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Stratifying the Space of Moduli

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Abstract. In the space of moduli of surfaces of genus g, the locus of surfaces with automorphisms is a complicated set which can be stratified according to topological conjugacy of the full automorphism groups of the surfaces. The strata are non-disjoint, and the intersections can contain surfaces admitting isomorphic subgroups of automorphisms which are topologically but not conformally equivalent. The paper is expository, with some explicit examples at the end.

1. Introduction

The space of moduli of compact Riemann surfaces of a given genus g has a large singular set corresponding to surfaces with automorphisms. A precise geometric understanding of the singular set requires a good deal of background material. The purpose of this expository paper is to present this material in a coherent and reasonably self-contained way. There are several possible approaches. Ours is essentially algebraic, involving Fuchsian groups whenever possible. A second purpose of the paper is to show how the distinction between topological and conformal conjugacy of group actions on surfaces is reflected in the branch locus in Teichmüller space. The main background references for the paper are [3,27,28]. Some general references for later sections are [6,31,16].

The contents of the paper are as follows. In Section 2. we define Fuchsian groups and their proper discontinuous action on the upper half plane. In Section 3. we define the Teichmüller space and modular group associated to a given Fuchsian group. In Section 4. we investigate the interaction between a quasiconformal map of the upper half plane and a given Fuchsian group, and prove, in Section 5., that every isomorphism between Fuchsian groups can be realized as conjugation by a quasiconformal map of the upper half plane. We sketch a proof of the existence and uniqueness of Teichmüller maps realizing such isomorphisms. This leads to the definition, in Section 6., of the Teichmüller metric on the Teichmüller space of a Fuchsian group. In Section 7. we show that the fixed point set of a finite subgroup of the modular group is itself a Teichmüller space, embedded in the ambient Teichmüller space on which the modular group acts. We state a version of the Nielsen-realization theorem, that every finite subgroup of the modular group has a non-empty fixed point set. In Section 8., we define the relative modular groups, which are set-wise stabilizers of fixed point sets of finite subgroups of the modular group, and the quotient groups of the relative modular groups which act effectively. Next we define the relative Riemann spaces parametrizing surfaces admitting a given symmetry group. In Section 9. we distinguish topological from conformal conjugacy of group actions on surfaces. This provides a precise way of describing the branch locus in Teichmüller space in terms of embedded Teichmüller spaces of smaller dimension. The branch locus corresponds to the singular set in the space of moduli. The stratification of this space is described in Section 10.. The intersections of the strata echo the corresponding relationships among the embedded spaces in Teichmüller space. We finish with some explicit examples in Section 11..

2. Uniformization and Fuchsian groups

According to the *uniformization theorem* (Klein, Poincaré, Koebe), a simply connected Riemann surface, up to conformal equivalence, is one of the following:

1. the complex plane;
2. the Riemann sphere;
3. the upper half plane $U = \{z \in \mathbb{C} | \mathrm{Im}(z) > 0\}$.

Each of these has a canonical complete metric of constant curvature. On U the metric is Poincarè's hyperbolic metric $|dz|/\mathrm{Im}(z)$, of constant curvature -1. A consequence of the uniformization theorem is that every Riemann surface can be represented as a quotient $\tilde{X}/\Gamma$, where $\tilde{X}$ is one of the three simply connected surfaces, and Γ is a discrete group of orientation-preserving isometries, acting *discontinuously* on $\tilde{X}$. This means that every point $x \in \tilde{X}$ is contained in an open set which does not meet any of its γ-translates, $\gamma \in \Gamma$, unless γ is the identity.

A slight weakening of the notion of discontinuity permits a covering theory of orbifolds (surfaces with cone points), analogous to the uniformization theorem for surfaces.

Definition 1. *A group of G of homeomorphisms of a topological space S acts* properly discontinuously *if*

1. *every $s \in S$ is contained in an open set $V \subset S$ such that, for $g \in G$, $gV \cap V \neq \emptyset$ implies $gs = s$;*
2. *the stabilizer of a point is finite.*

Definition 2. *A group G acts* effectively *on a set S if the (normal) subgroup $H = \{g \in G | gs = s \text{ for all } s \in S\}$ is trivial.*

If Γ acts effectively and properly discontinuously on $\tilde{X}$, the points in $\tilde{X}$ with non-trivial stabilizers form a discrete set $D \subset \tilde{X}$. The quotient map $\tilde{X} \to \tilde{X}/\Gamma$, restricted to $\tilde{X} - D$, is a covering which transfers the metric and conformal structures from $\tilde{X} - D$ to the quotient $(\tilde{X} - D)/\Gamma$, a surface punctured at a discrete set of points. The filled-in punctures become the cone points, and the metric and conformal structures extend in a canonical way to the resulting closed orbifold.

When $\tilde{X} = U$, Γ is a discrete subgroup of the real *Möbius group*

$$\mathcal{L} = \left\{ z \mapsto \frac{az+b}{cz+d}, \quad a, b, c, d \in \mathbb{R}, \quad ad - bc = 1 \right\} \simeq PSL(2, \mathbb{R}),$$

called a *Fuchsian group* (see [21], Chapter 5 or [22]). The boundary $\partial U = \mathbb{R} \cup \{\infty\}$ contains the *limit set* of accumulation points of Γ-orbits of points $z \in U$. A Fuchsian group is *elementary* if the limit set is finite. Here we consider only non-elementary Fuchsian groups *of the first kind*, for which the limit set is all of ∂U.

An element of $\mathcal{L}$ is *elliptic* if it has a single fixed point in U, *parabolic* if it has a single fixed point in ∂U, and *hyperbolic* if it has two fixed points in ∂U. Parabolic and hyperbolic elements have infinite order. An elliptic element can have infinite order, but, if it is contained in a Fuchsian group, by discreteness, its order must be finite. A hyperbolic element $t \in \mathcal{L}$ stabilizes a geodesic *axis* $l_t \subset U$. For $z \in l_t$ the hyperbolic distance between z and tz is an invariant of t called the *translation length*. The limits as $n \to \pm\infty$ of $t^n z$ are the endpoints of l_t on ∂U. They are called the *attracting* $(+)$ and *repelling* $(-)$ fixed points of t.

Lemma 1. *Non-identity elements in $\mathcal{L}$ commute if and only if they have the same fixed point set.*

Proof. [21], Theorem 5.2.4. One need only examine the centralizers of representatives of conjugacy classes. Parabolic elements are conjugate to $z \mapsto z \pm 1$, and the centralizer of these elements is $\{z \mapsto z + k | k \in \mathbb{R}\}$. All such elements fix $\{\infty\}$. Hyperbolic elements are conjugate to $z \mapsto \lambda z$, $\lambda > 0$, $\lambda \neq 1$, with centralizer $\{z \mapsto \mu z | \mu > 0\}$; the fixed point sets are $\{0, \infty\}$. After a standard conformal transformation from U to the interior of the unit disk, all elliptic elements are conjugate to rotations $w \mapsto e^{i\theta} w$, $\theta \in \mathbb{R}$, which are centralized by $\{w \mapsto e^{i\phi} w | 0 \leq \phi < 2\pi\}$. All such elements fix the origin. □

Corollary 2. *The centralizer in $\mathcal{L}$, and, in particular, the center, of a non-elementary Fuchsian group is trivial.*

Proof. The existence of a non-trivial element of $\mathcal{L}$ which commutes with every element of the group would imply that every element has the same (finite) fixed point set, making the group elementary. □

A finitely generated Fuchsian group has a fundamental polygon in $U \cup \partial U$ with finitely many sides [18]. If none of the sides is contained in ∂U, the polygon has finite hyperbolic area. If none of the vertices is contained in ∂U, Γ contains no parabolic elements, and the quotient U/Γ is compact. A Fuchsian group Γ of this type is called *co-compact*, and it has the following canonical presentation:

$$\Gamma = \left\langle a_1, b_1, \ldots, a_g, b_g, e_1, \ldots, e_n \,\middle|\, e_1^{\nu_1} = e_2^{\nu_2} = \cdots = e_n^{\nu_n} = 1, \prod_{i=1}^{g}[a_i, b_i] \prod_{j=1}^{n} e_j = 1 \right\rangle, \quad (1)$$

where the generators $\{e_i\}$ are elliptic and the other generators are hyperbolic. The $(n+1)$-tuple

$$(g; \nu_1, \nu_2, \ldots, \nu_n) \quad (2)$$

is called the *signature* of Γ, and it determines Γ uniquely if we assume, as we shall, that $\nu_1 \leq \nu_2 \leq \cdots \leq \nu_n$. The ν_i are called the *periods* of Γ, and they form a complete set of maximal orders of elliptic elements in Γ, one for each conjugacy class. A signature of the form (2) is said to have *type* (g, n).

The presentation (1) defines a Fuchsian group if and only if the number

$$\chi(\Gamma) = \chi(g; \nu_1, \nu_2, \ldots, \nu_n) = 2\pi \left\{ 2g - 2 + \sum_{j=1}^{n} \left(1 - \frac{1}{\nu_j}\right) \right\}, \tag{3}$$

is positive. $\chi(\Gamma)$ is the hyperbolic area of a fundamental polygon for Γ acting on U. If $\chi(\Gamma) > 0$, then $\chi(\Gamma) \geq \pi/21$, with equality if and only if Γ is the group with signature (0; 2, 3, 7).

Lemma 3 (The Riemann-Hurwitz relation). *If* $\Gamma \leq \Gamma^*$ *are co-compact Fuchsian groups, the ratio* $\chi(\Gamma)/\chi(\Gamma^*)$ *is finite and equal to the index* $[\Gamma^* : \Gamma]$.

Proof. See, e.g., [18], or [11], Theorem 1.2.7. □

For co-compact Γ, the compact quotient U/Γ is an *orbifold* of type (g, n), that is, a compact Riemann surface of genus g with n distinguished points, over which the quotient map $U \to U/\Gamma$, elsewhere a covering map, is ramified. All orbifolds with a finite number of distinguished points, except those of type $(g, n) = (0, 0)$, $(0, 1)$, $(0, 2)$, and $(1, 0)$, arise as quotients of co-compact Fuchsian groups acting on U. If Γ has presentation (1), the order of the isotropy subgroup of a point lying over the ith distinguished point is ν_i.

The *normalizer* of a subgroup G in a group H is the subgroup $N_H(G) = \{h \in H | hGh^{-1} = G\}$.

Lemma 4. *The normalizer* $N_{\mathcal{L}}(\Gamma)$ *of a co-compact Fuchsian group* Γ *is a co-compact Fuchsian group, and the index* $[N_{\mathcal{L}}(\Gamma) : \Gamma]$ *is finite.*

Proof. If $N_{\mathcal{L}}(\Gamma)$ is not discrete it contains an infinite sequence of distinct elements $\{\delta_i\}$ tending to the identity. For all $\gamma \in \Gamma$, there exists a positive integer m such that $\delta_i \gamma \delta_i^{-1} = \gamma$, $i \geq m$, otherwise $\{\delta_i \gamma \delta_i^{-1}\}$ would be an infinite sequence of distinct elements of Γ tending to γ, contradicting the discreteness of Γ. Thus, by Lemma 1, δ_i, $i \geq m$, has the same fixed point set as γ. Since Γ is nonabelian there exists $\gamma' \in \Gamma$ which does not commute with γ. Repeating the previous argument, there exists m' such that δ_i, $i \geq m'$, has the same fixed point set as γ'. But then γ, γ' have the same fixed point set and hence commute. This contradiction proves that $N_{\mathcal{L}}(\Gamma)$ is discrete. By the Riemann-Hurwitz relation, $\pi/21 \leq \chi(N_{\mathcal{L}}(\Gamma)) \leq \chi(\Gamma) < \infty$ whence the index $[N_{\mathcal{L}}(\Gamma) : \Gamma] < \infty$. Since a fundamental domain for $N_{\mathcal{L}}(\Gamma)$ is contained in a fundamental domain for Γ, which has no sides or vertices in ∂U, $N_{\mathcal{L}}(\Gamma)$ is co-compact. □

A co-compact Fuchsian group without elliptic elements (i.e., torsion-free) is called a *surface group*. If the signature is of type $(g, 0)$, the group is isomorphic to the fundamental group of a compact surface of genus g. We shall use Λ (Λ_g) to denote a surface group (of genus g). By the uniformization theorem, every compact surface of genus g is conformally equivalent to a quotient surface U/Λ', where Λ' is the image of an injective homomorphism $r : \Lambda_g \to \mathcal{L}$.

Theorem 5. *Surface groups Λ and Λ' of genus $g > 1$ are conjugate in $\mathcal{L}$ if and only if the surfaces U/Λ and U/Λ' are conformally equivalent.*

Proof. If $t^{-1}\Lambda t = \Lambda'$ for some $t \in \mathcal{L}$, then t induces the conformal map $\Lambda z \mapsto \Lambda' t^{-1} z$ between U/Λ and U/Λ', where Λz denotes the Λ-orbit of $z \in U$. Conversely, a conformal map $c : U/\Lambda \to U/\Lambda'$ lifts to a conformal map $\tilde{c}$ of the universal covering space U, i.e., an element $\tilde{c} \in \mathcal{L}$. Since $\tilde{c}(\Lambda z) = \Lambda'(\tilde{c} z)$, it follows that $\tilde{c}\Lambda_g \tilde{c}^{-1} = \Lambda'_g$. □

A subgroup G of a group H is called *characteristic* if all automorphisms of H preserve G. In particular, G is normal in H.

Theorem 6 (Bundgaard, Nielsen, Fox). *A co-compact Fuchsian group contains a surface group as a characteristic subgroup of finite index.*

Proof. [8,12]. The proof is based on two lemmas: (i) there exists a normal subgroup of finite index which contains no non-trivial power of any given elliptic element; (ii) the intersection of a finite number of subgroups of finite index is a subgroup of finite index. Since there are only finitely many conjugacy classes of elliptic elements, the theorem follows. □

3. Teichmüller spaces of Fuchsian groups

Henceforth Γ denotes a co-compact Fuchsian group with presentation (1).

Let $R(\Gamma)$ be the set of all injective homomorphisms $r : \Gamma \to \mathcal{L}$ such that the image $r(\Gamma)$ is Fuchsian. $R(\Gamma)$ is topologized as a subspace of the product of $2g + n$ copies of $\mathcal{L}$, by assigning to $r \in R(\Gamma)$ the point

$$(r(a_1), r(b_1), \dots, r(a_g), r(b_g), r(e_1), \dots r(e_n)) \in \mathcal{L}^{2g+n}.$$

The identity $\mathrm{id}_\Gamma : \Gamma \hookrightarrow \mathcal{L}$ may be taken as a base point in $R(\Gamma)$. $\mathrm{Aut}(\mathcal{L})$ acts on $R(\Gamma)$ by post-composition; since $\mathrm{Aut}(\mathcal{L}) \simeq \mathcal{L} \simeq \mathrm{Inn}(\mathcal{L})$, the action is by conjugation. $r_1, r_2 \in R(\Gamma)$ are called *equivalent* if their images are conjugate within $\mathcal{L}$. Equivalence classes $[r]$ are the points in $T(\Gamma)$, the *Teichmüller space* of Γ, which takes the quotient topology from $R(\Gamma)$.

Theorem 7 (Greenberg [17]). *Let Γ and Γ^* be co-compact Fuchsian groups. An injective homomorphism $i : \Gamma \to \Gamma^*$ induces a homeomorphism $\bar{i} : T(\Gamma^*) \to T(\Gamma)$ onto a closed subspace, defined by*

$$\bar{i} : [r] \mapsto [r \circ i]. \tag{4}$$

Proof. We sketch the proof given in [27], Theorem 7.11. Let $\delta_1, \delta_2, \dots, \delta_h$ be a canonical finite set of generators of Γ, as in (1). Let $\gamma_1, \gamma_2, \dots, \gamma_k \in \Gamma^*$ be a complete set of coset representatives of $i(\Gamma)$ in Γ^*. $k < \infty$ since, by the Riemann-Hurwitz relation, the inclusion $i : \Gamma \to \Gamma^*$ has finite index. It can be arranged that all the γ_i are hyperbolic elements ([27], Lemma 4.8). Γ^* is generated by the set

$$\{i(\delta_1), \ldots, i(\delta_h), \gamma_1, \ldots, \gamma_k\},$$

and $r \in R(\Gamma^*)$ corresponds to the point

$$(r \circ i(\delta_1), \ldots, r \circ i(\delta_h), r(\gamma_1), \ldots, r(\gamma_k)) \in \mathcal{L}^{h+k}.$$

For each $j = 1, \ldots, k$, there exists a smallest $n_j < \infty$ such that $\gamma_j^{n_j} \in i(\Gamma)$. Let $\gamma_j^{n_j} = i(e_j)$, $e_j \in \Gamma$. Then $r(\gamma_j)$ is the unique n_jth root of $r \circ i(e_j)$. Call this element $l_j \in \mathcal{L}$. Uniqueness of l_j follows from uniqueness of roots in infinite cyclic groups: $r \circ i(e_j)$ belongs to the infinite cyclic group generated by the hyperbolic element $r(\gamma_j)$. Then $r \circ i \in R(\Gamma)$ corresponds to the point

$$(r \circ i(\delta_1), \ldots, r \circ i(\delta_h), l_1, \ldots, l_k) \in \mathcal{L}^{h+k},$$

and r is uniquely and continuously recoverable from $r \circ i$. Hence i induces an embedding of $R(\Gamma^*)$ into $R(\Gamma)$, whose image is a closed subset. As r varies in its Teichmüller class, so does $r \circ i$ (the images are both conjugated by an element of $\mathcal{L}$). Thus the embedding descends to an embedding of the quotient Teichmüller spaces, with respect to the quotient topology. □

Let $\mathrm{Aut}^+(\Gamma)$ be the group of automorphisms of Γ which are both *type-* and *orientation-preserving*. Type-preserving automorphisms carry elliptic elements to elliptic elements, parabolic elements to parabolic elements, etc. (For co-compact Fuchsian groups, which contain only finite-order elliptic elements and infinite-order hyperbolic elements, all automorphisms are type-preserving.) Orientation-preserving automorphisms are induced by automorphisms of the free group on the generators of (1) which carry the final relator in (1) to a conjugate of itself but not of its inverse. Inner automorphisms are type- and orientation-preserving. Let $\mathrm{Inn}(\Gamma) \subseteq \mathrm{Aut}^+(\Gamma)$ be the normal subgroup of inner automorphisms. $\alpha \in \mathrm{Aut}^+(\Gamma)$ induces a homeomorphism of $T(\Gamma)$ defined by

$$[r] \mapsto [r \circ \alpha] \tag{5}$$

If $\alpha \in \mathrm{Inn}(\Gamma)$, there exists $\delta \in \Gamma$ such that $\alpha(\gamma) = \delta^{-1}\gamma\delta$. Then $r(\alpha(\Gamma)) = c^{-1}r(\Gamma)c$, where $c = r(\delta) \in \mathcal{L}$, so that $[r \circ \alpha] = [r]$. Thus the *effective* action is by the *Teichmüller modular group* of Γ

$$\mathrm{Mod}(\Gamma) = \{[\alpha]\} \simeq \frac{\mathrm{Aut}^+(\Gamma)}{\mathrm{Inn}(\Gamma)}.$$

$[\alpha]$ denotes the class in $\mathrm{Aut}^+(\Gamma)/\mathrm{Inn}(\Gamma)$ induced by α.

Theorem 8. *The action Mod*$(\Gamma) \times T(\Gamma) \to T(\Gamma)$ *defined by*

$$([\alpha], [r]) \mapsto [r \circ \alpha]$$

is properly discontinuous.

Proof. This was proved by Kravetz [24] in the case where Γ is surface group. We follow the proof in [28], Theorem 7, which assumes Kravetz′ result.

By Theorem 6, Γ contains a surface group Λ as a characteristic subgroup of finite index. The inclusion $i : \Lambda \hookrightarrow \Gamma$ induces the embedding $\bar{i} : T(\Gamma) \hookrightarrow T(\Lambda)$ defined by (4). Let $[r] \in T(\Gamma)$, and suppose $[r \circ i] \in V \subset T(\Lambda)$, where V is an open set satisfying property 1 of Definition 1 for the action of $\mathrm{Mod}(\Lambda)$ on $T(\Lambda)$ (Kravetz′ result is assumed here). Let $W = V \cap \bar{i}(T(\Gamma)) = \bar{i}^{-1}(V)$. W is an open set in $T(\Gamma)$ which is non-empty, since it contains $[r]$. Since Λ is characteristic in Γ, for every $\beta \in \mathrm{Aut}^+(\Gamma)$ there exists a unique $\alpha \in \mathrm{Aut}^+(\Lambda)$ such that $\beta \circ i = i \circ \alpha$. If $\beta \in \mathrm{Aut}^+(\Gamma)$ is such that $[\beta](W) \cap W \neq \emptyset$, then $[\alpha](V) \cap V \neq \emptyset$ and by property 1 for $\mathrm{Mod}(\Lambda)$ on $T(\Lambda)$, $[\alpha]$ fixes $[r \circ i]$, that is, $[r \circ i \circ \alpha] = [r \circ i]$. Equivalently, $[r \circ \beta \circ i] = [r \circ i]$ which implies $[r \circ \beta] = [r]$. Thus $W \subset T(\Gamma)$ is an open set containing $[r]$ which satisfies property 1 for $\mathrm{Mod}(\Gamma)$ acting on $T(\Gamma)$.

Property 2 of Definition 1 for $\mathrm{Mod}(\Gamma)$ acting on $T(\Gamma)$ follows from the next theorem and Lemma 4. □

Theorem 9. *The stabilizer of a point* $[r] \in T(\Gamma)$ *is isomorphic to the (finite) subgroup* $N_{\mathcal{L}}(r(\Gamma))/r(\Gamma)$ *of* $\mathrm{Mod}(\Gamma)$.

Proof. If $[\alpha] \in \mathrm{Mod}(\Gamma)$ fixes $[r]$, then $[r \circ \alpha] = [r]$ and there exists $t \in \mathcal{L}$ such that, for all $\gamma \in \Gamma$, $r \circ \alpha(\gamma) = tr(\gamma)t^{-1}$. It follows that $t \in N_{\mathcal{L}}(r(\Gamma))$. If $t \in r(\Gamma)$, $\alpha \in \mathrm{Inn}(\Gamma)$ and hence $[\alpha]$ is the identity in $\mathrm{Mod}(\Gamma)$. Thus the stabilizer of $[r]$ is isomorphic to a subgroup of $N_{\mathcal{L}}(r(\Gamma))/r(\Gamma)$. On the other hand, if $t \in N_{\mathcal{L}}(r(\Gamma))$, the map $\beta_t : r(\gamma) \mapsto tr(\gamma)t^{-1}$ is a type- and orientation-preserving automorphism of $r(\Gamma)$, whence $\alpha_t = r^{-1} \circ \beta_t \circ r$ is a type- and orientation-preserving automorphism of Γ. α_t is inner if and only if $t \in r(\Gamma)$. This establishes the isomorphism. □

4. Quasiconformal maps; compatibility

A homeomorphism $w : U \to U$ is called *quasiconformal*, or more precisely, *k-quasiconformal*, if there exists $k < 1$ such that $|w_{\bar{z}}/w_z| \leq k$ almost everywhere. Partial derivatives w_z and $w_{\bar{z}}$ exist in the sense of distribution theory ([3], §2), and obey the usual chain rules

$$(u \circ v)_z = (u_v \circ v)v_z + (u_{\bar{v}} \circ v)(\bar{v})_z; \tag{6}$$

$$(u \circ v)_{\bar{z}} = (u_v \circ v)v_{\bar{z}} + (u_{\bar{v}} \circ v)(\bar{v})_{\bar{z}}. \tag{7}$$

The expression $w_{\bar{z}}/w_z$ is called a *Beltrami differential.* A complex measurable function $\mu(z)$ defined on U determines a *Beltrami equation*

$$w_{\bar{z}} = \mu(z)w_z, \tag{8}$$

in which $\mu(z)$ is the *Beltrami coefficient.* If $|\mu(z)| < 1$, the Jacobian $(1 - \mu\bar{\mu})|w_z|^2$ of w is positive, and hence w is orientation-preserving.

Lemma 10.

(i) *If* $\mu(z)$ *is a Beltrami coefficient with* $|\mu(z)| \le k < 1$ *almost everywhere, and* w *is a solution of* (8)*, then* w *is* k*-quasiconformal; conversely, every* k*-quasiconformal self-homeomorphism of* U *is the solution of* (8) *for some Beltrami coefficient* $\mu(z)$, $|\mu(z)| \le k < 1$ *almost everywhere.*

(ii) *If* w *is a solution of* (8)*, and* $c \in \mathcal{L}$*, then* $w_1 = c \circ w$ *is also a solution; conversely, if* w*,* w_1 *are both solutions of* (8) *then there exists* $c \in \mathcal{L}$ *such that* $w_1 = c \circ w$.

Proof. [3], §4 E, §4 B. □

Lemma 11. *A quasiconformal self-homeomorphism of* U *has a unique extension to a self- homeomorphism of* $\bar{U} = U \cup \partial U$.

Proof. [3], §3 E. □

There is a definition of quasiconformality due to Ahlfors and Pfluger [1] which makes no reference to differentiability (distributional or otherwise): a conformal image of $\{(x, y) \in \mathbb{R}^2 | 0 \le x \le m, 0 \le y \le 1\}$ is called a *topological rectangle* R with *modulus* $m = m(R)$. By the Riemann mapping theorem, a continuous image of a topological rectangle is a topological rectangle (with a possibly different modulus). A homeomorphism $w : U \to U$ is k-quasiconformal if, for every topological rectangle R,

$$\frac{m(w(R))}{m(R)} \le \frac{(1+k)}{(1-k)}. \tag{9}$$

Lemma 12. *The composition of a* k_1*-quasiconformal map with a* k_2*-quasiconformal map is* k*-quasiconformal, where*

$$\frac{1+k}{1-k} = \frac{1+k_1}{1-k_1} \cdot \frac{1+k_2}{1-k_2}.$$

Proof. An immediate consequence of (9). □

For quasiconformal w, let $k(w) < 1$ be the smallest k such that w is k-quasiconformal. The *maximal dilatation* of w is

$$K(w) = \frac{1+k(w)}{1-k(w)}.$$

We have the following consequences of the preceding definitions and lemmas. Let w_1, w_2, $w : U \to U$ be quasiconformal maps, and let $c : U \to U$ be conformal. Then

$$K(c) = 1; \tag{10}$$

$$K(w_1 \circ w_2) \le K(w_1)K(w_2); \tag{11}$$

$$K(w) = K(w^{-1}); \tag{12}$$

$$K(w \circ c) = K(c \circ w) = K(w). \tag{13}$$

Let Γ be a Fuchsian group. A Beltrami coefficient $\mu(z)$ is Γ*-compatible* if it transforms according to the rule

$$\mu(\gamma(z))) = \mu(z)\gamma'(z)/\overline{\gamma'(z)} \quad \text{for all } \gamma \in \Gamma. \tag{14}$$

A holomorphic function $\phi(z)$ defined on U is called a Γ-2-form (automorphic form of weight 2 for Γ) if it transforms according to the rule

$$\phi(\gamma(z))) = \phi(z)/\gamma'(z)^2 \quad \text{for all } \gamma \in \Gamma. \tag{15}$$

A Γ-2-form induces an integrable *quadratic differential* on the quotient surface U/Γ, holomorphic except possibly for simple poles or removable singularities at the orbifold points. All holomorphic quadratic differentials on U/Γ arise in this way.

Lemma 13.

(i) *If ϕ is a Γ-2-form, then*

$$\mu(z) = k\overline{\phi(z)}/|\phi(z)|, \quad 0 \leq k < 1, \tag{16}$$

is a Γ-compatible Beltrami coefficient. Partially conversely,

(ii) *If $\phi(z)$ is a holomorphic function such that* (16) *is a Γ-compatible Beltrami coefficient, then, up to multiplication by a real character $\chi : \Gamma \to (\mathbb{R}^+, \cdot)$, ϕ is a Γ-2-form.*

Proof. [27], Lemma 5.18. (i) is a consequence of the chain rules. (ii) A *real character* is a homomorphism of a group into the multiplicative group of positive reals. If $\phi(z)$ has modulus $r(z) > 0$ and argument $\theta(z) \in \mathbb{R}$, then $\mu(z) = k\exp(-i\theta(z))$, and, since μ is Γ-compatible, for all $\gamma \in \Gamma$,

$$-\theta(\gamma z) - \arg\gamma'(z) = -\theta(z) + \arg\gamma'(z).$$

This implies that the number $\chi(\gamma) = \phi(\gamma(z))\gamma'(z)^2/\phi(z)$ is real $(\arg(\chi(\gamma) = 0)$. In addition $\chi(\gamma) = r(\gamma(z))|\gamma'(z)|^2/r(z) > 0$. The homomorphism property $\chi(\gamma_1\gamma_2) = \chi(\gamma_1)\chi(\gamma_2)$ is easily verified. □

Theorem 14. *Let Γ_1 be a subgroup of finite index in Γ. Let ϕ be a Γ_1-2-form. Then* (16) *is Γ-compatible if and only if ϕ is a Γ-2-form.*

Proof. [27], Theorem 5.20. By Lemma 13, (16) is Γ_1-compatible; if it is also Γ-compatible, then there is a real character $\chi : \Gamma \to (\mathbb{R}^+, \cdot)$ such that

$$\phi(\gamma(z)) = \chi(\gamma) \cdot \phi(z)/\gamma'(z)^2 \quad \text{for all } \gamma \in \Gamma.$$

For some finite t, $\gamma^t \in \Gamma_1$ since the index $[\Gamma : \Gamma_1]$ is finite. It follows that $\chi(\gamma^t) = (\chi(\gamma))^t = 1$. This implies $\chi(\gamma) = 1$, since $(\mathbb{R}^+, \cdot)$ has no non-trivial element of finite order. Hence ϕ is a Γ-2-form. Conversely, if ϕ is a Γ-2-form, then (16) is Γ-compatible by Lemma 13. □

5. Geometrical realizations

$r \in R(\Gamma)$ has a *geometrical realization* if there is an orientation-preserving homeomorphism $w : U \to U$ such that, for all $\gamma \in \Gamma$, $r(\gamma) = w\gamma w^{-1}$. It is a remarkable fact, not at all obvious, that *every* $r \in R(\Gamma)$ has a quasiconformal geometrical realization.

Let $\Omega(r, \Gamma)$, $r \in R(\Gamma)$, denote the set of quasiconformal geometrical realizations of r.

Lemma 15. *If $w \in \Omega(r, \Gamma)$, then the Beltrami differential $w_{\bar{z}}/w_z$ is Γ-compatible. Conversely, if $\mu(z)$ is a Γ-compatible Beltrami coefficient, and w is a solution of* (8), *there exists $r \in R(\Gamma)$ such that $w \in \Omega(r, \Gamma)$.*

Proof. [27], Theorem 5.14. Let $w \in \Omega(r, \Gamma)$. Then $w\Gamma w^{-1} = r(\Gamma)$ and for $\gamma \in \Gamma$, $z \in U$, $w(\gamma z) = r(\gamma)w(z)$. A computation using the chain rules (6), and the fact that γ and $r(\gamma)$ are conformal, shows that $w_{\bar{z}}/w_z$ satisfies (14). Conversely, if $\mu(z)$ is a Γ-compatible Beltrami differential, and w is a solution of (8), then $w \circ \gamma$ is also a solution and hence by statement (ii) of Lemma 10, there exists $c \in \mathcal{L}$ such that $w \circ \gamma = c \circ w$, or equivalently, $w\gamma w^{-1} = c \in \mathcal{L}$. Thus one can define $r : \Gamma \to \mathcal{L}$ by $r(\gamma) = w\gamma w^{-1}$ for all $\gamma \in \Gamma$. Clearly, $w \in \Omega(r, \Gamma)$. □

The next theorem is a consequence of a well-known result of Nielsen [29]: every automorphism of the fundamental group of compact surface ($= \Lambda_g$) is induced by a self-homemorphism of the surface. The homeomorphism can be taken to be piecewise linear [37]; as such it lifts to a quasiconformal homeomorphism of U. One can therefore restate Nielsen's result as follows: every automorphism $\alpha \in \mathrm{Aut}^+(\Lambda_g)$ has a quasiconformal geometrical realization.

Let $\Lambda = \Lambda_g$.

Theorem 16. *$\Omega(r, \Lambda)$ is non-empty for all $r \in R(\Lambda)$.*

Proof. Let $r \in R(\Lambda)$. There exists a piecewise linear homeomorphism $h : U/\Lambda \to U/r(\Lambda)$ which lifts to a quasiconformal homeomorphism $\tilde{h} : U \to U$ satisfying $h\{\Lambda z\} = r(\Lambda)\tilde{h}(z)$, $z \in U$. $\tilde{h}$ induces the isomorphism $r_0 : \Lambda \to r(\Lambda)$ defined by $\lambda \mapsto \tilde{h}\lambda\tilde{h}^{-1}$, $\lambda \in \Lambda$. $\alpha = r^{-1} \circ r_0 \in \mathrm{Aut}^+(\Lambda)$, so, by Nielsen's result, there is a quasiconformal map w realizing α. Hence $r = r_0 \circ \alpha^{-1}$ is geometrically realized by the quasiconformal map $\tilde{h} \circ w^{-1}$. □

Definition 3. *$w : U \to U$ is a* Teichmüller mapping *for Γ if it is either conformal or quasiconformal with Beltrami differential*

$$\frac{w_{\bar{z}}}{w_z} = k\frac{\overline{\phi(z)}}{|\phi(z)|}, \quad 0 < k < 1,$$

where ϕ is a Γ-2-form.

Theorem 17 (Uniqueness of Teichmüller mappings). *Let $w_0 \in \Omega(r, \Lambda)$, $r \in R(\Lambda)$, be a Teichmüller mapping for Λ. If $w \in \Omega(r, \Lambda)$, $w \neq w_0$, then $K(w_0) < K(w)$.*

Proof. [3], §11, 12. Teichmüller's original proof is in [33]. □

Let the surface group $\Lambda = \Lambda_g$ be generated by hyperbolic elements

$$\{a_1, b_1, \ldots, a_g, b_g\}, \tag{17}$$

subject to the relation $a_1b_1a_1^{-1}b_1^{-1} \ldots a_gb_ga_g^{-1}b_g^{-1} = 1$. $r \in R(\Lambda)$ is (a_1, b_1)-*normalized* if the repelling and attracting fixed points of $r(a_1)$ are $0, \infty$, respectively, and the repelling fixed point of $r(b_1)$ is 1. Let $Q(a_1, b_1, \Lambda) \subset R(\Lambda)$ be the subspace of (a_1, b_1)-normalized representations.

Let $\Omega(\Lambda)$ be the subspace of quasiconformal maps $w : \bar{U} \to \bar{U}$, topologized by uniform convergence, which realize some $r \in R(\Lambda)$ geometrically (i.e., such that $w\Lambda w^{-1}$ is Fuchsian) and which, in addition, fix $0, 1, \infty \in \partial U$. Let $B(\Lambda)$ be the space of Λ-compatible Beltrami coefficients $\mu(z)$ with $0 \leq |\mu(z)| < 1$, topologized by pointwise convergence at almost every $z \in U$.

Theorem 18. *$T(\Lambda)$ is homeomorphic to the open unit ball in $\mathbb{R}^{6g-6}$.*

Proof. [3], §14; [27], §6. We sketch the proof, omitting the continuity arguments. The starting point is the existence of a one-to-one correspondence between the space of Λ-2-forms and the vector space V of quadratic differentials on the quotient surface U/Λ, which has real dimension $6g - 6$, by the Riemann-Roch theorem ([11], §III.4). An inner product $(,)$ is defined on V by choosing a basis and taking it to be an orthonormal set. The continuous one-to-one map

$$\theta : \phi \mapsto \mu_\phi = (\phi, \phi)\frac{\bar{\phi}}{|\phi|}, \quad \phi \in V \tag{18}$$

has image $\theta(V) \subseteq B(\Lambda)$ by Lemma 13 and the fact that $(\phi, \phi) < 1$. By Lemmas 10 and 11, there is a (ϕ, ϕ)-quasiconformal solution to the Beltrami equation $w_{\bar{z}} = \mu_\phi(z) w_z$ whose continuous extension to ∂U fixes $0, 1, \infty$. Let $w_{[\phi]}$ be the solution; it is a Teichmüller mapping, hence, unique. The map

$$\sigma : \mu_\phi \mapsto w_{[\phi]} \tag{19}$$

from $\theta(V)$ to $\Omega(\Lambda)$ is therefore one-to-one. By the converse of Lemma 15, there exists $r_\phi \in R(\Lambda)$ defined by $r_\phi(\gamma) = w_{[\phi]}^{-1} \gamma\, w_{[\phi]}$, for all $\gamma \in \Lambda_g$. $w_{[\phi]}^{-1} 0 w_{[\phi]} = 0$ and $w_{[\phi]}^{-1} \infty w_{[\phi]} = \infty$; similarly, $w_{[\phi]}^{-1} 1 w_{[\phi]} = 1$. These are, respectively, the attracting and repelling fixed points of $r(a_1)$, and the repelling fixed point of $r(b_1)$. Thus $r_\phi \in Q(a_1, b_1, \Lambda)$. r_ϕ is unique since, by Lemma 10 (ii), another such solution would differ by a real Möbius transformation fixing $0, 1, \infty$, which must be the identity ([21], Corollary 2.5.3). A continuity argument ([27], 6.17) shows that the one-to-one map

$$\tau : w_{[\phi]} \mapsto r_\phi \tag{20}$$

is also onto $Q(a_1, b_1, \Lambda)$. This proves that $Q(a_1, b_1, \Lambda)$ is homeomorphic to the open unit ball in $\mathbb{R}^{6g-6}$. Finally, for every $r \in R(\Lambda)$ there exists a unique $c \in \mathcal{L}$ such that $c^{-1} r(\Lambda) c \in Q(a_1, b_1, \Lambda)$. Thus for every Teichmüller class $[r] \in T(\Gamma)$ there is a unique $r_0 \in Q(a_1, b_1, \Lambda)$ such that $[r] = [r_0]$. It follows that $T(\Lambda)$ is homeomorphic to $Q(a_1, b_1, \Lambda)$. □

Corollary 19. *$\Omega(r, \Lambda)$ contains a (unique) Teichmüller mapping.*

Proof. If $r \in Q(a_1, b_1, \Lambda)$, this follows immediately from the proof of Theorem 18. For the general $r \in R(\Lambda)$, there exists $c \in \mathcal{L}$ which maps the attracting and repelling fixed points of a_1, and the repelling fixed point of b, onto $0, \infty, 1$, respectively. Then the map $r' : \Lambda \to \mathcal{L}$ defined by $\lambda \mapsto c \circ r(\lambda) \circ c^{-1}$ belongs to $Q(a_1, b_1, \Lambda)$ and hence there is a unique Teichmüller mapping $t \in \Omega(r', \Lambda)$. It is not difficult to verify that $c^{-1} \circ t$ is a Teichmüller mapping in $\Omega(r, \Lambda)$, hence, the unique one. □

We now pass to the general co-compact Fuchsian group Γ.

Theorem 20. $\Omega(r, \Gamma)$ *is non-empty for all* $r \in R(\Gamma)$.

Proof. Let $r \in R(\Gamma)$. By Theorem 6, Γ contains a surface group Λ as a normal subgroup of finite index. By Theorem 16, the restriction $r|_\Lambda$ is realized by a unique Teichmuller mapping $t : U \to U$. Choose an arbitrary element $\gamma \in \Gamma$ and use it to define a map

$$t' = r(\gamma)t\gamma^{-1}, \quad t' : U \to U. \tag{21}$$

By (13), t' has the same maximal dilatation as t. For $\lambda \in \Lambda$,

$$\begin{aligned} t'\lambda t'^{-1} &= r(\gamma)t(\gamma^{-1}\lambda\gamma)t^{-1}r(\gamma^{-1}) \\ &= r(\gamma)r(\gamma^{-1}\lambda\gamma)r(\gamma^{-1}) \\ &= r(\lambda). \end{aligned}$$

Thus t' realizes $r|_\Lambda$ geometrically. Since it is also Teichmüller map for Λ, by uniqueness, $t' = t$. Then (21) shows that $t\gamma t^{-1} = r(\gamma)$. Since $\gamma \in \Gamma$ was arbitrary, $t \in \Omega(r, \Gamma)$. □

Theorem 21. *If* Γ_1 *is a subgroup of finite index in* Γ*, and* t *is a Teichmüller mapping for* Γ_1*, then* t *is also a Teichmüller mapping for* Γ.

Proof. Using Theorem 6 it is easily shown that there is a surface group Λ which is normal and of finite index in both Γ_1 and Γ. Let $r \in R(\Gamma)$, and let t be the unique Teichmüller mapping in $\Omega(r|_\Lambda, \Lambda)$. The proof of Theorem 20 shows that $t \in \Omega(r|_{\Gamma_1}, \Gamma_1) \cap \Omega(r, \Gamma)$. Thus t is both Γ_1- and Γ-compatible. By Theorem 14, t is a Teichmüller mapping for both Γ_1 and Γ. □

Corollary 22. $\Omega(r, \Gamma)$ *contains a (unique) Teichmüller mapping.*

Let Γ have signature (2).

Theorem 23. $T(\Gamma)$ *is homeomorphic to the open unit ball in* $\mathbb{R}^{6g-6+2n}$.

Proof. [27], Theorem 7.13. Let Λ be a surface group contained with finite index in Γ and generated by hyperbolic elements $a_1, b_1, \ldots a_h, b_h$. Define $Q(a_1, b_1, \Gamma)$ as the set of (a_1, b_1)-normalized representations of Γ. If $i : \Lambda \to \Gamma$ is the inclusion homomorphism, then $r \mapsto r \circ i$, $r \in Q(a_1, b_1, \Gamma)$, embeds $Q(a_1, b_1, \Gamma)$ as a closed subspace of $Q(a_1, b_1, \Lambda)$, by Theorem 7. Let $\psi = \tau\sigma\theta$ be the composition of the homeomorphisms (18), (19), (20). The image of the restriction of ψ^{-1} to $Q(a_1, b_1, \Gamma)$ is the open unit ball in a linear subspace, namely, the subspace of Λ-2-forms which are also Γ-2-forms. This subspace is linearly equivalent to the vector space of quadratic differentials on U/Γ having possible simple poles or removable singularities at the n orbifold points; by the Riemann-Roch theorem, the real dimension of this space is $6g - 6 + 2n$. □

6. The Teichmüller metric

If $r_1, r_2 \in R(\Gamma)$, then $r_2 \circ r_1^{-1} \in R(r_1(\Gamma))$. Let $t(r_2 \circ r_1^{-1})$ be the unique Teichmüller mapping in $\Omega(r_2 \circ r_1^{-1}, \Gamma)$.

Lemma 24. *The function*

$$d(r_1, r_2) = \log |K(t(r_2 \circ r_1^{-1}))| \tag{22}$$

defines a pseudo-metric on $R(\Gamma)$ *which induces a true metric on* $T(\Gamma)$.

Proof. $d(r_1, r_2) \geq 0$ since $K(t(r_2 \circ r_1^{-1})) \geq 1$. $d(r_1, r_2) = d(r_2, r_1)$ by (13). The triangle inequality follows from (12). $d(r_1, r_2) = 0$ if and only if $t(r_2 \circ r_1^{-1}) = c \in \mathcal{L}$, or, equivalently, for all $\gamma \in \Gamma$, $r_1(\gamma) = c^{-1} r_2(\gamma) c$. Thus $r_1(\Gamma)$ and $r_2(\Gamma)$ are conjugate in $\mathcal{L}$ and $[r_1] = [r_2]$ as points in $T(\Gamma)$. □

The metric on $T(\Gamma)$ is called the *Teichmüller metric*. It is not difficult to show that this metric induces the same topology on $T(\Gamma)$ as the quotient topology inherited from $R(\Gamma)$ ([27], Theorem 8.6).

Theorem 25. *With respect to the Teichmüller metrics, the embedding* $\bar{i} : T(\Gamma^*) \to T(\Gamma)$ *induced by the injective homomorphism* $i : \Gamma \to \Gamma^*$ *(cf. Theorem 7) is an isometry.*

Proof. Recall that the embedding is defined by $\bar{i} : [r] \mapsto [r \circ i]$. If d, d^* denote the Teichmüller pseudo-metrics on $R(\Gamma)$ and $R(\Gamma^*)$, respectively, and $r_1, r_2 \in R(\Gamma^*)$, then

$$d(r_1 \circ i, r_2 \circ i) = \log |K(t(r_2 \circ i \circ i^{-1} \circ r_1^{-1}))| = d^*(r_1, r_2).$$

It follows that the Teichmüller distance between $[r_1]$ and $[r_2]$ is the same as the Teichmüller distance between $[r_1 \circ i]$ and $[r_2 \circ i]$. □

Corollary 26. *Mod*(Γ) *acts as a group of isometries of* $T(\Gamma)$.

Proof. This is the special case of Theorem 25 in which $\Gamma^* = \Gamma$, $d^* = d$ and $i = \alpha \in \mathrm{Aut}^+(\Gamma)$. □

There is a complex structure on $T(\Gamma)$, compatible with the Teichmüller metric, making $T(\Gamma)$ biholomorphically equivalent to a bounded domain in $\mathbb{C}^{3g-3+n}$ [2]. We shall not need this structure, so we omit the details.

7. Fixed point sets

Let H be a subgroup of Mod(Γ) and let $F(H) \subset T(\Gamma)$ be the set of points fixed by H under the action

$$\mathrm{Mod}(\Gamma) \times T(\Gamma) \to T(\Gamma). \tag{23}$$

Theorem 27 (Nielsen-Kerckhoff). $F(H)$ *is non-empty if and only if* H *is a finite group.*

Proof. H is necessarily finite by Theorem 9 and Lemma 4. That every finite subgroup of Mod(Γ) has a non-empty fixed point set was a long-standing conjecture, known as the *Nielsen realization problem*. It was first proved for restricted classes of groups such as cyclic and solvable, and finally in full generality by S. Kerckhoff [23] using one-parameter families of deformations of hyperbolic structures known as *earthquakes*. □

Lemma 28. *A finite subgroup $H < Mod(\Gamma)$ exists if and only if there is a short exact sequence*

$$1 \to \Gamma \xrightarrow{i} \Gamma^* \xrightarrow{\rho} H \to 1, \tag{24}$$

where $\Gamma^* \leq N_{\mathcal{L}}(i(\Gamma))$.

Proof. Let the sequence (24) be given. The natural homomorphism

$$j : N_{\mathcal{L}}(i(\Gamma)) \to \mathrm{Aut}^+(i(\Gamma)) \simeq \mathrm{Aut}^+(\Gamma) \tag{25}$$

is injective because $i(\Gamma)$ has trivial centralizer in $\mathcal{L}$; the restriction $j|_{i(\Gamma)}$ is an isomorphism between $i(\Gamma)$ and $\mathrm{Inn}(i(\Gamma))$, because $i(\Gamma)$ has trivial center (Corollary 2). Hence j induces an injective homomorphism

$$j_* : \frac{N_{\mathcal{L}}(i(\Gamma))}{i(\Gamma)} \to \mathrm{Mod}(\Gamma). \tag{26}$$

Of course, the restriction of j_* to $\Gamma^*/i(\Gamma)$ is also injective, and the image in $\mathrm{Mod}(\Gamma)$ is isomorphic to H.

Conversely, suppose H is a finite subgroup of $\mathrm{Mod}(\Gamma)$. Let $[\alpha_1], [\alpha_2], \ldots, [\alpha_k]$ be the elements of H. Assume $[\alpha_1] = 1$. By the Nielsen-Kerckhoff theorem there exists $[r] \in T(\Gamma)$ which is fixed by all of the $[\alpha_i]$, i.e., $[r \circ \alpha_i] = [r]$, for $1 \leq i \leq k$. Equivalently, there exist $c_i \in \mathcal{L}$, $1 \leq i \leq k$, such that, for all $\gamma \in \Gamma$, $r(\alpha_i(\gamma)) = c_i^{-1} r(\gamma) c_i$. Each c_i is unique because the centralizer of $r(\Gamma)$ is trivial. In particular, $c_1 = 1$. Let $\Gamma^* \leq N_{\mathcal{L}}(r(\Gamma))$ be the Fuchsian group generated by the set $r(\Gamma) \cup \{c_1, \ldots, c_k\}$. Every element of Γ^* can be written uniquely in the form $c_j r(\gamma)$, $\gamma \in \Gamma$. The group operation is $c_j r(\gamma_1) \cdot c_l r(\gamma_2) = c_j c_l r(\gamma_1)^{c_l} r(\gamma_2)$, where $r(\gamma_1)^{c_l} = c_l^{-1} r(\gamma) c_l$. The map $\rho : \Gamma^* \to H$ defined by by $c_i\{r(\gamma)\} \mapsto [\alpha_i]$, $i = 1, \ldots, k$, for all $\gamma \in \Gamma$, is an epimorphism. Thus there is a short exact sequence of the form (24), except that we have $r(\Gamma)$ in place of Γ. There is no loss of generality, since we may assume that $r = \mathrm{id}_\Gamma$. □

The next theorem characterizes the fixed point set of $H < \mathrm{Mod}(\Gamma)$ as the embedded image of Teichmüller space.

Theorem 29. *If $H < Mod(\Gamma)$ belongs to the short exact sequence* (24)*, then*

$$F(H) = \bar{i}(T(\Gamma^*)), \tag{27}$$

where $\bar{i} : T(\Gamma^*) \to T(\Gamma)$ is the embedding induced by $i : \Gamma \to \Gamma^*$.

Proof. [27], Theorem 9.11. Let $[s] \in \bar{i}(T(\Gamma^*)) \subset T(\Gamma)$. Then there exists $[\sigma] \in T(\Gamma^*)$ such that $s = \sigma \circ i$. For every $[\alpha] \in H$, there is a unique $c \in \Gamma^* \leq N_{\mathcal{L}}(\Gamma) < \mathcal{L}$ such that, for all $\gamma \in \Gamma$, $s(\alpha(\gamma)) = c^{-1} s(\gamma) c$. Thus $[s \circ \alpha] = [s]$. It follows that $s \in F(H)$. Conversely, suppose $[s] \in F(H) \subset T(\Gamma)$. We may assume $s = \mathrm{id}_\Gamma$. Let $\sigma = \mathrm{id}_{\Gamma^*}$. Clearly, $s = \sigma \circ i$ and hence, $[s] \in \bar{i}(T(\Gamma^*))$. □

Remark 1. The embedding $\bar{i} : T(\Gamma^*) \hookrightarrow T(\Gamma)$ can be a surjection even if $i(\Gamma) \neq \Gamma^*$. For this to occur, the dimensions of $T(\Gamma)$ and $T(\Gamma^*)$ (computed using Theorem 23) must be equal. This implies that $1 < \Gamma^*/i(\Gamma) \simeq H < \mathrm{Mod}(\Gamma)$ fixes *every point* in $T(\Gamma)$, so that the action of $\mathrm{Mod}(\Gamma)$ on $T(\Gamma)$ is not effective. The complete list of subgroup pairs $\Gamma < \Gamma^*$ for which the Teichmüller dimensions are equal, together with the corresponding signatures, and the indices of the inclusions, is given in [32] (see also [17]).

Let $\mathrm{Stab}_{\mathrm{Mod}(\Gamma)}(F(H))$ denote the set-wise stabilizer of $F(H)$ in $\mathrm{Mod}(\Gamma)$.

Theorem 30. *$Stab_{Mod(\Gamma)}(F(H)) = N_{Mod(\Gamma)}(H)$*

Proof. [28], §4. If $[\alpha]$ belongs to the normalizer $N_{\mathrm{Mod}(\Gamma)}(H)$ of H, then $[\alpha]F(H) = F(H)$, by a simple calculation. Thus

$$N_{\mathrm{Mod}(\Gamma)}(H) \subseteq \mathrm{Stab}_{\mathrm{Mod}(\Gamma)}(F(H)).$$

Conversely, let $\alpha \in \mathrm{Aut}^+(\Gamma)$, and suppose $[\alpha] \in \mathrm{Stab}_{\mathrm{Mod}(\Gamma)}(F(H))$. We may assume H belongs to the short exact sequence (24). Since $F(H) = \bar{i}(T(\Gamma^*))$, for every $[r] \in T(\Gamma^*)$, there exists $r' \in T(\Gamma^*)$ such that

$$[r \circ i \circ \alpha] = [r' \circ i].$$

This implies the existence of $\beta \in \mathrm{Aut}^+(\Gamma^*)$ such that

$$i \circ \alpha = \beta \circ i.$$

There is a unique element $c \in N_{\mathcal{L}}(\Gamma^*)$ such that

$$\beta(\delta) = c\delta c^{-1},$$

for all $\delta \in \Gamma^*$. In particular, $\alpha(i(\gamma)) = \beta(i(\gamma)) = ci(\gamma)c^{-1}$. It follows that $c \in N_{\mathcal{L}}(i(\Gamma))$ and $j_*(c) = [\alpha] \in \mathrm{Mod}(\Gamma)$, where j_* is the injective homomorphism (26). Then

$$H = j_*(\Gamma^*/i(\Gamma)) = j_*(c\Gamma^* c^{-1}/i(\Gamma)) = [\alpha]H[\alpha]^{-1},$$

which shows that $[\alpha] \in N_{\mathrm{Mod}(\Gamma)}(H)$. □

8. Relative modular groups

The action of $N_{\mathrm{Mod}(\Gamma)}(H)$ on $F(H)$ is properly discontinuous and isometric (being a restriction of (23)). Since $H < N_{\mathrm{Mod}(\Gamma)}(H)$ acts trivially on $F(H)$ by definition, the effective action is by $N_{\mathrm{Mod}(\Gamma)}(H)/H$, or, possibly, a quotient of this group.

Definition 4. *$N_{Mod(\Gamma)}(H)/H$ is called the* relative modular group of Γ^* with respect to $i(\Gamma)$.

There are other ways to define the relative modular group (see, e.g., [14,10]). One useful alternative definition is as a subgroup of $\mathrm{Mod}(\Gamma^*)$.

Lemma 31. *The subgroup*

$$Mod(\Gamma^*, i(\Gamma)) = \{[\beta] \in Mod(\Gamma^*) | \beta(i(\Gamma)) = i(\Gamma)\} \leq Mod(\Gamma^*)$$

is isomorphic to the relative modular group $N_{Mod(\Gamma)}(H)/H$.

Proof. [28], Theorem 10. Let $\mathrm{Aut}^+(\Gamma^*, i(\Gamma)) = \{\beta \in \mathrm{Aut}^+(\Gamma^*) | \beta(i(\Gamma)) = i(\Gamma)\} \leq \mathrm{Aut}^+(\Gamma^*)$. Note that $\mathrm{Inn}(\Gamma^*, i(\Gamma)) = \mathrm{Inn}(i(\Gamma))$ since $i(\Gamma)$ is normal in Γ^*. The restriction homomorphism $f : \mathrm{Aut}^+(\Gamma^*, i(\Gamma)) \to \mathrm{Aut}^+(i(\Gamma))$ defined by $f : \beta \mapsto \beta|_{i(\Gamma)}$ is injective: if $\beta'|_{i(\Gamma)} = \beta|_{i(\Gamma)}$, then, for all $\delta \in \Gamma^*$, $\beta'(\delta)^{-1}\beta(\delta)$ is in the (trivial) centralizer of $\beta(i(\Gamma))$ and therefore $\beta = \beta'$. Let $[\alpha] = [\beta|_{i(\Gamma)}]$. If $[r] \in T(\Gamma^*)$, $[r \circ i] \in \bar{i}(T(\Gamma^*))$ and

$$[\alpha][r \circ i] = [r \circ i \circ \alpha] = [r \circ \beta \circ i] = \bar{i}([r \circ \beta]) \in \bar{i}(T(\Gamma^*)).$$

Thus $[\alpha]$ stabilizes $\bar{i}(T(\Gamma^*))$, and hence, by Theorem 30, $[\alpha] \in N_{\mathrm{Mod}(\Gamma)}(H)$. $\beta \in \mathrm{Inn}(\Gamma^*, i(\Gamma))$ if and only if $[\alpha] \in H$. Thus f induces an injective homomorphism $\tilde{f} : \mathrm{Mod}(\Gamma^*, i(\Gamma)) \to N_{\mathrm{Mod}(\Gamma)}(H)/H$ defined by $[\beta] \to [\alpha]$. On the other hand, if $[\alpha] \in N_{\mathrm{Mod}(\Gamma)}(H)/H$, then $[\alpha]$ stabilizes $F(H)$ and by the proof of Theorem 30 there exists $\beta \in \mathrm{Aut}^+(\Gamma^*)$ such that $\beta|_{i(\Gamma)} = \alpha$. Thus $\beta \in \mathrm{Aut}^+(\Gamma^*, i(\Gamma))$, $\tilde{f}([\beta]) = [\alpha]$, and $\tilde{f}$ is surjective. □

If $i(\Gamma)$ is characteristic in Γ^*, $\mathrm{Mod}(\Gamma^*, i(\Gamma)) = \mathrm{Mod}(\Gamma^*)$; otherwise, the index $[\mathrm{Mod}(\Gamma^*) : \mathrm{Mod}(\Gamma^*, i(\Gamma))]$ is equal to the number of distinct images $[\beta](i(\Gamma))$, $[\beta] \in M(\Gamma^*)$. This number is finite since each $[\beta](i(\Gamma))$ is the kernel of a homomorphism from a finitely generated group (G^*) onto a finite group (H), of which there are just finitely many. (See [14].)

Even the action of the relative modular group

$$\mathrm{Mod}(\Gamma^*, i(\Gamma)) \times \bar{i}(T(\Gamma^*)) \to \bar{i}(T(\Gamma^*)), \tag{28}$$

is not always effective. Let H_1/H be the largest subgroup of $N_{\mathrm{Mod}(\Gamma)}(H)/H$ which fixes every point in $F(H)$. Then the effective action is by the group

$$\frac{N_{\mathrm{Mod}(\Gamma)}(H)/H}{H_1/H} \simeq N_{\mathrm{Mod}(\Gamma)}(H)/H_1. \tag{29}$$

9. Topological versus conformal conjugacy

The orbit space

$$\mathcal{R}(\Gamma^*, i(\Gamma)) = \bar{i}(T(\Gamma^*))/\mathrm{Mod}(\Gamma^*, i(\Gamma))$$

of the action (28) is called the *relative Riemann space* of $(\Gamma^*, i(\Gamma))$ [10], [14]. If $i(\Gamma) = \Gamma^*$ it is called simply the *Riemann space of* Γ. The Riemann space of Λ_g (a surface group) is the *space of moduli* of compact Riemann surfaces of genus g, denoted $\mathcal{R}_g$. Of course, one obtains the same relative Riemann space if one replaces $\mathrm{Mod}(\Gamma^*, i(\Gamma))$ with the *effective* relative modular group (29).

Let H_0 be a finite group. A *surface with* H_0*-symmetry*, or, briefly, a *surface-symmetry pair*, is a pair (X, H) in which X is a compact Riemann surface and $H \simeq H_0$ is a finite group of automorphisms of X. It is convenient to define two equivalence relations on the set of surfaces with H_0 symmetry.

Definition 5. *Surfaces with H_0 symmetry* (X_1, H_1) *and* (X_2, H_2) *are* conformally equivalent *(resp.* topologically equivalent*) if there exists a conformal map (resp. orientation-preserving homeomorphism) $t : X_1 \to X_2$ such that $H_1 = tH_2t^{-1}$. The H_i-actions are called* conformally *(resp.* topologically*) conjugate.*

Theorem 32. *If H belongs to the short exact sequence* (24)*, there is a bijection between $\mathcal{R}(\Gamma^*, i(\Gamma))$ and the set of conformal equivalence classes of surfaces with H-symmetry.*

Proof. Let $[r_1], [r_2] \in \bar{i}(T(\Gamma^*) \subset T(\Gamma)$. Then there exist $[r'_j] \in T(\Gamma^*)$, $j = 1, 2$ such that $[r'_j] = [r_j \circ i]$. Put $X_j = U/r_j(\Gamma)$, and

$$H_j = r'_j(\Gamma^*)/r_j(\Gamma) \simeq H, \quad j = 1, 2. \tag{30}$$

The H_j action on X_j is defined by

$$r'_j(\gamma)r_j(\Gamma) : \; r_j(\Gamma)z \;\mapsto\; r'_j(\gamma)r_j(\Gamma)z, \quad \gamma \in \Gamma^*, z \in U. \tag{31}$$

If $[r_1], [r_2]$ are in the same $\mathrm{Mod}(\Gamma^*, i(\Gamma))$- orbit, there exists $\beta \in \mathrm{Aut}^+(\Gamma^*, i(\Gamma))$ such that $[r'_1 \circ \beta] = [r'_2]$. Moreover, β restricts to $\alpha \in \mathrm{Aut}^+(\Gamma)$ such that $[r_1 \circ \alpha] = [r_2]$. Equivalently, there exists $c \in \mathcal{L}$ such that

$$r'_1(\beta(\gamma)) = cr'_2(\gamma)c^{-1}, \quad \gamma \in \Gamma^*. \tag{32}$$

In particular, $r_1(\Gamma) = cr_2(\Gamma)c^{-1}$. The map

$$r_1(\Gamma)z \mapsto c^{-1}r_1(\Gamma)z, \quad z \in U,$$

induces a conformal map $\tilde{c} : X_1 \to X_2$, since by (32), $c^{-1}r_1(\Gamma)z = r_2(\Gamma)c^{-1}z$.

For $\gamma \in \Gamma^*$, let $h_i(\gamma) : X_i \to X_i$ denote the map (31). To show that $\tilde{c}$ commutes with the H_j-actions, we show that the diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{h_1(\gamma)} & X_1 \\ \tilde{c} \downarrow & & \tilde{c} \downarrow \\ X_2 & \xrightarrow{h_2(\gamma)} & X_2 \end{array} \tag{33}$$

commutes. For $z \in U$, let $p_z = r_1(\Gamma)z \in X_1$. On the one hand,

$$\tilde{c} \circ h_1(\gamma)(p_z) = r_2(\Gamma)c^{-1}r'_1(\gamma)z, \tag{34}$$

and on the other,

$$h_2(\gamma) \circ \tilde{c}(p_z) = r_2(\Gamma)r'_2(\gamma)c^{-1}z; \tag{35}$$

the diagram commutes if and only if the right-hand sides of (34) and (35) represent the same point in X_2. Using (32), the right -hand side of (34) is equal to

$$r_2(\Gamma)r'_2(\beta^{-1}(\gamma))c^{-1}z.$$

Since $r'_2(\beta^{-1}(\gamma))c^{-1}$ and $r'_2(\gamma)c^{-1}$ represent the same coset of $r_2(\Gamma)$, it follows that the right-hand sides of (34) and (35) both represent the the same point in X_2, namely, $r_2(\Gamma)c^{-1}z$.

We have shown that there is a one-to-one map from the relative Riemann space $\mathcal{R}(\Gamma^*, \Gamma)$ to the set of conformal equivalence classes of surfaces with H symmetry. The proof that this map is surjective is straightforward and we omit it. $\square$

Suppose $i_j : \Gamma \to \Gamma^*$, $j = 1, 2$, are two different injections of Γ into Γ^* such that the image $i_j(\Gamma)$ is normal in Γ^*. Let $[r_1] \in \bar{i}_1(T(\Gamma^*)$ and $[r_2] \in \bar{i}_2(T(\Gamma^*)$. There exist $[r'_j] \in T(\Gamma^*)$, $j = 1, 2$ such that $[r_j] = [r'_j \circ i_j]$, $j = 1, 2$. Put $X_j = U/r_j(\Gamma)$ and define H_j, and H_j-actions on X_j, exactly as at (30), (31), respectively. The isomorphism

$$r'_2 \circ r'^{-1}_1 : r'_1(\Gamma^*) \to r'_2(\Gamma^*)$$

has a quasiconformal geometrical realization $t : U \to U$ by Theorem 20. We may take t to be the unique Teichmüller map in $\Omega(r'_2 \circ r'^{-1}_1, \Gamma_1)$. t induces an orientation-preserving homeomorphism $\tilde{t} : X_1 \to X_2$ which (in place of $\tilde{c}$) satisfies the diagram (33). Thus the surface-symmetry pairs (X_1, H_1) and (X_2, H_2) are topologically conjugate.

One way to obtain distinct injections $i_j : \Gamma \to \Gamma^*$, $j = 1, 2$ is to precompose i_1 with $\beta \in \mathrm{Aut}^+(\Gamma)$, obtaining $i_2 = i_1 \circ \beta$. Then $[r_1] = [r_2 \circ \beta]$, i.e., $[r_1], [r_2]$ are in the same $\mathrm{Mod}(\Gamma)$- orbit and the surfaces X_1 and X_2 are conformally equivalent. The surface-symmetry pairs (X_j, H_j), $j = 1, 2$, however, need not be, since β need not extend to an automorphism of G^*.

Under the action of $\mathrm{Mod}(\Gamma)$, the images of an embedded Teichmüller space $\bar{i}(T(\Gamma^*)) = F(H) \subset T(\Gamma)$ can intersect without coinciding (provided the embedded space has positive Teichmüller dimension). In this case, we shall argue, the intersection is composed of embedded Teichmüller spaces of strictly smaller dimension.

Let $[\beta] \in \mathrm{Mod}(\Gamma)$, $[r_1], [r_2] \in F(H) \cap [\beta]F(H)$, $[r_1] \neq [r_2]$. Suppose $[r_2] = [\beta][r_1]$, so that $[r_1]$ and $[r_2]$ project to the same point in $X \in \mathcal{R}(\Gamma)$. If $[r_1]$ and $[r_2]$ are not in the same $\mathrm{Mod}(\Gamma^*, i(\Gamma))$-orbit, i.e., if they determine distinct points in the relative Riemann space $\mathcal{R}(\Gamma^*, i(\Gamma))$, the surface-symmetry pairs (X, H_1) and (X, H_2) are not conformally equivalent. Thus $X \in \mathcal{R}(\Gamma)$ is a surface admitting two H-actions which are topologically but not conformally conjugate. Equivalently, the full automorphism group $G = \mathrm{Aut}(X)$ contains two *non-conjugate* copies of H_1. The inclusion $i : \Gamma \to \Gamma^*$ extends to

$$\Gamma \xrightarrow{i} \Gamma^* \xrightarrow{j} \Gamma_1, \tag{36}$$

where j is a strict normal inclusion, $j \circ i : \Gamma \to \Gamma_1$ is also a normal inclusion, and $G \simeq \Gamma_1/\Gamma$. Then

$$F(G) = \bar{i} \circ \bar{j}(T(\Gamma_1)) \subseteq F(H) \cap [\beta]F(H).$$

If the dimension of $F(G)$ is positive, and there are $\mathrm{Mod}(\Gamma)$-images of $F(G)$ which intersect without coinciding, the procedure just described can be repeated beginning with $F(G)$ instead of $F(H)$. After finitely many repetitions the procedure must terminate: we reach a finite group $G' > G$ such that all $\mathrm{Mod}(\Gamma)$-images of $F(G')$ are either disjoint or identical. Then topological conjugacy implies conformal conjugacy and there is a bijection between the relative Riemann space and the image of $F(G')$ in $\mathcal{R}(\Gamma)$. This is always the case if the Teichmüller dimension of $F(G')$ is 0.

For further discussion of these matters from an algebraic geometric view point, see [16].

10. Stratification

Let $\mathrm{Mod}_g = \mathrm{Mod}(\Lambda_g)$. Let (H) denote the conjugacy class of a finite subgroup $H < \mathrm{Mod}_g$. There is a bijection between the set

$$\{(H) | H < \mathrm{Mod}_g,\, H \text{ non-trivial, finite}\}$$

and the set of topological equivalence classes of surface-symmetry pairs (X, H), where $X \in \mathcal{R}_g$ is a surface of genus g (see, e.g. [31]).

For $X \in \mathcal{R}_g$, let $\Sigma(X)$ denote the conjugacy class $(\mathrm{Aut}(X))$ in Mod_g. We call $\Sigma(X)$ the *symmetry type* of X. The (H)*-equisymmetric stratum* of $\mathcal{R}_g$ is the set

$$\mathcal{R}_g^{(H)} = \{X \in \mathcal{R}_g | \Sigma(X) \geq (H)\} \tag{37}$$

(the terminology is due to S. A. Broughton [6]). The *stratification* of $\mathcal{R}_g$ is the union of the equisymmetry strata over all conjugacy classes of non-trivial finite subgroups of Mod_g. The stratification is the image of the *branch locus* in $T(\Lambda_g) = \mathcal{T}_g$ under the action of Mod_g and is thus also the locus of surfaces with automorphisms. Each stratum is the image of an embedded Teichmüller space in $\mathcal{T}_g$. One can obtain the stratification by considering all normal inclusions of the form $i : \Lambda_g \to \Gamma^*$ where Λ_g is fixed and Γ^* varies over all co-compact Fuchsian groups containing Λ_g as a normal subgroup of finite index, and i varies over all normal inclusions $i : \Lambda_g \to \Gamma^*$. From this it is clear that the number of strata is finite: there are finitely many Γ^* such that the ratio $\chi(\Lambda_g)/\chi(\Gamma^*)$ is an integer, as required by the Riemann-Hurwtiz relation (Lemma 3), and each such Γ^*, being finitely generated, admits finitely many homomorphisms with kernel isomorphic to Λ_g.

It can happen that $\mathcal{R}_g^{(G)} = \mathcal{R}_g^{(H)}$. This occurs when $G \supset H$ and the dimensions of the covering embedded Teichmüller spaces are equal (cf. Remark 1 following Theorem 29). In this case, H is not the full automorphism group of any surface in $\mathcal{R}_g^{(H)}$. Naturally, it is preferable to associate the largest possible group with a given stratum.

It is possible to make the strata disjoint, and avoid the sort of duplication just described (allowing some empty strata), by defining

$$\tilde{\mathcal{R}}_g^{(H)} = \mathcal{R}_g^{(H)} - \bigcup_{G \supset H} \mathcal{R}_g^{(G)}.$$

On the other hand, for some purposes, it is precisely the intersection of different strata which are of most interest: the intersection and nesting relationships between the strata echo those of the covering branch locus.

The strata (37) are known to be irreducible complex algebraic varieties ([4], §7).

11. Examples

1. The equisymmetric stratification of $\mathcal{R}_2$ is simple enough to describe in words: $\mathcal{R}_2$ itself has complex dimension 3. It contains a 2-dimensional stratum $\mathcal{R}_2^{(D_2)}$, where D_2 is the Klein 4-group. This stratum contains two 1-dimensional substrata associated with the dihedral group D_4 of order 8 and the product $C_2 \times S_3$ (cyclic by symmetric). These two strata, in turn, intersect in two distinct 0-dimensional strata, one corresponding to the group $GL_2(F_3)$ of order 48 and the other to a certain group of order 24. Finally, there is an isolated 0-dimensional stratum, not contained in $\mathcal{R}_2^{(D_2)}$, associated with the cyclic group C_{10}. (See [6,26], and also, [25].)
2. For all $g > 2$, $\mathcal{R}_g$ itself is properly regarded as the stratum associated with the trivial group, but this is not the case in genus 2. In fact $\mathcal{R}_2 = \mathcal{R}_2^{(C_2)}$. The explanation is that all surfaces of genus 2 admit the hyperelliptic involution. Equivalently: there is a normal inclusion $i : \Lambda_2 \to \Gamma^*$, where Γ^* is the group with signature $(0; 2, 2, 2, 2, 2, 2)$. It is easily verified that the dimension of the Teichmüller space of Γ^* is equal to the dimension of $\mathcal{T}_2$. Thus M_2 acts ineffectively on $\mathcal{T}_2$ because $\Gamma^*/\Lambda_2 \simeq C_2 < \mathrm{Mod}_2$ fixes every point.
3. The equisymmetric stratification of $\mathcal{R}_3$ can be constructed from the information in [6]; see the author's Ph.D. dissertation [34] for a (complicated) picture.
4. Let $\Gamma^{(n)}$ denote the Fuchsian group with signature $(n; 2, \ldots, 2)$, where the number of 2's is $2g + 2 - 4n$. For $g \geq 2n - 1$ and $g > 1$, there is a short exact sequence

$$1 \to \Lambda_g \xrightarrow{i} \Gamma^{(n)} \to C_2 \to 1. \tag{38}$$

Let $\Gamma^{(n)}/\Lambda_g = \langle J_n | J_n^2 = 1\rangle \simeq C_2$. J_n is known as an n-hyperelliptic involution since it acts on the surface U/Λ_g with quotient a surface of genus n. (J_0 is the classical hyperelliptic involution.) A short argument shows that all surface-symmetry pairs $(U/\Lambda_g, \langle J_n\rangle)$ are topologically conjugate. By a theorem of Farkas and Kra [11], §V.1.9, if $g > 4n + 1$, J_n is unique (hence central) in $\mathrm{Aut}(U/\Lambda_g)$. Restated in terms of Fuchsian groups, this theorem says that whenever there is a sequence of injective homomorphisms

$$\Lambda_g \xrightarrow{i} \Gamma^{(n)} \xrightarrow{j} N_{\mathcal{L}}(\Lambda_g), \tag{39}$$

the image $j(\Gamma^{(n)})$ is normal. (The image $i(\Lambda_g)$ is normal in $\Gamma^{(n)}$ because the index is 2.) Hence any automorphism group Γ^*/Λ_g of U/Λ_g which contains $\Gamma^{(n)}/\Lambda_g = \langle J_n\rangle$, contains it as a normal subgroup, so J_n is a central element. As a consequence, all pairs $(U/\Lambda_g, \langle J_n\rangle)$ are conformally conjugate.

In the intermediate range $(g - 1)/4 \leq n < (g + 1)/2$, the theorem of Farkas and Kra does not hold and there may be $\langle J_n\rangle$ actions that are topologically but not conformally conjugate. We show this is the case when $g = 3$ and $n = 1$. In genus 3 the Klein 4-group $D_2 = \langle J_0, J_1\rangle$ acts with signature $(0; 2, 2, 2, 2, 2, 2)$. The hyperelliptic involution J_0 is unique, so the other two nontrivial elements in D_2, namely J_1 and $J_0 J_1$, acting with signature $(1; 2, 2, 2, 2)$, are a pair of non-conjugate 1-hyperelliptic involutions.

5. (4., continued.) The theorem of Farkas and Kra has another consequence: if $g > 4n+1$, and Λ_g is contained in $\Gamma^{(n)}$, then $N_{\mathcal{L}}(\Lambda_g) \leq N_{\mathcal{L}}(\Gamma^{(n)})$. A finite subgroup $H < \mathrm{Mod}_g$ may be represented as Γ^*/Λ_g, $\Gamma^* \leq N_{\mathcal{L}}(\Lambda_g)$. By assumption, G^*/Λ_g contains $\Gamma^{(n)}/\Lambda_g = \langle J_n \rangle$ as a normal subgroup. It follows that every finite subgroup $H < \mathrm{Mod}_g$ stabilizes the fixed point set $F(\langle J_n \rangle) \subset \mathcal{T}_g$. In other words, the embedded Teichmüller space
$$\bar{i}(T(\Gamma^{(n)}) \subset \mathcal{T}_g,$$
where i is the injective homomorphism in (38), is invariant under the action of Mod_g. The n-hyperelliptic stratum $\mathcal{R}_g^{(\langle J_n \rangle)}$ is therefore disjoint from all other strata except those contained in it. This is an advantage: to determine the substrata of $\mathcal{R}_g^{(\langle J_n \rangle)}$ one need only consider the (finite) class of finite groups which are central extensions of C_2 by groups which act in genus n. See the author's paper [35] for the case $n = 0$.

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