# Dessins and curves 

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## Definition

A map is an imbedding of a finite connected graph $\mathcal{G}$ on a compact oriented surface $X$ such that the complement $X \backslash \mathcal{G}$ is a union of 2-cells, called faces.

- $\mathcal{G}$ may have loops and multiple edges;
- a directed edge is called a dart;
- sometimes, "free edges" (with only one dart) are allowed:

free edge

Here is a map on the sphere:


$$
\begin{gathered}
V-E+F=2 \\
\Longrightarrow g=0
\end{gathered}
$$

1 vertex, 1 edge, 2 faces

More generally,

## Definition

A dessin d'enfant is an imbedding of a finite connected bipartite graph $\mathcal{G}$ on a compact oriented surface $X$ such that the complement $X \backslash \mathcal{G}$ is a union of 2-cells.

Note:

- Every map is a dessin by insertion of a "white" vertex at the midpoint of every edge (or at the free end of a free edge).
- Then darts correspond to "white-to-black" edges.

map

dessin

Conversely, a dessin with white valencies $\leq 2$ can be made into a map by

- inserting arrows (white-to-black) on each edge;
- erasing white vertices.

In this talk, l'll work almost exclusively with maps.

Here is a regular map on a torus:


8 vertices; 12 edges ( 24 darts); 4 faces (centered at [A], [B], [C], [D] with indicated identifications)

## Maps as transitive permutation groups

Let $\mathcal{M}$ be a map with $n$ darts. Label the darts with the symbols $0,1,2,3, \ldots, n-1$ (in some convenient order).

## Definition

The monodromy group of $\mathcal{M}$ is the subgroup
$G_{\mathcal{M}}=\langle x, y\rangle \subseteq S_{n}$, where

- $x \equiv$ product of dart cycles at the vertices;
- $y \equiv$ product of dart pairs on the edges;
the darts surrounding a vertex acquire a cyclic ordering by the orientation of the ambient surface.


## In our example,



9
orientation
the generators of the monodromy group ( $\leq S_{24}$ ) are

- $x=\left(\begin{array}{ll}0 & 1\end{array}\right)(345)(678) \ldots \quad$ (Eight 3-cycles $\leftrightarrow$ vertices)
- $y=(23)(46) \ldots \quad$ (Twelve 2-cycles $\leftrightarrow$ edges)

It is not hard to see that, for any map, the monodromy group $\langle x, y\rangle$ acts transitively on the darts:

- connectivity of the underlying graph $\Longrightarrow$ there is an edgepath $e_{1} e_{2} \ldots e_{k}$ between any two vertices $v_{1}$ and $v_{2}$;
- a dart $\delta_{1}$ at $v_{1}$ can be rotated to lie on $e_{1}$ (using a power of $x)$;
- then reversed to point toward the initial vertex of $e_{2}$ (using $y)$;
- etc.

It is also not hard to see that, for any map, the cycles of $(x y)^{-1}$ correspond to "walks" around the oriented boundaries of the map's faces, keeping the face 'on the left.' This is because $(x y)^{-1}$ acts on a dart as follows: (i) rotate to the previous edge in the cyclic order; then, (ii) reverse.


orientation

Starting at dart 5 in our example, we get the cycle
(569191614), which describes the boundary of the central face.

Starting again at an unused dart, and continuing until all darts have appeared, we obtain the other cycles of $(x y)^{-1}$. In our example,

$$
\begin{gathered}
(x y)^{-1}=(0313222011)(1108231215) \\
(217182174)(569191614)
\end{gathered}
$$

The four 6-cycles correspond to the four hexagonal faces.

## Map equivalence

## Definition

Two maps $\mathcal{M}_{1}, \mathcal{M}_{2}$ with $n$ darts are equivalent if their monodromy groups $G_{1}, G_{2}$ are strongly conjugate in $S_{n}$.

This means there exists a single permutation in $S_{n}$ which simultaneously conjugates the generators $x_{1}, y_{1} \in G_{1}$ to the corresponding generators $x_{2}, y_{2} \in G_{2}$.

Remark: there is a weaker equivalence relation on dessins, defined by conjugacy (but not strong conjugacy) of the monodromy groups. This is the appropriate equivalence relation for studying the faithful action of the absolute Galois group $\operatorname{Gal}(\overline{\mathbb{Q}} / \mathbb{Q})$ on dessins.

I won't pursue that very interesting topic here.

## Map automorphisms

## Definition

The automorphism group $\operatorname{Aut}(\mathcal{M})$ of a map $\mathcal{M}$ with $n$ darts is the centralizer of its monodromy group in $S_{n}$.
(Rationale: automorphisms are self-equivalences of $\mathcal{M}$, i.e., permutations in $S_{n}$ which commute with both monodromy generators.)

Note: $\operatorname{Aut}(\mathcal{M})$ is well-defined on equivalence classes of maps (conjugate subgroups have conjugate centralizers).

## The type of a map

## Definitions

If $m$ is the Icm of the vertex valencies of a map $\mathcal{M}$, and $r$ is the Icm of the face valencies, we say that M has type ( $m, r$ ).

- $\mathcal{M}$ is uniform if all vertices have valence $m$ and all faces have valence $r$;
- $\mathcal{M}$ is regular if $\operatorname{Aut}(\mathcal{M})$ is transitive on the darts.

Lemma: Regular $\Longrightarrow$ Uniform. Proof: transitivity of $\operatorname{Aut}(\mathcal{M})$ on darts implies that all darts have the same local incidence relations.

## Maps via uniformization

Let $\Gamma=\Gamma(m, r)$ be the triangle group with presentation

$$
\Gamma(m, r)=\left\langle\xi_{1}, \xi_{2}, \xi_{3} \mid \xi_{1}^{m}=\xi_{2}^{2}=\xi_{3}^{r}=\xi_{1} \xi_{2} \xi_{3}=1\right\rangle .
$$

Geometrically, $\Gamma$ is the orientation-preserving subgroup of the group of isometries generated by reflections in the sides of a triangle contained in $\mathcal{U}=\mathbb{P}^{1}, \mathbb{C}$, or $\mathbb{H}$.


C

Iterating the reflections produces a triangular tessellation of $\mathcal{U}$ which contains the universal map of type ( $m, r$ )

If $\mathcal{M}$ is a map of type $(m, r)$, there is an obvious surjective homomorphism $\theta: \Gamma(m, r) \rightarrow G_{\mathcal{M}}=\langle x, y\rangle$, namely

$$
\theta: \xi_{1} \mapsto x, \quad \xi_{2} \mapsto y, \quad \xi_{3} \mapsto(x y)^{-1} .
$$

( $\theta=i d$ if $\mathcal{M}$ is the universal map of type $(m, r)$.)

Let $G_{\delta, \mathcal{M}} \leq G_{\mathcal{M}}$ be the isotropy subgroup of a dart $\delta \in \mathcal{M}$.

## Definition

The canonical map subgroup $M$ for a map $\mathcal{M}$ of type $(m, r)$ is

$$
M \equiv \theta^{-1}\left(G_{\delta, \mathcal{M}}\right) \leq \Gamma(m, r)
$$

Lemma: $M$ is well-defined up to conjugacy in $\Gamma(m, r)$ (i.e., independent of the choice of $\delta$ ). Proof: by the transitivity of $G_{\mathcal{M}}$ on the darts, all $G_{\delta, \mathcal{M}}$ are conjugate.

Let

- $M^{*} \equiv \bigcap_{\gamma \in \Gamma} \gamma^{-1} M \gamma \leq \Gamma \quad\left(M^{*}=\right.$ the core of $M$ in $\left.\Gamma\right)$;
- $|\Gamma / M| \equiv$ the set of cosets $M$ in $\Gamma$;
- $D \equiv$ the set of darts of $\mathcal{M}$.


## Lemma (Jones, Singerman, '78)

The permutation groups $\left(G_{\mathcal{M}}, D\right)$ and $\left(\Gamma / M^{*},|\Gamma / M|\right)$ are isomorphic.

That is,

$$
\begin{array}{ccccc}
G_{\mathcal{M}} & \times & D & \rightarrow & D \\
\simeq \downarrow & & b i j \downarrow & & b i j \downarrow \\
\frac{\Gamma}{M^{*}} & \times & \left|\frac{\Gamma}{M}\right| & \rightarrow & \left|\frac{\Gamma}{M}\right|
\end{array}
$$

is commutative.

Via the canonical map subgroup $M \leq \Gamma(m, r)$ we obtain, for a map $\mathcal{M}$,

$$
X_{\mathcal{M}} \equiv \frac{\mathcal{U}}{M} \equiv \text { the canonical Riemann surface of } \mathcal{M},
$$

where $\mathcal{U} \equiv \mathbb{C}, \mathbb{P}^{1}$, or $H^{2}$. Rationale: $M \leq \Gamma(m, r)$ acts properly discontinuously by isometries on $\mathcal{U}$, hence the quotient $\mathcal{U} / M$ inherits a metric which can be completed to a complex structure.

It follows that

## Geometrization of the map

- The canonical Riemann surface $X_{\mathcal{M}}$ contains the map $\mathcal{M}$ geometrically: edges are geodesics; face-centers and edge midpoints are well-defined.


## The geometric-algebraic-conformal dictionary

$\{$ subgroups of $\Gamma(m, r)\} \longleftrightarrow\{$ maps of type $(m, r)\}$.
$M \longleftrightarrow \mathcal{M}$
Moreover
$\left\{\begin{array}{l}\text { conj. classes } \\ \text { of subgp }\end{array}\right\} \longleftrightarrow\left\{\begin{array}{l}\text { equiv. classes } \\ \text { of maps }\end{array}\right\} \longleftrightarrow\left\{\begin{array}{l}\text { conformal } \\ \text { equiv. classes } \\ \text { of surfaces }\end{array}\right\}$
$[M] \longleftrightarrow[\mathcal{M}] \longleftrightarrow\left[X_{\mathcal{M}}\right]$

Furthermore, for $M \leftrightarrow \mathcal{M}$,

- $M$ is torsion-free $\Longleftrightarrow \mathcal{M}$ is uniform;
- $M$ is torsion-free and normal $\Longleftrightarrow \mathcal{M}$ is regular;
- $\operatorname{Aut}(\mathcal{M}) \simeq N_{\Gamma}(M) / M$.

The last statement implies that map automorphisms are also conformal automorphisms of $X_{\mathcal{M}}$.

The torus example is a regular map of type $(6,3)$ :

$\Gamma(6,3)=\mathbb{Z}_{6} \ltimes(\mathbb{Z} \oplus \mathbb{Z})$, the infinite Euclidean triangle group acting on $\mathbb{C}$ by isometries; $M=2 \mathbb{Z} \oplus 2 \mathbb{Z}$, the normal, torsion-free subgroup generated by the squares of the translations $[A] \mapsto[B],[A] \mapsto[C] . X_{\mathcal{M}}=\mathbb{C} / M$, and Aut $\left(X_{\mathcal{M}}\right) \geq \mathbb{Z}_{6} \ltimes(\mathbb{Z} / 2 \mathbb{Z} \oplus \mathbb{Z} / 2 \mathbb{Z})$, a nonabelian group of order 24.

## Maps via Bely̆'s theorem

## Bely̌'s theorem

A Riemann surface $X$ is defined over a number field (finite extension of $\mathbb{Q}$ ) if and only if there is a meromorphic function

$$
f: X \rightarrow \mathbb{P}^{1}
$$

with at most three critical points.
If $f: X \rightarrow \mathbb{P}^{1}$ is a Bely̆i function, with critical points in $\{0,1, \infty\} \subset \mathbb{P}^{1}$, then

$$
f^{-1}\left(\begin{array}{ll}
0 \\
0 & 0
\end{array}\right)
$$

is a dessin lying geometrically on $X$. \{face centers $\}=f^{-1}(\infty)$.

Example: the Fermat curve $F_{n}$, defined by $x^{n}+y^{n}=1$.

- $F_{n}$ has genus $(n-1)(n-2) / 2$;
- $f:(x, y) \mapsto x^{n}$ is a Bely̆i function of degree $n^{2}$;
- $f^{-1}\left(\begin{array}{l}\bullet \\ 0\end{array}\right.$
(This is the minimum-genus imbedding of $K_{n, n}$. E.g.: $K_{3,3}$ imbeds on the torus but not on the sphere, since one crossing is needed.)
$\operatorname{Aut}\left(F_{n}\right)=S_{3} \ltimes\left(\mathbb{Z}_{n} \oplus \mathbb{Z}_{n}\right) \geq \mathbb{Z}_{n} \oplus \mathbb{Z}_{n}=\operatorname{Aut}\left(K_{n, n}\right)$.

Recall: a map automorphism is conformal automorphism of the canonical Riemann surface. As such, it acts properly discontinuously; in particular, there is a finite set of points with a non-trivial cyclic isotropy subgroup. These points must be among the geometric points of the map:

- vertices;
- edge midpoints;
- face centers.

The orbit of an edge midpoint can only be fixed by an automorphism of order 2. If the map has type ( $m, r$ ), the orbit of a face centers can only be fixed by an automorphism whose order divides $r$; a vertex orbit by an automorphism of order dividing $m$.

This information (branching data) gives a 'signature' for the action of a cyclic group of (map) automorphisms.

## One-vertex maps

I now specialize to one-vertex maps with $k$ non-free edges.

Motivation: these maps characterize several well-known families of curves, which are recovered as the corresponding canonical Riemann surfaces.

To obtain all equivalence classes of one-vertex maps with $k$ non-free edges, start with a star map consisting of $2 k$ labelled darts ('free edges'), and pair them off in all possible ways.
E.g., with $k=3$, the star map

has $\frac{1}{3!}\binom{6}{2}\binom{4}{2}\binom{2}{2}=15$ possible pairings.

## Here are three of them:

(a)


$$
\begin{aligned}
& x=\left(\begin{array}{llll}
0 & 1 & 2 & 3
\end{array} 45\right) \\
& y=\left(\begin{array}{ll}
0 & 1
\end{array}\right)\left(\begin{array}{ll}
2 & 3
\end{array}\right)\left(\begin{array}{l}
4
\end{array}\right) \\
& y x^{-1}=\left(\begin{array}{lll}
0 & 4 & 2
\end{array}\right)(1)(3)(5)
\end{aligned}
$$

(b)


$$
\begin{aligned}
& x=\left(\begin{array}{lllll}
0 & 1 & 2 & 3 & 4
\end{array}\right) \\
& y=(0) 3)(14)(25) \\
& y x^{-1}=\left(\begin{array}{llll}
0 & 2 & 4
\end{array}\right)\left(\begin{array}{ll}
1 & 3
\end{array}\right)
\end{aligned}
$$

(c)

$x=(012345)$
$y=(12)(34)(50)$
$y x^{-1}=(0)(2)(4)(135)$

One-vertex maps are equivalent (by duality) to unicellular (one-face) maps, which have been extensively studied, e.g.,

- Harer and Zagier ('86) enumerated them by means of a recurrence relation on the genus;
- Chapuy (2011) enumerated them by a direct bijective approach;
Unicellular maps of genus 0 are plane trees, which have a large literature (Shabat, Voevodsky, Zvonkin, others).

One-vertex maps with $k$ non-free edges can be considered (again by duality) as side-pairings of regular $2 k-$ gon. - this point of view makes the underlying Riemann surface explicit.

In my approach, maps are counted according to the size of their automorphism group.

## Counting one-vertex maps

Let $\mathcal{M}$ be a one-vertex map with $k \geq 1$ non-free edges. The monodromy group of $\mathcal{M}$ is generated by

- $x$, a $2 k$-cycle which we fix as ( $012 \ldots 2 k-1$ ); and
- $y$, a free involution (product of $k$ disjoint transpositions), which may be chosen in

$$
\frac{1}{k!} \prod_{i=0}^{k-1}\binom{2 k-2 i}{2}=(2 k-1) \cdot(2 k-3) \cdots \cdot 3 \cdot 1=(2 k-1)!!
$$

ways.

## Lemma (W)

(1) $\operatorname{Aut}(\mathcal{M}) \leq\langle x\rangle=\left\langle\left(\begin{array}{lll}0 & 1 & 2 \ldots 2 k-1)\rangle \text {. }\end{array}\right.\right.$
(2) If $\mathcal{M}$ is equivalent to $\mathcal{M}^{\prime}$, with monodromy groups $G_{\mathcal{M}}=\langle x, y\rangle$ and $G_{\mathcal{M}^{\prime}}=\left\langle x, y^{\prime}\right\rangle$, respectively, then $y^{\prime}=x^{-s} y x^{s}$ for some $s, 0 \leq s \leq p-1$, where $p$ is a divisor of $2 k$ such that $\operatorname{Aut}(\mathcal{M})=\operatorname{Aut}\left(\mathcal{M}^{\prime}\right)=\left\langle x^{p}\right\rangle$.

## Proof sketch:

(1) An abelian, transitive permutation group is its own centralizer. Hence $\operatorname{Aut}(\mathcal{M}) \leq\langle x\rangle$.
(2) If $\langle x, y\rangle$ and $\left\langle x, y^{\prime}\right\rangle$ are strongly conjugate $\left(y \neq y^{\prime}=\sigma^{-1} y \sigma, \sigma \in S_{2 k}\right)$, and $\operatorname{Aut}(\mathcal{M})=\operatorname{Aut}\left(\mathcal{M}^{\prime}\right)=\left\langle x^{p}\right\rangle$, then $\sigma \in\langle x\rangle \backslash\left\langle x^{p}\right\rangle$. Since $x^{p}$ commutes with $y$, all possible $y^{\prime}$ are obtained by taking $\sigma \in\left\{x, x^{2}, \ldots, x^{p-1}\right\}$.

## Prescribing the automorphism group:

Constructing $\mathcal{M}$ such that $\operatorname{Aut}(\mathcal{M})=\left\langle x^{p}\right\rangle$ is equivalent to choosing a free involution, $y$, which commutes with $x^{p}$, but not with $x^{s}, 0<s<p$.
$x^{p}$ is a product of $p$ cycles of length $2 k / p$ :

$$
(0, p \ldots)(1, p+1 \ldots) \ldots(p-1,2 p-1 \ldots) .
$$

If $y$ commutes with $x^{p}$, then conjugation by $y$

- fixes a $2 k / p$-cycle or transposes two of them;
- preserves the internal cyclic order in each $2 k / p$-cycle.
(A fixed cycle is possible only if $2 k / p$ is even.)


## Lemma (W)

The number of free involutions in $S_{2 k}$ commuting with $x^{p}$ is

$$
\bar{\nu}_{p}= \begin{cases}\sum_{m=0}^{q}\left(\frac{d}{2}\right)^{m} \frac{p!}{m!(p-2 m)!} & \text { if } d \text { is even } \\ \left(\frac{d}{2}\right)^{q} \frac{p!}{q!} & \text { if } d \text { is odd }\end{cases}
$$

where $d=2 k / p$, and $q=\left\lfloor\frac{p}{2}\right\rfloor$.
Two special cases:

- $\bar{\nu}_{2 k}=(2 k-1)!$ !, as expected, since every free involution commutes with $x^{2 k}=$ id;
- $\bar{\nu}_{1}=1$, since $x^{k}$ is the unique (free) involution in $\operatorname{Cent}_{S_{2 k}}(x)=\langle x\rangle$.

A inclusion/exclusion argument yields a formula for $\nu_{p}$, the number of one-vertex maps with $k$ edges whose full automorphism group is $\left\langle x^{p}\right\rangle, p$ divisor of $2 k$.

## Lemma (W)

The number of free involutions commuting with $x^{p}$, but not with $x^{s}, 0<s<p$, is

$$
\nu_{p}=\bar{\nu}_{p}+\sum_{i=1}^{s}(-1)^{i} \sigma_{i},
$$

where

$$
\begin{aligned}
& \sigma_{1}=\sum_{1 \leq j \leq s} \bar{\nu}_{p / p_{j}}, \sigma_{2}=\sum_{1 \leq j<k \leq s} \bar{\nu}_{p / p_{j} p_{k}}, \\
& \sigma_{3}=\sum_{1 \leq j<k<l \leq s} \bar{\nu}_{p / p_{j} p_{k} p_{l},}, \sigma_{4}=\ldots,
\end{aligned}
$$

and $p_{i}, i=1, \ldots, s$, are the prime divisors of $p$.

Finally, combining these two results,

## Theorem 2 (W)

$\nu_{p} / p$ is the number of equivalence classes of one-vertex maps with $k$ edges whose automorphism group is exactly $\left\langle x^{p}\right\rangle, p$ a divisor of $2 k$.

Special cases: $\nu_{1} / 1=1 ; \quad$ and

$$
\nu_{2} / 2= \begin{cases}k / 2 & \text { if } k \text { is even } \\ (k-1) / 2 & \text { if } k \text { is odd }\end{cases}
$$

These special cases (with maximal and second maximal automorphism group) were shown previously for $k=3$ :

(b)


$$
\begin{aligned}
& x=\left(\begin{array}{llll}
0 & 1 & 2 & 3
\end{array} 4\right) \\
& y=(03)(14)(25) \\
& y x^{-1}=\left(\begin{array}{llll}
0 & 2 & 4
\end{array}\right)\left(\begin{array}{ll}
1 & 3
\end{array}\right)
\end{aligned}
$$

(c)


$$
\begin{aligned}
& x=\left(\begin{array}{llll}
0 & 1 & 2 & 3
\end{array} 4\right) \\
& y=(12)(34)(50) \\
& y x^{-1}=(0)(2)(4)(135)
\end{aligned}
$$

(a) and (c) are equivalent, with $\operatorname{Aut}(\mathcal{M}) \simeq \mathbb{Z}_{3}$
(b) is regular with $\operatorname{Aut}(\mathcal{M}) \simeq \mathbb{Z}_{6}$

## Some classical curves

In the 1890's, A. Wiman identified the curves (genus $g>1$ )

$$
\begin{array}{cc}
w^{2}=z^{2 g+1}-1 & \text { (type I) } \\
w^{2}=z\left(z^{2 g}-1\right) & \text { (type II) } \\
w^{3}=z^{4}+1 \quad(g=3) & \text { (type III). }
\end{array}
$$

Their maximal cyclic automorphism groups are, resp.,

$$
\begin{gathered}
(w, z) \mapsto\left(-w, z e^{2 \pi i / 2 g+1}\right) \quad \text { of order } 4 g+2 \quad \text { (type II) } \\
(w, z) \mapsto\left(w e^{\pi i / 2 g}, z e^{\pi i / g}\right) \quad \text { of order } 4 g \quad \text { (type II) } \\
(w, z) \mapsto\left(w e^{2 \pi i / 3}, z e^{2 \pi i / 4}\right) \quad \text { of order } 4 g=12 \quad \text { (type III). }
\end{gathered}
$$

The Wiman curves of type I and II are characterized in terms of regular one-vertex maps.

## Theorem (D. Singerman, 2001)

Let $\mathcal{M}$ be a regular, one-vertex map on a surface $X$ of genus $g>0$. Then $X$ is the Wiman curve of type I and $\mathcal{M}$ has $2 g+1$ edges, or $X$ is the Wiman curve of type II and $\mathcal{M}$ has $2 g$ edges.
(The theorem was stated in terms of the dual "unifacial dessins".)

Note:

- For the type I surface, $\operatorname{Aut}(X)=\operatorname{Aut}(\mathcal{M})=\mathbb{Z}_{4 g+2}$;
- For the type II surface, $\operatorname{Aut}(X)=S D_{8 g} \geq \operatorname{Aut}(\mathcal{M})=\mathbb{Z}_{4 g}$.

Remark: the natural $g=1$ analogues of the Wiman curves of types I and II are the elliptic curves with moduli $e^{2 i \pi / 3}$ and $i$.

Here they are, with their regular one-vertex maps:

(These are the elliptic curves admitting automorphisms with fixed points of maximal and second-maximal finite order.)

Remark:


The left side is the "geometrization" of the dessin at right.

## Edge-transitive maps

Goal: a similar characterization, in terms of well-known curves, of the surfaces underlying one-vertex maps with second maximal automorphism group.

## Definition

A map $\mathcal{M}$ is strictly edge-transitive if $\operatorname{Aut}(\mathcal{M})$ acts transitively on the edges, but not on the darts.

Henceforth $\mathcal{M}$ denotes a strictly edge-transitive one-vertex map with $k$ edges and monodromy group $\langle x, y\rangle \leq S_{2 k}$

Then:

- $\operatorname{Aut}(\mathcal{M})=\mathbb{Z}_{k}=\left\langle x^{2}\right\rangle ;$
- the edge-midpoints are permuted transitively in a $k$-cycle;
- there are at most two distinct face-valences (since a dart and its "reversal" may have distinct incidence relations).
- Let $l_{1}$ and $l_{2}$ denote the two face-valences of $\mathcal{M}$ (possibly $I_{1}=I_{2}$ );
- Write $\mathbb{Z}_{k}=(\{0,1, \ldots, k-1\},+)$ additively $(\bmod k)$, with generator 1 .


## Lemma (W)

The action of $\operatorname{Aut}(\mathcal{M})=\mathbb{Z}_{k}$ on the canonical Riemann surface has

- signature $\left(0 ; k, l_{1}, l_{2}\right)$ and
- generating vector $\langle 1, t, k-(t+1)\rangle$, for a unique integer $t, 0<t<k / 2, t \neq(k-1) / 2$. Moreover,

$$
I_{1}=k /(t, k) \quad \text { and } \quad I_{2}=k /(t+1, k) .
$$

(1) The free involution $y$ commutes with $x^{2}$ but not with $x$, hence it transposes the two $k$-cycles comprising $x^{2}$,

$$
C_{0}=(02 \ldots 2 k-2) \quad \text { and } \quad C_{1}=(13 \ldots 2 k-1),
$$

while preserving their internal cyclic orderings.
(2) This is done by pairing symbols in the first cycle with symbols in the second cycle with a forward shift by $t$ symbols, $0 \leq t \leq k-1$. (lf $k$ is odd, $t=(k-1) / 2$ is excluded, since in that case, $y=x^{k}$ and the map is regular.)

Example $(k=5)$ :

$$
\begin{aligned}
& x^{2}=(02468)(13579) \\
& y=\left(\begin{array}{ll}
0 & 1
\end{array}\right)(23) \ldots \quad[t=0] \text { or } \\
& y=(03)(25) \ldots \quad[t=1] \text { or } \\
& y \neq(05)(27) \cdots=x^{k} \quad[t=(k-1) / 2] \\
& y=(07)(29) \ldots \quad[t=3] \text { or } \\
& y=(09)(21) \ldots \quad[t=4] .
\end{aligned}
$$

Finding cycle-lengths in $(x y)^{-1}$ : there are at most two different cycle lengths ( $l_{1}$ and $l_{2}$ ).
(1) If a cycle contains an odd symbol $2 c+1$, then

$$
2 c+1 \stackrel{x-1}{\rightarrow} 2 c \xrightarrow{\text { y }} 2 c+2 t+1,
$$

which implies a cycle-length $I_{1}$ where $I_{1}$ is the minimal positive integer for which $2 l_{1} t \equiv 0(\bmod 2 k)$. It follows that

$$
I_{1}=k /(t, k) .
$$

(2) A similar argument starting with an even symbol yields

$$
I_{2}=k /(t+1, k) .
$$

It follows that $(x y)^{-1}$ is a product of $(t, k)$ cycles of length $I_{1}$ and $(t+1, k)$ cycles of length $I_{2}$.

## End of proof sketch

(1) Finally, $\langle x, y\rangle$ is strongly conjugate to $\left\langle x, x^{-1} y x\right\rangle$, and $x^{-1} y x$ is obtained from $y$ by replacing the "shift-parameter" $t$ by $k-(t+1)$. Hence, up to map equivalence, we may assume $t<k / 2$.
(2) The Icm of $l_{1}$ and $l_{2}$ is $k$, so (by the "LCM condition" of Maclachlan/Harvey) a ( $0 ; k, l_{1}, l_{2}$ )-generating vector for $\mathbb{Z}_{k}$ exists, and may be taken in the normal form $\langle 1, t, k-(t+1)\rangle$.

## Extendability

- Certain group actions (with non-finitely maximal signatures) extend to actions by larger groups. (Bujalance, Cirré, Conder 1999, 2002).
- Extendability of an action depends in part on the form of the generating vector.
- EXAMPLE: A $\mathbb{Z}_{k}$-action with signature ( $0 ; k, k, u$ ) has an extension to a (non-abelian, non-dihedral) $G$-action with signature $(0 ; 2, k, 2 u)$ if and only if $\mathbb{Z}_{k}$ has an automorphism $\alpha$ of order 2 such that $\alpha(1) \neq-1$. (This is the case iff $k \neq 2,4, p^{s}, 2 p^{s}, p$ an odd prime). Then the $\mathbb{Z}_{k}$-action extends to an action by $G \simeq \mathbb{Z}_{2} \ltimes_{\alpha} \mathbb{Z}_{k}$, iff the generating vector of the $\mathbb{Z}_{k}$ action is $\langle 1, \alpha(1), k-(1+\alpha(1))\rangle$.


## Let $X$ be the canonical Riemann surface of $\mathcal{M}$.

If the generating vector $\langle 1, t, k-(t+1)\rangle$ of the $\mathbb{Z}_{k}$ action is NOT extendable (e.g., in particular, if the signature $\left(0 ; k, l_{1}, l_{2}\right)$ is finitely maximal), then

- $\operatorname{Aut}(\mathcal{M})=\operatorname{Aut}(X)=\mathbb{Z}_{k}$;
- $X$ has equation $w^{k}=z^{k / 1}(z-1)^{k / l_{2}}$; and
- $\operatorname{Aut}(X)$ is generated by

$$
(w, z) \mapsto\left(w, e^{2 i \pi / k} z\right)
$$

## Summary of the main result

If $\operatorname{Aut}(X)>\operatorname{Aut}(\mathcal{M})$ (i.e., if the signature $\left(0 ; k, l_{1}, l_{2}\right)$ is not finitely maximal, and $t$ is such that the generating vector $\langle 1, t, k-(t+1)\rangle$ of the $\mathbb{Z}_{k}$-action is extendable), then $X$ belongs to one of several families of well-known ("classical") curves,

- the Accola-Maclachlan curves;
- the Kulkarni curves;
- the Wiman curves (again);
- two "nameless" families whose conformal automorphism groups are metacyclic $\simeq \mathbb{Z}_{2} \ltimes_{\alpha} \mathbb{Z}_{k}$ or $\simeq \mathbb{Z}_{3} \ltimes_{\beta} \mathbb{Z}_{k}$;
or $X$ is the Klein quartic, or one of two other exceptional curves of genus 4 or 10.


## Proof sketch

If $\operatorname{Aut}(\mathcal{M})>\operatorname{Aut}(X)$, the signature $\left(0 ; k, l_{1}, l_{2}\right)$ must be one of the extendable, cyclic-admissible signatures:

| Case | $\sigma$ | $\sigma^{\prime}$ | $\left[\Lambda\left(\sigma^{\prime}\right): \Lambda(\sigma)\right]$ | Conditions |
| ---: | :---: | :---: | :---: | :---: |
| N6 | $(0 ; k, k, k)$ | $(0 ; 3,3, k)$ | 3 | $k \geq 4$ |
| N8 | $(0 ; k, k, u)$ | $(0 ; 2, k, 2 u)$ | 2 | $u \mid k, k \geq 3$ |
| T1 | $(0 ; 7,7,7)$ | $(0 ; 2,3,7)$ | 24 | - |
| T4 | $(0 ; 8,8,4)$ | $(0 ; 2,3,8)$ | 12 | - |
| T8 | $(0 ; 4 k, 4 k, k)$ | $(0 ; 2,3,4 k)$ | 6 | $k \geq 2$ |
| T9 | $(0 ; 2 k, 2 k, k)$ | $(0 ; 2,4,2 k)$ | 4 | $k \geq 3$ |
| T10 | $(0 ; 3 k, k, 3)$ | $(0 ; 2,3,3 k)$ | 4 | $k \geq 3$ |

Table : Cyclic-admissible signatures ( $\sigma$ ) and possible extensions ( $\sigma^{\prime}$ )
(Singerman, 1972)

## Proof sketch, cont.

In addition, generating vectors must be appropriate for extension (Bujalance, Cirré, Conder, 1999). On the "classical" curves, most generating vectors are unique (up to a normal form), and in most cases, extendable:

| Curve | Group | Signature | Gen. Vector |
| :--- | :--- | :--- | :--- |
| Wiman I | $\mathbb{Z}_{4 g+2}$ | $(0 ; 4 g+2,2 g+1,2)$ | $\langle 1,2 g, 2 g+1\rangle$ |
| Wiman II | $\mathbb{Z}_{4 g}$ | $(0 ; 4 g, 4 g, 2)$ | $\langle 1,2 g-1,2 g\rangle$ |
| Acc-Maclac | $\mathbb{Z}_{2 g+2}$ | $(0 ; 2 g+2,2 g+2, g+1)$ | $\langle 1,1,2 g\rangle$ |
| Kulkarni | $\mathbb{Z}_{2 g+2}$ | $(0 ; 2 g+2,2 g+2, g+1)$ | $\langle 1, g+2, g-1\rangle$ |
| Wiman III | $\mathbb{Z}_{12}$ | $(0 ; 12,4,3)$ | $\langle 1,3,8\rangle$ |
| Klein | $\mathbb{Z}_{7}$ | $(0 ; 7,7,7)$ | $\langle 1,2,4\rangle$ |

Table : Maximal cyclic actions on the classical curves

## Theorem $(\mathrm{W})$ : If $\operatorname{Aut}(X)>\operatorname{Aut}(\mathcal{M})$, then:

(1) $k=12, t=3$, and $X$ is the Wiman type III curve;
(2) $k=2 g+1, t=1$, and $X$ is the Wiman type I curve;
(3) $k=2 g+2, t=1$, and $X$ is the Accola-Maclachlan curve;
(9) $k=2 g+1, t=\beta(1)$, and $\operatorname{Aut}(X) \simeq \mathbb{Z}_{3} \ltimes_{\beta} \mathbb{Z}_{k} ;$ except

- if $k=7, t=\beta(1)=2, X$ is the Klein quartic.
(3) $2 g+2 \leq k \leq 4 g, t=\alpha(1)$, and $\operatorname{Aut}(X) \simeq \mathbb{Z}_{2} \ltimes_{\alpha} \mathbb{Z}_{k}$; except
- if $k=2 g+2, g \equiv-1(\bmod 4), \alpha(1)=g+2, X$ is the Kulkarni curve; or
- if $k=12$ or $24, g=4$ or $10, \alpha(1)=7$ or $19(r e s p),. \operatorname{Aut}(X)$ contains $\mathbb{Z}_{2} \ltimes_{\alpha} \mathbb{Z}_{k}$ with index 3.

In case 5 , with $k=4 g$ and $\alpha(1)=2 g-1, \mathbb{Z}_{2} \ltimes_{\alpha} \mathbb{Z}_{k} \simeq \mathrm{SD}_{8 g}$ and $X$ is the Wiman curve of type II.

- Melekoğlu and Singerman (2008) characterized
- \{Wiman I, II, Acc-Maclac\}, as curves of genus $g>1$ admitting double-star maps (2-sheeted covers of a one-vertex map with ALL free edges on $\mathbb{P}^{1}$ );
- \{Wiman II, Acc-Maclac\}, as Platonic M- and ( $M-1$ )-surfaces (admitting an anticonformal involution with maximal, resp., second maximal number of fixed ovals).
- Singerman (2001) obtained the special case $I_{1}=l_{2}$ of our main Theorem.
- Kulkarni (1997) considered "large" cyclic actions with a fixed point in terms of side-pairings of hyperbolic polygons.

The paper on which this talk is based is
A. Weaver, Classical curves via one-vertex maps, Geometriae Dedicata (2013) 163, 141-158.

Also available at:
arXiv:1201.1646v3 [math.NT]

Thanks for your attention!

