchsian group $\Lambda = \Lambda_{\alpha,\beta,\mu}$ with the signature $[2(\alpha+1), 3(\beta+1), 6(\mu+1)]$ generated by x_1, x_2, x_3, and an epimorphism $\theta : \Lambda \to G$ defined by $\theta(x_1) = c^3x, \theta(x_2) = c^2y, \theta(x_3) = c$ or $\theta(x_1) = c^3wv^{-1}, \theta(x_2) = c^2v, \theta(x_3) = c$, respectively, where R is the set of relations making ρ central, n is a positive integer and $\varepsilon \in \{0, 1\}$. All nonequivalent actions are listed in Table 4.

Theorem 4.3. *The topological type of the triangular symmetric $(3, 3, 3)$-action on elliptic–hyperelliptic Riemann surface is determined by a finite group of automorphisms $G = E_\varepsilon^n$ or $G = F_{\varepsilon\varepsilon'}^n$ with the presentation*

$$E_\varepsilon^n = \langle x, y, c, \rho : \rho^2, c^3\rho^\mu, x^n\rho^\varepsilon, cxc^{-1}y^{-1}x, cyc^{-1}x\rho^\beta, [x, y]\rho^\gamma, R\rangle \quad (13)$$

or

$$F_{\varepsilon\varepsilon'}^n = \langle w, v, c, \rho : \rho^2, c^3\rho^\mu, v^n\rho^\varepsilon, w^{n/3}\rho^{\varepsilon'}, cvc^{-1}wv^{-1}\rho^\beta, cwc^{-1}w^2v^{-3}\rho^\beta, [v, w]\rho^\gamma, R\rangle, \quad (14)$$

a Fuchsian group $\Lambda_{\beta,\gamma,\mu}$ with the signature $[(\mu+1)3, (\beta+1)3, (|\gamma-\beta-\mu|+1)3]$ generated by x_1, x_2, x_3 and an epimorphism $\theta : \Lambda \to G$ defined by $\theta(x_1) = c, \theta(x_2) = c^{-2}x, \theta(x_3) = x^{-1}c$ or $\theta(x_1) = c, \theta(x_2) = c^{-2}wv^{-1}, \theta(x_3) = vw^{-1}c$, respectively, where R is the set of relations making ρ central, n is a positive integer and $\varepsilon, \varepsilon' \in \{0, 1\}$. All nonequivalent actions are listed in Table 5.

Proof. The proofs of above three theorems are similar and therefore we give the argument concerning the $(2, 3, 6)$-action only. By [17], Theorem 4.1, such an action arises by the lifting an automorphism group $\tilde{G}$ of an elliptic curve with a presentation which differs from $\tilde{G}_C$ in Table 2 by having the additional relation $\tilde{x}^m = \tilde{y}^{mk}$, for some integers m, k such that m divides n and $k^2-k+1 \equiv 0\ (n/m)$. We shall prove that the symmetric character of X requires $m = n$ or $m = n/3$, which leads to the presentations (11) or (12), respectively. Here $\theta(x_1) = c^3x, \theta(x_2) = c^2y, \theta(x_3) = c$ and it is easy to check that the assignment $\theta(x_i) \to \theta(x_j)^{-1}, \theta(x_j) \to \theta(x_i)^{-1}$ does not induce an automorphism of G for any two distinct $i, j \in \{1, 2, 3\}$. So assume that $\varphi : G \to G$ is induced by the assignment $\theta(x_1) \to \theta(x_1)^{-1}$ and $\theta(x_2) \to \theta(x_2)^{-1}$. Then $\varphi(x) = x^{-1}, \varphi(y) = x^{-1}y\rho^\beta$ and $\varphi(c) = c^{-1}y^{-1}x\rho^\alpha$. Thus $\varphi(x)^m = \varphi(y)^{mk}\rho^\delta$ implies $(\tilde{x})^{-m} = (\tilde{x}^{-1}\tilde{y})^{mk} = \tilde{x}^{-mk}\tilde{y}^{mk} = \tilde{x}^{m-mk}$, where $\tilde{x}$ is the image of x in $\tilde{G} = G/\langle\rho\rangle$. Since $\tilde{x}^{2m} = \tilde{x}^{mk}$, it follows that $k \equiv 2\ (n/m)$. Furthermore, $k^2-k+1 \equiv 0\ (n/m)$ which implies $3 \equiv 0\ (n/m)$. Thus $m = n$ or $m = n/3$. If $m = n$ then

G has the presentation (11) and $\tilde{G} \cong \tilde{G}_C$. In the case $m = n/3$ it is convenient to replace the generators x, y by $w = xy$ and $v = y$. Since $x^{n/3} = y^{2n/3}\rho^\delta$, it follows that $x^n = \rho^\delta$. On the other hand, $x^n = cx^nc^{-1} = y^n\rho^{n\alpha} = \rho^{\varepsilon+n\alpha}$ which implies that $\rho^\delta = \rho^{\varepsilon+n\alpha}$. Thus $w^{n/3} = x^{n/3}y^{n/3}\rho^{\gamma n/3(n/3-1)/2} = \rho^{\delta+\varepsilon+\gamma n/3(n/3-1)/2} = \rho^{n\alpha+\gamma n/3(n/3-1)/2}$. Consequently G has the presentation (12) and $\tilde{G} \cong \tilde{G}_D$.

The expression for the signature of Λ in terms of the parameters α, β, μ is an easy computation following from $a_i = \varepsilon_i - 1$, $i = 1, 2, 3$ and the relations $\alpha = a_1$, $\beta = a_2$, $\mu = a_3$ and $\gamma = a_1 + a_3 = \alpha + \mu$, which were proved in [17]. □

5. Full conformal actions: A corrigendum

A Fuchsian group $\Lambda(\tau)$ with signature τ is *finitely maximal* if there is no Fuchsian group $\Lambda(\tau')$ with signature τ' containing it with finite index. Most Fuchsian groups are finitely maximal, but there a few infinite families of signatures, together with a few other individual signatures, for which the corresponding Fuchsian groups are not finitely maximal (see [15]). Suppose $\Lambda(\tau)$ is not finitely maximal. Let $\Lambda(\tau')$ be a Fuchsian group such that $\Lambda(\tau) < \Lambda(\tau')$ with finite index. Let $G < G'$ be finite groups such that $[G' : G] = [\Lambda(\tau') : \Lambda(\tau)]$. An epimorphism $\theta : \Lambda(\tau) \to G$ (with kernel a surface group of genus g) determines a G-action on $X = \mathcal{H}/\ker(\theta)$. θ may extend to an epimorphism $\theta' : \Lambda(\tau') \to G'$ (with the same kernel). If it does, G is not the full automorphism group of the surface $X = \mathcal{H}/\ker(\theta)$. When τ, τ' are both triangular, the G-action and the G'-action both determine the same exceptional point X.

Clearly, if G acts on an elliptic–hyperelliptic surface with a signature that is either finitely maximal or extends with finite index only to signatures which are not elliptic–hyperelliptic, it must be the full group of conformal automorphisms of the surface. It turns out that the converse is also true: no other triangular signature is the signature of the full group of conformal automorphisms of an elliptic–hyperelliptic surface. This is not obvious, but is a consequence of the following theorem, which is a corrigendum of Theorem 8.1 in [17].

Theorem 5.1. *A group $G = \Lambda/\Gamma$ is the full group of conformal automorphisms of an elliptic–hyperelliptic Riemann surface $X = \mathcal{H}/\Gamma$ of genus $g > 5$ if and only if G is one of the groups listed in the Theorems* 3.1–7.1 *of* [17] *and Λ has a finitely maximal signature or Λ has one of the following nonmaximal signatures:* $[2, 2, 4, 4]$, $[4, 4, 8]$, $[2, 4, 8]$, $[2, 2, 8, 8]$, $[2, 6, 6]$, $[2, 2, 6, 6]$ *corresponding to the cases* 4.1, $t = 1$; 4.5, $t = 0$; 4.4, $t = 0$; 4.6, $t = 1$; 6.2, $t = 0$; 6.2, $t = 1$, *respectively, in* [17].

6. Exceptional points

In $\mathcal{M}_g^0$, $g > 30$, there are exactly three exceptional points [18], and Singerman's results [16] imply that these three exceptional points are also symmetric. We show in this section that the elliptic–hyperelliptic locus $\mathcal{M}_g^1$ can contain an arbitrarily large number of exceptional points, but no more than four of them are also symmetric.

Let G be the full group of conformal automorphisms of a surface $X \in \mathcal{M}_g^1$, and let ρ denote the 1-hyperelliptic involution. Suppose $G/\langle\rho\rangle$ is of the form (6) and G acts with a triangular signature, so that X is an exceptional point. Then G has order $2nmt$ and $t > 2$. Let $\mathcal{R}_t$ denote the ring $\mathcal{G}$ if $t = 4$ and $\mathcal{E}$ if $t = 3$ or 6. We recall that the prime factors of n/m must split in $\mathcal{R}_t$.

Lemma 6.1. *Let $p > 3$ be a splitting prime factor of n/m with multiplicity $\mu > 0$ (i.e, p^μ is the highest power of p dividing n/m). Then the p-Sylow subgroup of* (6)*, and hence of G, has $1 + \lfloor \mu/2 \rfloor$ distinct possible isomorphism types.*

Proof. Since p splits in $\mathcal{R}_t$, $p = \pi \cdot \pi'$, and since $p > 3$, π, π' are nonassociate primes which therefore generate distinct ideals $\pi\mathcal{R}_t, \pi'\mathcal{R}_t$. Thus the p-Sylow subgroup of (6), which is also the p-Sylow subgroup of G, could be any one of the following:

$$\frac{\mathcal{R}_k}{\pi^l(\pi')^{\mu-l}\mathcal{R}_k} \simeq \frac{\mathcal{R}_k}{\pi^l\mathcal{R}_k} \oplus \frac{\mathcal{R}_k}{(\pi')^{\mu-l}\mathcal{R}_k} \simeq \mathbb{Z}_{p^l} \oplus \mathbb{Z}_{p^{\mu-l}}, \quad 0 \le l \le \lfloor \mu/2 \rfloor. \tag{15}$$

Theorem 6.2. *There exist infinite sequences of genera in which the number of exceptional points in $\mathcal{M}_g^1$ is larger than any preassigned positive integer.*

Proof. If G is the full group of conformal automorphisms of an exceptional point $X \in \mathcal{M}_g^1$, it acts with one of signatures [2, 4, 8], [4, 4, 8], [2, 6, 6], [2, 3, 12], [4, 3, 6], [4, 6, 6] or [4, 6, 12], and by (5), the ratio $[nm : (g-1)]$ is

$$[a : b] \in \{[2 : 1], [2 : 3], [1 : 1], [2 : 3], [2 : 5], [1 : 3]\}.$$

These ratios place restrictions on the multiplicities of prime divisors of $g-1$. If p is a nonsplitting prime in $\mathcal{R}_t$, the p-Sylow subgroup of $\mathbb{Z}_n \oplus \mathbb{Z}_m$ is of the form $\mathbb{Z}_{p^k} \oplus \mathbb{Z}_{p^k}$ by Theorem 3.1. Thus if $p > \max\{a, b\}$, it must have even (possibly 0) multiplicity in $g-1$. There is no restriction on the multiplicity of a splitting prime $p > \max\{a, b\}$ in $g-1$. If the multiplicity of such a splitting prime is μ, by Lemma 6.1, there are at least $1 + \lfloor \mu/2 \rfloor$ different isomorphism types for G, hence at least that many distinct exceptional points in $\mathcal{M}_g^1$.

Given a, b, t, an $\mathcal{R}_t$-splitting prime $p > 5$, and an arbitrary positive integer N, there exist infinite sequences of genera g such that the multiplicity of p in $g-1$ is greater than N. It follows that there are infinitely many genera g in which the number of exceptional points in the elliptic–hyperelliptic locus is larger than N. □

It is easy to construct infinite sequences of genera in which the elliptic–hyperelliptic locus contains no exceptional points. For example, $g_n = 1 + p^{2n+1}$, $n = 0, 1, \ldots$, where p is any prime such that $p \equiv -1 \bmod 12$, is such a sequence, since p has odd multiplicity in $g_n - 1$, and does not split in $\mathcal{G}$ or in $\mathcal{E}$.

We now restrict attention to exceptional points which are also symmetric.

Theorem 6.3. *The genus g of a symmetric exceptional point $X \in \mathcal{M}_g^1$ is $ka^2 + 1$ for some integer a and $k \in \{1, 2, 3, 6, 10, 30\}$. For such g there are the following nonequivalent triangular symmetric actions on X; those in bold type correspond to the full automorphism groups.*

1. $k = 1$ *and* $a \equiv 0$ (2) : $4.2.B_{00}^a$, B_{01}^a, $4.3.A_0^a$, A_1^a, $\mathbf{4.4.B_{00}^{2a}}$, $\mathbf{B_{01}^{2a}}$, $4.6.A_0^a$, A_1^a, $\mathbf{6.2.C_0^a}$, $3.3.E_0^a$; *and* $\mathbf{6.8.D_{00}^a}$ *if* $a \equiv 0$ (3),
2. $k = 1$ *and* $a \equiv 1$ (2) : $\mathbf{6.2.C_1^a}$, $3.3.E_0^a$; *and* $\mathbf{6.8.D_{01}^a}$ *if* $a \equiv 0$ (3),
3. $k = 2$ *and* $a \equiv 0$ (2) : $4.2.A_0^a$, A_1^a, $4.3.B_{00}^{2a}$, B_{01}^{2a}, $\mathbf{4.4.A_0^{2a}}$, $\mathbf{A_1^{2a}}$, $4.6.B_{00}^{2a}$, B_{01}^{2a}, $\mathbf{6.3.C_0^{2a}}$, $6.7.C_0^a$, $3.2.E_0^{2a}$; *and* $6.4.D_{00}^{2a}$, $\mathbf{6.5.D_{00}^{2a}}$, $3.4.F_{00}^{2a}$ *if* $a \equiv 0$ (3),
4. $k = 2$ *and* $a \equiv 1$ (2) : $4.2.A_0^a$, A_1^a, $4.3.B_{11}^{2a}$, B_{10}^{2a}, $\mathbf{4.4.A_0^{2a}}$, $\mathbf{A_1^{2a}}$, $4.6.B_{10}^{2a}$, B_{11}^{2a}, $6.7.C_1^a$, $\mathbf{6.3.C_1^{2a}}$ $3.2.E_1^{2a}$; *and* $6.4.D_{11}^{2a}$, $\mathbf{6.5.D_{11}^{2a}}$, $3.4.F_{11}^{2a}$ *if* $a \equiv 0$ (3),
5. $k = 3$ *and* $a \equiv 0$ (2) : $\mathbf{4.5.B_{00}^{2a}}$, $\mathbf{B_{01}^{2a}}$, $\mathbf{6.8.C_0^a}$, $\mathbf{6.2.D_{00}^{3a}}$, $3.3.F_{00}^{3a}$,
6. $k = 3$ *and* $a \equiv 1$ (2) : $\mathbf{6.8.C_0^a}$, $\mathbf{6.2.D_{10}^{3a}}$, $3.3.F_{01}^{3a}$,
7. $k = 6$ *and* $a \equiv 0$ (2) : $\mathbf{4.5.A_0^{2a}}$, $\mathbf{A_1^{2a}}$, $\mathbf{6.3.D_{00}^{6a}}$, $6.4.C_0^{2a}$, $\mathbf{6.5.C_0^{2a}}$, $6.7.D_{00}^{3a}$, $3.2.F_{00}^{6a}$, $3.4.E_0^{2a}$,
8. $k = 6$ *and* $a \equiv 1$ (2) : $\mathbf{4.5.A_0^{2a}}$, $\mathbf{A_1^{2a}}$, $\mathbf{6.3.D_{11}^{6a}}$, $6.4.C_1^{2a}$, $\mathbf{6.5.C_1^{2a}}$, $6.7.D_{11}^{3a}$, $3.2.F_{11}^{6a}$, $3.4.E_1^{2a}$,
9. $k = 10$: $\mathbf{6.6.C_0^{2a}}$ *or* $\mathbf{6.6.C_1^{2a}}$ *according to a being even or odd,*
10. $k = 30$: $\mathbf{6.6.D_{00}^{6a}}$ *or* $\mathbf{6.6.D_{11}^{6a}}$ *according to a being even or odd.*

Proof. Let $G = \Lambda/\Gamma$ be a group of conformal automorphisms acting with a triangular signature on a symmetric elliptic–hyperelliptic surface $X = \mathcal{H}/\Gamma$ of genus g. Thus Γ is a surface group of genus g and Λ has one of the signatures (7). Let ρ be the 1-hyperelliptic involution of X. Theorems 4.1–4.3 show that the reduced group $G/\langle\rho\rangle$ is isomorphic to one of groups listed in Table 2. The genus of X is determined as in the following example. Suppose $G/\langle\rho\rangle \cong \tilde{G}_D$. Then the order of G is $12n^2/3 = 4n^2$ and the Riemann–Hurwitz relation (5) where Λ has signature $[2\varepsilon_1, 3\varepsilon_2, 6\varepsilon_3]$ with at least one ε_i equal to 2, is $g = 1 + 2n^2(1 - 1/2\varepsilon_1 - 1/3\varepsilon_2 - 1/6\varepsilon_3)$. The expression in parentheses is either 1/4, 5/12, 1/2, 1/6, 1/4 or 1/12. Since g is an integer, n must be divisible by 2, 6, 1, 3, 2, 6, respectively. Thus there exists an integer a such that $g = ka^2 + 1$, for $k = 2, 30, 1, 3, 6$, respectively. Similar computations using the other reduced groups $G/\langle\rho\rangle$ in Table 2, and compatible signatures, yield the general result $g = ka^2 + 1$ for some $k \in \{1, 2, 3, 6, 10, 30\}$.

The actions are determined as follows: Given a signature and a reduced group $G/\langle\rho\rangle$ (compatible with the signature), we use the tables in Theorems 4.1–4.3, together with the value of k calculated as above. For example, actions with signature [4, 6, 6] have reduced group $\tilde{G}_D$ or $\tilde{G}_C$, and correspond to Case 6.6 in Table 4. The Riemann–Hurwitz relation using $\tilde{G}_D$ implies n is divisible by 6 and $k = 30$. Putting $n = 6a$, we have $n \equiv 0$ or $\equiv 6 \bmod 12$ according as a is even or odd. Thus the actions are $6.6D_{00}^{6a}$ if a is even and $6.6.D_{11}^{6a}$ if a is odd. Similarly, the Riemann–Hurwitz relation using $\tilde{G}_C$ implies n is divisible by two and $k = 10$. Thus $n \equiv 0, \pm 2, \pm 4$ or 6 mod

Table 6
Extensions of triangular actions on symmetric surfaces in $\mathcal{M}_g^1$

g	τ	G	τ'	G'	$\tau' : \tau$
a^2+1	[4, 8, 8]	$4.2.B_{00}^a, B_{01}^a$	[2, 8, 8]	$4.6.A_0^a, A_1^a$	2
$2a^2+1$	[4, 8, 8]	$4.2.A_0^a, A_1^a$	[2, 8, 4]	$4.4.A_0^{2a}, A_1^{2a}$	4
$2a^2+1$	[4, 8, 8]	$4.2.A_0^a, A_1^a$	[2, 8, 8]	$4.6.B_{00}^{2a}, B_{01}^{2a}$	2
a^2+1	[4, 4, 4]	$4.3.A_0^a, A_1^a$	[2, 4, 8]	$4.4.B_{00}^{2a}, B_{01}^{2a}$	2
$2a^2+1$	[4, 4, 4]	$4.3.B_{00}^{2a}, B_{01}^{2a}$	[2, 4, 8]	$4.4.A_0^{2a}, A_1^{2a}$	2
a^2+1	[2, 8, 8]	$4.6.A_0^a, A_1^a$	[2, 8, 4]	$4.4.B_{00}^{2a}, B_{01}^{2a}$	2
$2a^2+1$	[2, 8, 8]	$4.6.B_{00}^{2a}, B_{01}^{2a}$	[2, 8, 4]	$4.4.A_0^{2a}, A_1^{2a}$	2
$2a^2+1$	[2, 6, 12]	$6.4.D_{00}^{2a}$	[2, 3, 12]	$6.3.C_0^{2a}$	3
$6a^2+1$	[2, 6, 12]	$6.4.C_0^{2a}$	[2, 3, 12]	$6.3.D_{00}^{6a}$	3
$6a^2+1$	[4, 3, 12]	$6.7.D_{00}^{3a}$	[2, 3, 12]	$6.3.D_{00}^{6a}$	4
$2a^2+1$	[4, 3, 12]	$6.7.C_0^a$	[2, 3, 12]	$6.3.C_0^{2a}$	4
$2a^2+1$	[6, 3, 3]	$3.2.E_0^{2a}$	[2, 3, 12]	$6.3.C_0^{2a}$	2
$6a^2+1$	[6, 3, 3]	$3.2.F_{00}^{6a}$	[2, 3, 12]	$6.3.D_{00}^{6a}$	2
$3a^2+1$	[3, 6, 6]	$3.3.F_{00}^{3a}$	[2, 6, 6]	$6.2.D_{00}^{3a}$	2
a^2+1	[3, 6, 6]	$3.3.E_0^a$	[2, 6, 6]	$6.2.C_0^a$	2
$2a^2+1$	[6, 6, 6]	$3.4.F_{00}^{2a}$	[6, 3, 3]	$3.2E_0^{2a}$	3

12. From Table 4 the actions are either $6.6.C_0^{2a}$ or $6.6.C_1^{2a}$. Since [4, 6, 6] does not extend to an elliptic–hyperelliptic signature, these actions are full actions. This yields statements 9 and 10 of the theorem. The other statements are derived similarly. We appeal to Theorem 5.1 to determine extensions of G-actions with signature τ to G'-actions with signature τ'. These are given in Table 6. □

Corollary 6.4. *If $\mathcal{M}_g^1$ contains exceptional points, at most four of them are symmetric.*

Proof. Full actions are in bijection with exceptional points. If $g-1 = ka^2 = k'a'^2$, for $k, k' \in \{1, 2, 3, 6, 10, 30\}$, then $k = k'$ and hence also $a = a'$. Thus the ten enumerated cases in Theorem 6.3 are mutually exclusive. One merely observes that the maximum number of full actions in any one case is four.

7. Symmetries of exceptional points

Let $X = \mathcal{H}/\Gamma$ be a symmetric elliptic–hyperelliptic Riemann surface whose full group of conformal automorphisms $G = \Lambda/\Gamma$ acts with a triangular signature $[k, l, m]$. The existence of a symmetry on X means that Λ is the canonical Fuchsian group of a proper NEC group $\widetilde{\Lambda}$, containing Λ with index 2, and containing Γ as a normal subgroup. Then $\widetilde{\Lambda}/\Gamma = \mathcal{A}$ is the full group of conformal and anticonformal automorphisms of X. Comparison of (1) and (3) yields two possibilities for the signature of $\widetilde{\Lambda}$: $(0; +; [-]; (k, l, m))$, and, if $k = l$, $(0; +; [k]; \{(m)\})$; we shall see that, in this context, the second possibility does not occur.

Let $\widetilde{\theta}$ be the canonical epimorphism $\widetilde{\Lambda} \to \mathcal{A}$ with kernel Γ. A symmetry $\phi \in \mathcal{A}$ is the image under $\widetilde{\theta}$ of an element d from the subset $\widetilde{\Lambda} \setminus \Lambda$ of orientation-reversing elements of $\widetilde{\Lambda}$. If d cannot be chosen as a reflection then ϕ has no ovals. Otherwise d is conjugate to one of the reflection generators c in the canonical system of generators of $\widetilde{\Lambda}$. The number of ovals $\|\phi\|$ is the number of empty period cycles in the group $\widetilde{\Gamma} = \widetilde{\theta}^{-1}(\langle\phi\rangle)$. A formula for $\|\phi\|$ is given in [9] in terms of orders of centralizers:

$$\|\phi\| = \sum |C(\mathcal{A}, \widetilde{\theta}(c_i))|/|\widetilde{\theta}(C(\widetilde{\Lambda}, c_i))|, \tag{16}$$

where c_i runs over pairwise nonconjugate canonical reflection generators in $\widetilde{\Lambda}$ whose images are conjugate to ϕ, and $C(A, a)$ denotes the centralizer of the element a in the group A. In [14] (see also [16]) it is proved that the centralizer

Table 7
Conformal group G and ovals

k	G	Ovals	Conditions	k	G	Ovals	Conditions
1	$4.4.B_{00}^{2a}$	$a, 2, a$	$a \equiv 0\ (2)$	3	$4.5.B_{01}^{2a}$	$2a, a, a$	$a \equiv 0\ (2)$
	$4.4.B_{01}^{2a}$	$a, 2, a$	$a \equiv 0\ (2)$		$4.5.B_{00}^{2a}$	$2a, a, a$	$a \equiv 0\ (2)$
	$6.2.C_0^a$	$2, a, a$	$a \equiv 0\ (2)$		$6.2.D_{00}^{3a}$	$2, a, a$	$a \equiv 0\ (2)$
	$6.2.C_1^a$	$1, a, a$	$a \equiv 1\ (2)$		$6.2.D_{10}^{3a}$	$1, a, a$	$a \equiv 1\ (2)$
	$6.8.D_{00}^a$	$a, \frac{2}{3}a$	$a \equiv 0\ (6)$		$6.8.C_0^a$	$a, 2a$	None
	$6.8.D_{01}^a$	$a, \frac{2}{3}a$	$a \equiv 3\ (6)$	6	$4.5.A_0^{2a}$	$4a, 2a, a$	None
2	$4.4.A_0^{2a}$	$2a, 4, a$	None		$4.5.A_1^{2a}$	$4a, 2a, a$	None
	$4.4.A_1^{2a}$	$2a, 1, a$	None		$6.3.D_{00}^{6a}$	$a, 3a$	$a \equiv 0\ (2)$
	$6.3.C_\varepsilon^{2a}$	a, a	$\varepsilon \equiv a\ (2)$		$6.3.D_{11}^{6a}$	$a, 3a$	$a \equiv 1\ (2)$
	$6.5.D_{00}^{2a}$	$\frac{1}{3}a, a$	$a \equiv 0\ (6)$		$6.5.C_\varepsilon^{2a}$	a, a	$\varepsilon \equiv a\ (2)$
	$6.5.D_{11}^{2a}$	$\frac{1}{3}a, a$	$a \equiv 3\ (6)$	30	$6.6.D_{00}^{6a}$	$3a, 3a$	$a \equiv 0\ (2)$
10	$6.6.C_\varepsilon^{2a}$	$a, 3a$	$\varepsilon \equiv a\ (2)$		$6.6.D_{11}^{6a}$	$3a, 3a$	$a \equiv 1\ (2)$

of a reflection c in an NEC group Λ is isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}$ if the associated period cycle in Λ is empty or consists of odd periods only; otherwise it is isomorphic to $\mathbb{Z}_2 \oplus (\mathbb{Z} * \mathbb{Z})$, where $*$ denotes the free product.

Using these results we obtain the following classification of the centralizers of reflections in an NEC group whose canonical Fuchsian group is a triangle group. The notation $c_1 \sim c_2$ denotes conjugacy in $\widetilde{\Lambda}$.

Lemma 7.1. (a) *Let $\widetilde{\Lambda}$ be an NEC group with signature $(0; +; [-]; (k', l', m'))$ and let c_0, c_1, c_2 be the canonical system of generators of $\widetilde{\Lambda}$. Then*

(i) *for $k' = 2k+1, l' = 2l+1, m' = 2m+1, c_0 \sim c_1 \sim c_2$ and $C(\widetilde{\Lambda}, c_0) = \langle c_0 \rangle \oplus (\langle (c_2c_0)^m (c_1c_2)^l (c_0c_1)^k \rangle)$,*
(ii) *for $k' = 2k, l' = 2l+1, m' = 2m+1, c_0 \sim c_1 \sim c_2$ and $C(\widetilde{\Lambda}, c_0) = \langle c_0 \rangle \oplus (\langle (c_0c_1)^k \rangle * \langle (c_2c_0)^m (c_1c_2)^l (c_1c_0)^k (c_2c_1)^l (c_0c_2)^m \rangle)$,*
(iii) *for $k' = 2k, l' = 2l, m' = 2m+1, c_0 \sim c_2$ and $C(\widetilde{\Lambda}, c_0) = \langle c_0 \rangle \oplus (\langle (c_0c_1)^k \rangle * \langle (c_2c_0)^m (c_2c_1)^l (c_0c_2)^m \rangle)$, $C(\widetilde{\Lambda}, c_1) = \langle c_1 \rangle \oplus (\langle (c_0c_1)^k \rangle * \langle (c_1c_2)^l \rangle)$,*
(iv) *for $k' = 2k, l' = 2l, m' = 2m$ $C(\widetilde{\Lambda}, c_0) = \langle c_0 \rangle \oplus (\langle (c_0c_1)^k \rangle * \langle (c_0c_2)^m \rangle)$, $C(\widetilde{\Lambda}, c_1) = \langle c_1 \rangle \oplus (\langle (c_0c_1)^k \rangle * \langle (c_1c_2)^l \rangle)$, $C(\widetilde{\Lambda}, c_2) = \langle c_2 \rangle \oplus (\langle (c_0c_2)^m \rangle * \langle (c_1c_2)^l \rangle)$.*

(b) *Let $\widetilde{\Lambda}$ be an NEC group with signature $(0; +; [k]; \{(m)\})$ and let x, e, c_0, c_1 be a canonical system of generators of $\widetilde{\Lambda}$. Then $c_0 \sim c_1$ and*

$$C(\widetilde{\Lambda}, c_0) = \begin{cases} < c_0 > \oplus < (c_0c_1)^{m/2} > * < e(c_0c_1)^{m/2}e^{-1} > & \text{if } m \text{ is even,} \\ < c_0 > \oplus (< e(c_0c_1)^{(m-1)/2} >) & \text{if } m \text{ is odd.} \end{cases}$$

Theorem 7.2. *Let $\mathcal{A}$ be the full group of conformal and anticonformal automorphisms of a symmetric exceptional point $X \in \mathcal{M}_g^1$ with genus $g = ka^2 + 1$, $k \in \{1, 2, 3, 6, 10, 30\}$. Then $\mathcal{A}$ acts with NEC signature $(0; +; [-]; (k', l', m'))$ and is a semidirect product $G \rtimes \langle \tau : \tau^2 \rangle$, where G is the full group of conformal automorphisms of X, listed in Table 7 according to the value of k, τ is a symmetry of X, and the action of τ on the generators of G is given in Table 8. $\mathcal{A}$ contains two or three conjugacy classes of symmetries with fixed points, the number of whose ovals is given in Table 7.*

Proof. Let $\mathcal{A} = \tilde{\Lambda}/\Gamma$ and let $\tilde{\theta} : \widetilde{\Lambda} \to \mathcal{A}$ be the canonical epimorphism with kernel Γ. We first show that $\tilde{\Lambda}$ cannot have the signature $(0; +; [k]; \{(m)\})$. Suppose it had such a signature, and let x, e, c_0, c_1 be a canonical system of generators. Then $x_1 = x, x_2 = c_0x^{-1}c_0, x_3 = c_0c_1$ is a system of canonical generators of the canonical Fuchsian group Λ, which has signature $[k, k, m]$. The full group of conformal automorphisms $\Lambda/\Gamma = G$ is generated by $g_1 = \tilde{\theta}(x_1), g_2 = \tilde{\theta}(x_2)$, both of order k. Since $c_0x_1c_0 = x_2^{-1}$ and $c_0x_2c_0^{-1} = x_1^{-1}$, it follows, for $\tau = \tilde{\theta}(c_0)$, that

$$\tau g_1 \tau = g_2^{-1} \quad \text{and} \quad \tau g_2 \tau = g_1^{-1}. \tag{17}$$

Table 8
Action of τ on G

G	$\tau c\tau$	$\tau x\tau$	$\tau y\tau$	$\tau w\tau$	$\tau v\tau$	$\tau\rho\tau$
A_ε	c^{-1}	$x\rho^\alpha$	y^{-1}			ρ
$B_{\varepsilon\delta}$	c^{-1}			$v^{-2}w^{-1}\rho^{\alpha+\gamma}$	$v\rho^\alpha$	ρ
C_ε	$y^{-1}c^{-1}\rho^{\alpha+\beta}$	x^{-1}	cyc^{-1}			ρ
$D_{\varepsilon\varepsilon'}$	$v^{-1}c^{-1}\rho^{\alpha+\beta}$			$v^3w^{-2}\rho^\beta$	$v^2w^{-1}\rho^{\alpha+\beta+\mu}$	ρ

Thus $g_1 \mapsto g_2^{-1}$, $g_2 \mapsto g_1^{-1}$ is an outer automorphism of G. This is false if G has type 4.5, 6.2 or 6.6. On the other hand, these are the only cases where the action is full and the signature has the form $[k, k, m]$.

Thus we may assume $\tilde{\Lambda}$ has signature $(0; +; [-]; (k', l', m'))$. Let c_0, c_1, c_2 be the system of canonical generators of $\tilde{\Lambda}$. Then $x_1 = c_0c_1$, $x_2 = c_1c_2$ and $x_3 = c_2c_0$ is a system of canonical generators of the canonical Fuchsian group Λ. Since $c_1x_1c_1 = x_1^{-1}$ and $c_1x_2c_1 = x_2^{-1}$, it follows that

$$\tau g_1\tau = g_1^{-1} \quad \text{and} \quad \tau g_2\tau = g_2^{-1}, \tag{18}$$

for $g_1 = \tilde{\theta}(x_1)$, $g_2 = \tilde{\theta}(x_2)$ and $\tau = \tilde{\theta}(c_1)$. Thus $\mathcal{A}$ is a semidirect product $G \rtimes \mathbb{Z}_2 = \langle g_1, g_2\rangle \rtimes \langle \tau : \tau^2\rangle$, where τ is a symmetry of X. G is the full group of conformal automorphisms of X, hence it must have one of the actions listed in bold in Theorem 6.3. G has generators x, y, c such that $g_1, g_2, (g_1g_2)^{-1}$ are equal to $c^{-2}x, c, y^{-1}c$, respectively, if G has a $(2, 4, 4)$-action, or to c^3x, c^2y, c, if G has a $(2, 3, 6)$-action. (18) induces the action of τ on x, y, c given in Table 8.

A symmetry of X with fixed points is conjugate to one of $\tilde{\theta}(c_0) = g_1\tau$, $\tilde{\theta}(c_1) = \tau$ or $\tilde{\theta}(c_2) = \tau g_2$ and we shall calculate the number of its ovals using (16). Let υ_i and $\tilde{\upsilon}_i$ denote the orders of $\tilde{\theta}(C(\tilde{\Lambda}, c_i))$ and $C(\mathcal{A}, \tilde{\theta}(c_i))$, respectively. Any element $h \in \mathcal{A}$ has the unique presentation

$$h = x^r y^s c^t \tau^p \rho^q, \tag{19}$$

determined by a sequence (r, s, t, p, q) of integers satisfying $0 \le p, q \le 1$, $0 \le t < 4$ (or 6), $0 \le s < n$ and $0 \le r < d$, where $d = n$ for A_ε^n and C_ε^n; $d = n/2$ for $B_{\varepsilon\delta}^n$; and $d = n/3$ for $D_{\varepsilon\varepsilon'}^n$. Furthermore, for any integer κ

$$(x^r y^s)^\kappa = x^{r\kappa} y^{s\kappa} \rho^{\gamma\kappa(\kappa-1)/2}. \tag{20}$$

Assume G has $(2, 4, 4)$-action. Then we check that any element of $C(\mathcal{A}, g_1\tau)$ is determined by

$$(r, s, 0, p, q), \quad \text{with } x^{2r} = \begin{cases} \rho^{s(\alpha+\gamma)} & \text{if } p = 0, \\ \rho^{\alpha(s+1)+\mu+\gamma s} & \text{if } p = 1; \end{cases}$$

or

$$(r, s, 2, p, q), \quad \text{with } x^{2(r+1)} = \begin{cases} \rho^{s(\alpha+\gamma)} & \text{if } p = 1, \\ \rho^{\alpha(s+1)+\mu+\gamma s} & \text{if } p = 0. \end{cases}$$

If $G = A_\varepsilon^n$ with even n, we obtain the following possible sequences:

1. $(0, s, 0, 0, q), (-1, s, 2, 1, q) : s(\gamma + \alpha) \equiv 0\ (2)$,
2. $(0, s, 0, 1, q), (-1, s, 2, 0, q) : s(\gamma + \alpha) + \alpha + \mu \equiv 0\ (2)$
3. $(n/2, s, 0, 0, q), (\frac{n}{2} - 1, s, 2, 1, q) : s(\gamma + \alpha) \equiv \varepsilon\ (2)$,
4. $(n/2, s, 0, 1, q), (\frac{n}{2} - 1, s, 2, 0, q) : s(\gamma + \alpha) + \alpha + \mu \equiv \varepsilon\ (2)$

whose total number, given particular values of α, γ, μ and ε, is $\tilde{\upsilon}_0 = 8n$. For $G = A_\varepsilon^n$ with n odd and also for $G = B_{\varepsilon\delta}^n$, we have only the first two possibilities, so that $\tilde{\upsilon}_0 = 4n$.

$C(\mathcal{A}, \tau) = \{x^r y^s \tau^p \rho^q : y^{2s} = \rho^{r\alpha}\} \cup \{x^r y^s c^2 \tau^p \rho^q : y^{2s} = \rho^{r\alpha+\mu}\}$, so its elements are determined by the sequences

1. $(r, 0, 0, p, q) : \alpha r \equiv 0\ (2)$; $(r, \frac{n}{2}, 0, p, q) : \alpha r \equiv \varepsilon\ (2)$;
2. $(r, 0, 2, p, q) : \alpha r + \mu \equiv 0\ (2)$; $(r, \frac{n}{2}, 2, p, q) : \alpha r + \mu \equiv \varepsilon\ (2)$.

In all cases but 4.4, $\tilde{\upsilon}_1 = \tilde{\upsilon}_0$ and in the exceptional case $\tilde{\upsilon}_1 = 16n$ or $8n$ according to G being A_0^n or A_1^n, $B_{0\delta}^n$, respectively.

$C(\mathcal{A}, \tau g_2) = \{x^r y^s \rho^q, x^r y^s c^3 \tau \rho^q : x^{r-s} y^{s-r} = \rho^{s\gamma+\alpha(r+s)}\} \cup \{x^r y^s c^2 \rho^q, x^r y^s c \tau \rho^q : x^{r-s} y^{s-r} = \rho^{s\gamma+\mu+\alpha(r+s)}\}$. Thus for $G = A_\varepsilon^n$, $\tilde{\upsilon}_2$ is the total number of sequences of the form $(r, r, 0, 0, q)$, $(r, r, 3, 1, q)$, $r\gamma \equiv 0$ (2) and $(r, r, 2, 0, q)$, $(r, r, 1, 1, q)$, $r\gamma + \mu \equiv 0$ (2). According to the particular values of γ and μ, $\tilde{\upsilon}_2 = 4n$, except the case 4.3, where $\tilde{\upsilon}_2 = 8n$. For $G = B_{\varepsilon\delta}^n$, only half of the listed sequences are possible; however, we must now include sequences of the form $(r, \frac{n}{2} + r, 0, 0, q)$, $(r, \frac{n}{2} + r, 3, 1, q)$ with $\alpha n/2 + \gamma r + \delta \equiv 0$ (2) and $(r, \frac{n}{2} + r, 2, 0, q)$, $(r, \frac{n}{2} + r, 1, 1, q)$ with $\alpha n/2 + r\gamma + \mu + \delta \equiv 0$ (2). In all cases, except 4.3, there are $2n$ additional sequences and in the exceptional case, $4n$ or 0 according to $\varepsilon + \delta$ being even or odd. Consequently, $\tilde{\upsilon} = 4n$, except the case 4.3 with $\varepsilon + \delta \equiv 0$ (2), where $\tilde{\upsilon} = 8n$.

Let k, l, m be integers such that $k' = 2k$, $l' = 2l$ and $m' = 2m$. Then by Lemma 7.1,

$$\upsilon_0 = 4 \cdot \mathrm{ord}(g_1^k (g_1 g_2)^m), \qquad \upsilon_1 = 4 \cdot \mathrm{ord}(g_1^k g_2^l) \quad \text{and} \quad \upsilon_2 = 4 \cdot \mathrm{ord}((g_1 g_2)^m g_2^l). \tag{21}$$

Thus we obtain the following values of (ν_0, ν_1, ν_2): 4.2 (4, 4, 4); 4.3 $(8, 8, 2n \cdot 2^{\varepsilon+\delta\ (2)})$ for $G = B_{\varepsilon\delta}^n$ and $(8, 8, 4n)$ for $G = A_\varepsilon^n$; 4.4 $(8, 4n \cdot 2^\varepsilon, 8)$; 4.5 (4, 8, 8) and 4.6 (8, 8, 4).

Now we check that $h\tau g_2 h^{-1} \neq g_1\tau$ and $h\tau g_2 h^{-1} \neq \tau$ for any element h of the form (19) while τ and $g_1\tau$ are conjugate only for odd n via $x^{(n-1)/2}c^3$ or $x^{(n-1)/2}c$ according to $\varepsilon = 0$ or $\varepsilon = 1$. Thus by (16), there are two or three conjugacy classes of symmetries with fixed points and the numbers of their ovals are equal to $\tilde{\upsilon}_0/\upsilon_0$, $\tilde{\upsilon}_1/\upsilon_1$, $\tilde{\upsilon}_2/\upsilon_2$ or $\tilde{\upsilon}_0/\upsilon_0 + \tilde{\upsilon}_1/\upsilon_1$, $\tilde{\upsilon}_2/\upsilon_2$ according to n being even or odd.

The arguments are similar in the case that G has a (2, 3, 6) action, and we omit them. □

Acknowledgements

The first author was supported by BW 5100-5-0289-7. The second author was supported by PSC CUNY Research Award 35-9370.

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